3.1: Symmetry and Coordinate Graphs

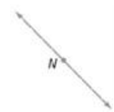
Essential Questions:

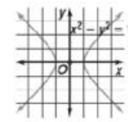
- How do we determine symmetry using algebra?
- How do we classify functions as even or odd?

Point Symmetry

Two distinct points P and P' are symmetric with respect to point M if and only if M is the midpoint of $\overline{PP'}$. Point M is symmetric with respect to itself.

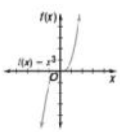
- Each point ***P*** in the set must have an ***image point P'*** that is also in the set. A figure that is symmetric with respect to a given point can be **rotated 180°** about that point and appear unchanged.





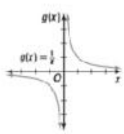
Point Symmetry

- The origin is a common point of symmetry.



$f(x) = x^3$			
x	$f(x)$	$f(-x)$	$-f(x)$
1	1	−1	−1
2	8	−8	−8
3	27	−27	−27
4	64	−64	−64

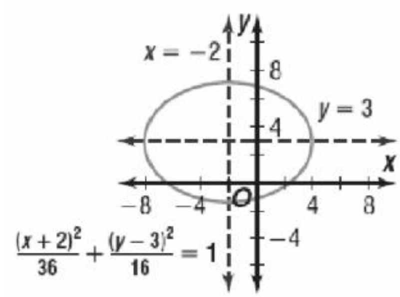
Note that $f(-x) = -f(x)$.



$g(x) = \frac{1}{x}$			
x	$g(x)$	$g(-x)$	$-g(x)$
1	1	−1	−1
2	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
3	$\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
4	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$

Note that $g(-x) = -g(x)$.

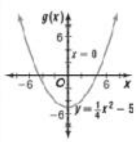
Line Symmetry

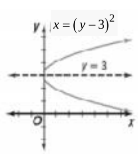


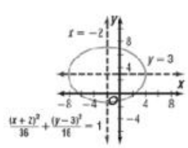
Line Symmetry

Two distinct points P and P' are symmetric with respect to a line ℓ if and only if ℓ is the perpendicular bisector of $\overline{PP'}$. A point P is symmetric to itself with respect to line ℓ if and only if P is on ℓ.

- Each graph below has line symmetry. The equation of each line of symmetry is given. Graphs that have line symmetry **can be folded along the line of symmetry so that the two halves match exactly**.







Symmetry

x-axis

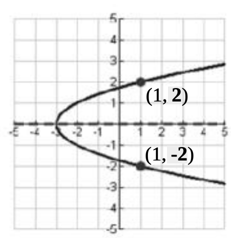
What we do: keep x the same but negate y

$$x = y^2 - 3$$

$$x = (-y)^2 - 3$$

$$x = y^2 - 3$$

Equals original function



Symmetry
y-axis

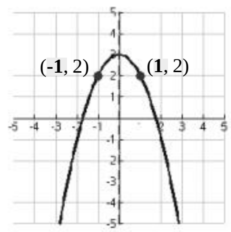
What we do: keep y the same but negate x

$$\boxed{y=-x^2+3}$$

$$\Rightarrow y=-(-x)^2+3$$

$$y=-x^2+3$$

Equals original function



Symmetry
y = *x*

What we do: interchange (swap) x and y

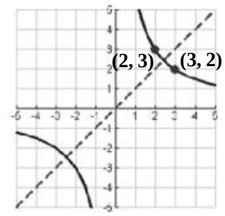
$$\boxed{y=\frac{6}{x}}$$

$$\Rightarrow x=\frac{6}{y}$$

$$xy=6$$

$$y=\frac{6}{x}$$

Equals original function



Symmetry
y = -*x*

What we do: interchange AND negate x and y

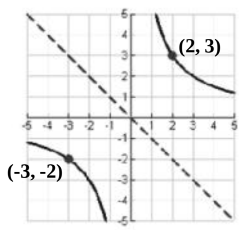
$$\boxed{y=\frac{6}{x}}$$

$$\Rightarrow -x=\frac{6}{-y}$$

$$xy=6$$

$$y=\frac{6}{x}$$

Equals original function



Symmetry
origin

What we do: negate x AND negate y

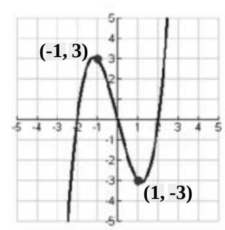
$$\boxed{y=x^3-4x}$$

$$\Rightarrow (-y)=(-x)^3-4(-x)$$

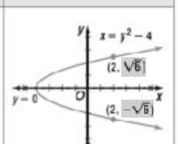
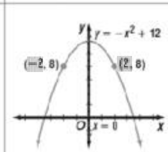
$$-y=-x^3+4x$$

$$y=x^3-4x$$

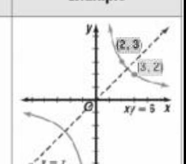
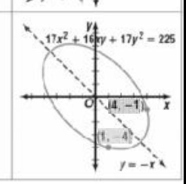
Equals original function



Line Symmetry

Symmetry with Respect to the:	Definition and Test	Example
***x*-axis** ***Rule*:** **(x, y) ➡ (x, -y)**	$(a, -b) \in S$ if and only if $(a, b) \in S$. *Example:* $(2, \sqrt{6})$ and $(2, -\sqrt{6})$ are on the graph. *Test:* Substituting (a, b) and $(a, -b)$ into the equation produces equivalent equations.	
***y*-axis** ***Rule*:** **(x, y) ➡ (-x, y)**	$(-a, b) \in S$ if and only if $(a, b) \in S$. *Example:* (2, 8) and (−2, 8) are on the graph. *Test:* Substituting (a, b) and $(-a, b)$ into the equation produces equivalent equations.	

Line Symmetry

Symmetry with Respect to the Line:	Definition and Test	Example
y* = *x ***Rule*:** **(x, y) ➡ (y, x)**	$(b, a) \in S$ if and only if $(a, b) \in S$. *Example:* (2, 3) and (3, 2) are on the graph. *Test:* Substituting (a, b) and (b, a) into the equation produces equivalent equations. c	
y* = -*x ***Rule*:** **(x, y) ➡ (-y, -x)**	$(-b, -a) \in S$ if and only if $(a, b) \in S$. *Example:* (4, −1) and (1, −4) are on the graph. *Test:* Substituting (a, b) and $(-b, -a)$ into the equation produces equivalent equations.	

Common types of symmetry are:

- with respect to the ***x*-axis**
- with respect to the ***y*-axis**
- with respect to the **origin**
- with respect to the line ***y = x***
- with respect to the line ***y = -x***

Table of Symmetric Relationships

Symmetry with respect to:	What do we do?	Example
***x*-axis**	**keep *x* the same but negate *y***	
***y*-axis**	**Keep *y* the same but negate *y***	
y = x	**Interchange *x* and *y***	
y = -x	**Interchange AND negate *x* and *y***	
origin	**Negate *x* and negate *y***	

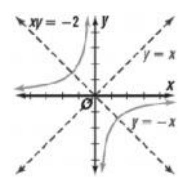
Test the symmetry

with respect to:	*x* (*x*, -*y*)	*y* (-*x*, *y*)	*y = x* (*y*, *x*)	*y = -x* (-*y*, -*x*)	*Origin* (-*x*, -*y*)
1) $y = 3x^2 + 4$					
2) $5x^2 - 6y^2 = 8$					
3) $x^3 + y^2 = 4$					
4) $xy = -2$					
5) $y^2 = \frac{4x^2}{9} - 4$					

Example 1

Determine the types of symmetry for the graph of ***xy* = -2**

- x-axis — No
- y-axis — No
- origin — Yes
- y = x — Yes
- y = -x — Yes



Example 2

Determine the types of symmetry for the graph of $y^2 = \frac{4x^2}{9} - 4$

- x-axis — yes
- y-axis — yes
- origin — yes
- y = x — no
- y = -x — no

Even and Odd Functions

A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and

$$f(-x) = f(x)$$

For an **even** function, for every point (***x*, *y***) on the graph, the point **(-*x*, *y*)** is also on the graph.

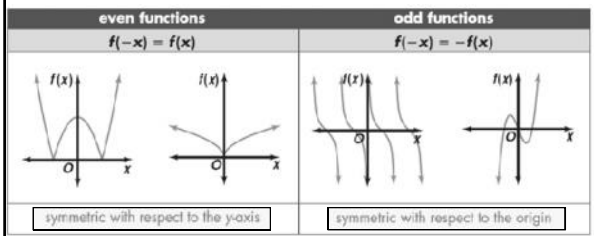
Even and Odd Functions

A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and

$$f(-x) = -f(x)$$

For an **odd** function, for every point (***x, y***) on the graph, the point (***-x, -y***) is also on the graph.

Even and Odd Functions



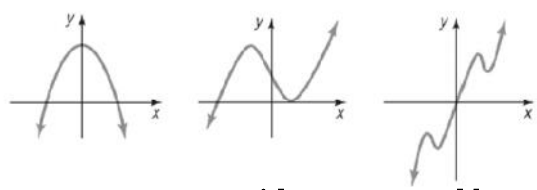
Classifying Functions as **Even** or **Odd**

- **EVEN** functions are symmetric with respect to the **y-axis**
- **ODD** functions are symmetric with respect to the **origin**

Even and Odd Functions

EXAMPLE

Determine whether each graph given is an even function, an odd function, or a function that is neither even nor odd.



even **neither** **odd**

Even and Odd Functions

EXAMPLE

Identifying Even and Odd Functions

Use a graphing utility to conjecture whether each of the following functions is even, odd, or neither. Verify the conjecture algebraically. Then state whether the graph is symmetric with respect to the y-axis or with respect to the origin.

1) $f(x) = -3x^4 - x^2 + 2$ **even; *y*-axis**

2) $g(x) = 5x^3 - 1$ **neither**

3) $h(x) = 2x^3 - x$ **odd; origin**

Determine whether the graph of each equation is symmetric with respect to the *x*-axis, the *y*-axis, the line $y = x$, the line $y = -x$, or none of these.

1. $4x^2 + 9y^2 = 36$

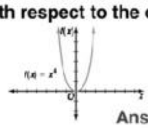
Answer: x-axis, y-axis, origin

2. $2x + \frac{y}{3} = 4$

Answer: none of these

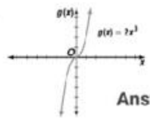
Determine whether each graph is symmetric with respect to the origin.

3.



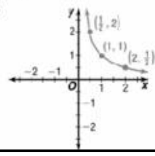
Answer: no

4.

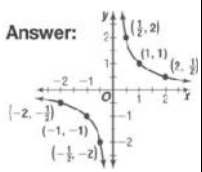


Answer: yes

5. Complete the graph so that it is the graph of an odd function.



Answer:



CW 3.1

DO: evens 14-26, 32-36, p. 134

Exercises

Determine whether the graph of each function is symmetric with respect to the origin.

6. $f(x) = x^6 - 9x$ **7.** $f(x) = \frac{1}{5x} - x^{19}$

Determine whether the graph of each equation is symmetric with respect to the *x*-axis, *y*-axis, the line $y = x$, the line $y = -x$, or none of these.

8. $6x^2 = y - 1$ **9.** $x^3 + y^3 = 4$

Determine whether the graph of each equation is symmetric with respect to the *x*-axis, the *y*-axis, both, or neither. Use the information about symmetry to graph the relation.

11. $y = \sqrt{2 - x^2}$ **12.** $|y| = x^3$

Exercises

Determine whether the graph of each function is symmetric with respect to the origin.

14. $f(x) = 3x$ **15.** $f(x) = x^3 - 1$ **16.** $f(x) = 5x^2 + 6x + 9$

17. $f(x) = \frac{1}{4x^2}$ **18.** $f(x) = -7x^5 + 8x$ **19.** $f(x) = \frac{1}{x} - x^{100}$

20. Is the graph of $g(x) = \frac{x^2 - 1}{x}$ symmetric with respect to the origin? Explain how you determined your answer.

Determine whether the graph of each equation is symmetric with respect to the *x*-axis, *y*-axis, the line $y = x$, the line $y = -x$, or none of these.

21. $xy = -5$ **22.** $x + y^2 = 1$ **23.** $y = -8x$

24. $y = \frac{1}{x^2}$ **25.** $x^2 + y^2 = 4$ **26.** $y^2 = \frac{4x^2}{9} - 4$

27. Which line(s) are lines of symmetry for the graph of $x^2 = \frac{1}{y^2}$?

CW 3.1 DO: evens 14-26, 32-36, p. 134

Exercises

Determine whether the graph of each equation is symmetric with respect to the *x*-axis, the *y*-axis, both, or neither. Use the information about symmetry to graph the relation.

31. $y^2 = x^2$ **32.** $|x| = -3y$ **33.** $y^2 + 3x = 0$

34. $|y| = 2x^2$ **35.** $x = \pm\sqrt{12 - 8y^2}$ **36.** $|y| = xy$

37. Graph the equation $|y| = x^3 - x$ using information about the symmetry of the graph.

Exercises

Determine whether the graph of each function is symmetric with respect to the origin.

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