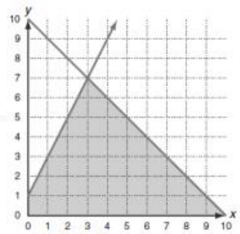
2.7: *Linear Programming*

Maximize or minimize each objective function.

1. Maximize $P = 5x + 2y$

for the constraints $\begin{cases} y \geq 0 \\ x \geq 0 \\ y \leq -x + 10 \\ y \leq 2x + 1 \end{cases}$



(10, 0)

Max. value = 50

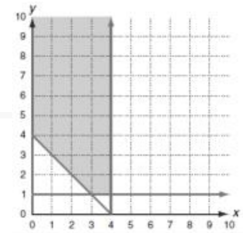
1

2.7: *Linear Programming*

Maximize or minimize each objective function.

2. Minimize $P = 4x + 6y$

for the constraints $\begin{cases} 0 \leq x \leq 4 \\ y \geq 1 \\ y \geq -x + 4 \end{cases}$



(3, 1)

Min. value = 18

2

2.7: *Linear Programming*

3. A grocer buys cases of almonds and walnuts. Almonds are packaged 20 bags per case. The grocer pays \$30 per case of almonds and makes a profit of \$17 per case. Walnuts are packaged 24 bags per case. The grocer pays \$26 per case of walnuts and makes a profit of \$15 per case. He orders no more than 300 bags of almonds and walnuts together at a maximum cost of \$400.

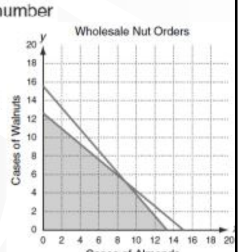
a. Write the constraints. Use x for the number of cases of almonds ordered and y for the number of cases of walnuts ordered.

b. Graph the constraints.

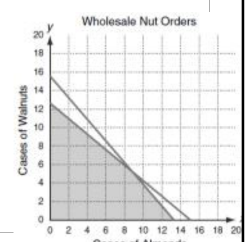
c. Write the objective function for the profit.

$P = 17x + 15y$

$\begin{cases} x \geq 0 \\ y \geq 0 \\ 20x + 24y \leq 300 \\ 30x + 26y \leq 400 \end{cases}$



2.7: *Linear Programming*



2.7: *Linear Programming*

3. A grocer buys cases of almonds and walnuts. Almonds are packaged 20 bags per case. The grocer pays \$30 per case of almonds and makes a profit of \$17 per case. Walnuts are packaged 24 bags per case. The grocer pays \$26 per case of walnuts and makes a profit of \$15 per case. He orders no more than 300 bags of almonds and walnuts together at a maximum cost of \$400.

d. How many cases of almonds and walnuts maximize the grocer's profit?

9 cases of almonds, 5 cases of walnuts

5

2.7: Linear Programming

Linear programming is a strategy for finding the **optimum** value – either ***maximum*** or ***minimum*** - of a linear function that is subject to certain constraints.

These **constraints**, or restrictions, are stated as a system of linear inequalities.

Example: Find the maximum value of ***z***, given:

$$\boldsymbol{z} = 2x + 3y \quad and \quad \begin{cases} 2x+5y \le 25 \\ 3x+2y \le 21 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

objective function

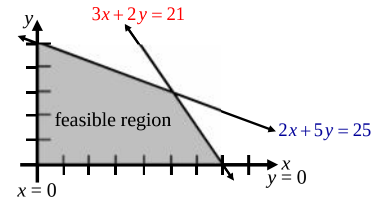
constraints

Example continued

7

The system of linear inequalities determines a set of **feasible solutions**. The graph of this set is the **feasible region**.

Example continued: Graph the feasible region determined by the system $\begin{cases} 2x+5y \le 25 \\ 3x+2y \le 21 \\ x \ge 0 \\ y \ge 0 \end{cases}$ of constraints.

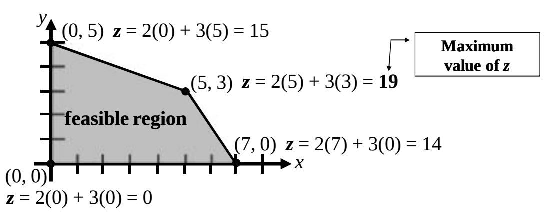


Example continued

8

If a linear programming problem has a solution, then the solution is at a *vertex* of the feasible region.

Example continued: Maximize the value of $\boldsymbol{z} = 2x + 3y$ over the feasible region.



Test the value of ***z*** at each of the vertices. The maximum value of ***z*** is **19**. This occurs at the point (5, 3) or when $x = 5$ and $y = 3$.

9

Solving Linear Programming Problems Graphically

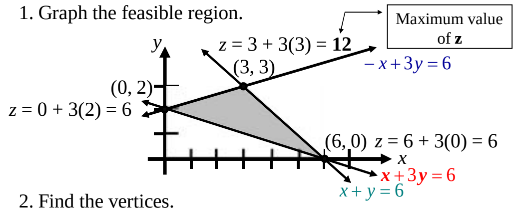
1. Graph the feasible region.
2. Find the vertices of the region.
3. Evaluate the objective function at each vertex.
4. Select the vertices that optimize the objective function.
   a) If the feasible region is ***bounded*** the objective function will have both a maximum and a minimum.
   b) If the feasible region is ***unbounded*** and the objective function has an optimal value, the optimal value will occur at a vertex of the feasible region.

Note: If an objective function has a maximum or minimum value, it ***MUST*** occur at one or more of the vertices of the feasible region.

10

Example: Find the maximum and minimum value of $z = x + 3y$ subject to the constraints $-x + 3y \le 6$, $x + 3y \ge 6$, and $x + y \le 6$.

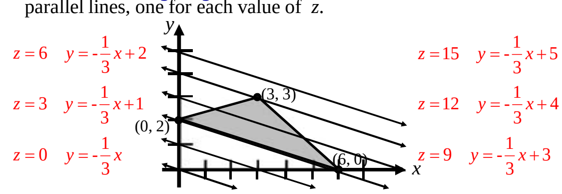
1. Graph the feasible region.



2. Find the vertices.
3. Evaluate the objective function at each vertex.
4. The maximum value of z is 12 and occurs at (3, 3). The minimum value of z is 6 and occurs at both (0, 2) and (6, 0) and at every point along the line joining them.

11

Rewriting the objective function $z = x + 3y$ in slope-intercept form gives $y = -\frac{1}{3}x + \frac{z}{3}$. This equation represents a family of parallel lines, one for each value of z.



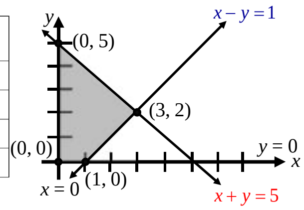
As z increases through values 0, 3, 6, 9, 12, and 15, the corresponding line passes through the feasible region.

The point at which the family of lines first meets the feasible region gives the minimum value of z, and the point at which the family of lines leaves the feasible region gives the maximum.

12

Example: Minimize $z = 3x - y$ subject to
$x - y \le 1$, $x + y \le 5$, $x \ge 0$, and $y \ge 0$.

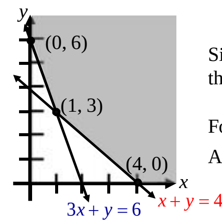
vertex	value of z at vertex
(0, 0)	$z = 3(0) - (0) = 0$
(1, 0)	$z = 3(1) - (0) = 3$
(3, 2)	$z = 3(3) - (2) = 7$
(0, 5)	$z = 3(0) - (5) = -5$



The minimum value of z is –5 and this occurs at (0, 5).

13

Example: **Maximize** $z = 2x + y$ subject to $3x + y \ge 6$, $x + y \ge 4$, $x \ge 0$, and $y \ge 0$.



Since the feasible region is unbounded there may be no maximum value of z.

For $x \ge 4$, $(x, 0)$ is a feasible solution. At $(x, 0)$, $z = 2x$.

Therefore as x increases without bound, z increases without bound and there is *no maximum value* of z.

14

Example: Sarah makes bracelets and necklaces to sell at a craft store. Each bracelet makes a profit of \$7, takes 1 hour to assemble, and costs \$2 for materials. Each necklace makes a profit of \$12, takes 2 hour to assemble, and costs \$3 for materials.

Sarah has 48 hours available to assemble bracelets and necklaces. If she has \$78 available to pay for materials, how many bracelets and necklaces should she make to maximize her profit?

To formulate this as a linear programming problem:

1. Identify the variables.
2. Write the objective function.
3. Write the constraints.

Example continued

15

Example continued:

1. Let x = the number of bracelets Sarah makes
   Let y = the number of necklaces Sarah makes

2. Express the profit as a function of x and y.

   $p = 7x + 12y$ ⟷ Function to be maximized

3. Express the constraints as inequalities.

   Cost of materials: $2x + 3y \le 78$.
   Time limitation: $x + 2y \le 48$.

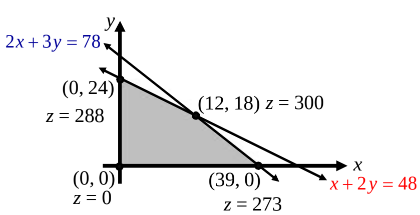
Since Sarah cannot make a negative number of bracelets or necklaces, $x \ge 0$ and $y \ge 0$ must also hold.

Example continued

16

Example continued:

Maximize $p = 7x + 12y$ subject to the constraints
$2x + 3y \le 78$, $x + 2y \le 48$, $x \ge 0$, and $y \ge 0$.



Sarah should make 12 bracelets and 18 necklaces for a maximum profit of \$300.

17