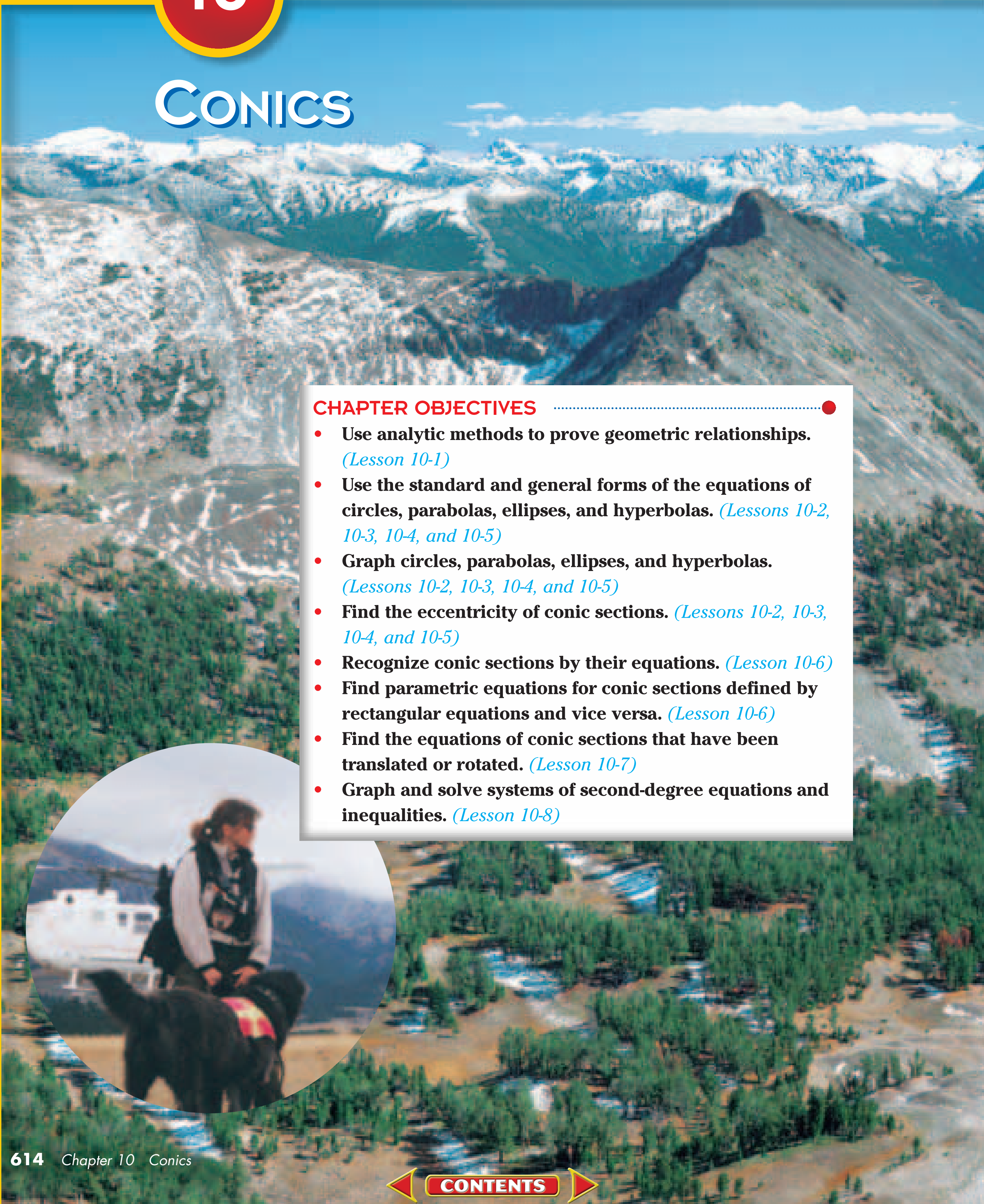
Chapter 10

CONICS



CHAPTER OBJECTIVES

- **Use analytic methods to prove geometric relationships.** *(Lesson 10-1)*
- **Use the standard and general forms of the equations of circles, parabolas, ellipses, and hyperbolas.** *(Lessons 10-2, 10-3, 10-4, and 10-5)*
- **Graph circles, parabolas, ellipses, and hyperbolas.** *(Lessons 10-2, 10-3, 10-4, and 10-5)*
- **Find the eccentricity of conic sections.** *(Lessons 10-2, 10-3, 10-4, and 10-5)*
- **Recognize conic sections by their equations.** *(Lesson 10-6)*
- **Find parametric equations for conic sections defined by rectangular equations and vice versa.** *(Lesson 10-6)*
- **Find the equations of conic sections that have been translated or rotated.** *(Lesson 10-7)*
- **Graph and solve systems of second-degree equations and inequalities.** *(Lesson 10-8)*

10-1

Introduction to Analytic Geometry

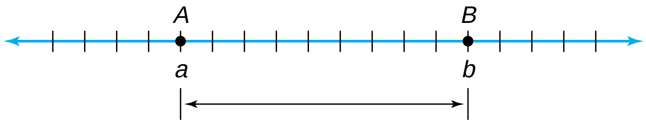
OBJECTIVES

- Find the distance and midpoint between two points on a coordinate plane.
- Prove geometric relationships among points and lines using analytical methods.

Real World Application

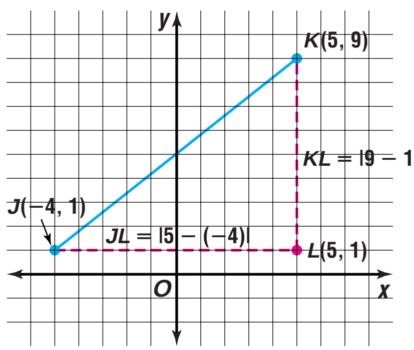
SEARCH AND RESCUE The *Absaroka Search Dogs* has provided search teams, each of which consists of a canine and its handler, to assist in lost person searches throughout the Montana and Wyoming area since 1986. While the dogs use their highly sensitive noses to detect the lost individual, the handlers use land navigation skills to insure that they do not become lost themselves. A handler needs to be able to read a map and calculate distances. *A problem related to this will be solved in Example 2.*

The distance between two points on a number line can be found by using absolute value. Let A and B be two points with coordinates a and b, respectively.



Distance between A and B $\Rightarrow$ $|a - b|$ or $|b - a|$

The distance between two points in the coordinate plane can also be found. Consider points $J(-4, 1)$ and $K(5, 9)$. To find JK, first choose a point L such that $\overline{JL}$ is parallel to the x-axis and $\overline{KL}$ is parallel to the y-axis. In this case, L has coordinates $(5, 1)$. Since K and L lie along the line $x = 5$, KL is equal to the absolute value of the difference in the y-coordinates of K and L, $|9 - 1|$. Similarly, JL is equal to the absolute value of the difference in the x-coordinates of J and L, $|5 - (-4)|$.

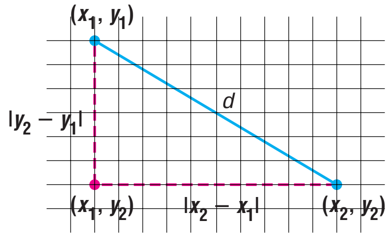


Since $\triangle JKL$ is a right triangle, JK can be found using the Pythagorean Theorem.

$(JK)^2 = (KL)^2 + (JL)^2$	*Pythagorean Theorem*
$JK = \sqrt{(KL)^2 + (JL)^2}$	*Take the positive square root of each side.*
$JK = \sqrt{\lvert 9 - 1\rvert^2 + \lvert 5 - (-4)\rvert^2}$	*$KL = \lvert 9 - 1\rvert$, $JL = \lvert 15 - (-4)\rvert$*
$JK = \sqrt{8^2 + 9^2}$	*$\lvert 9 - 1\rvert^2 = 8^2$, $\lvert 15 - (-4)\rvert^2 = 9^2$*
$JK = \sqrt{145}$ or about 12	

$\overline{JK}$ is about 12 units long.

From this specific case, we can derive a formula for the distance between any two points. In the figure, assume (x_1, y_1) and (x_2, y_2) represent the coordinates of any two points in the plane.



$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$ *Pythagorean Theorem*

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Why does $|x_2 - x_1|^2 + |y_2 - y_1|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$?*

Distance Formula for Two Points

The distance, d units, between two points with coordinates (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Examples

1 Find the distance between points at (−3, 7) and (2, −5).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Distance Formula*

$d = \sqrt{(2 - (-3))^2 + (-5 - 7)^2}$ *Let $(x_1, y_1) = (-3, 7)$ and $(x_2, y_2) = (2, -5)$.*

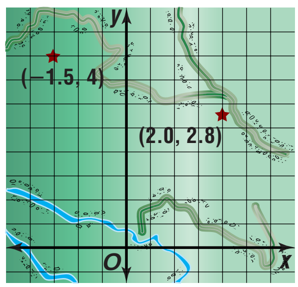
$d = \sqrt{5^2 + (-12)^2}$

$d = \sqrt{169}$ or 13

The distance is 13 units.

Real World Application

2 SEARCH AND RESCUE Refer to the application at the beginning of the lesson. Suppose a backpacker lies injured in the region shown on the map at the right. Each side of a square on the grid represents 15 meters. An Absaroka team searching for the missing individual is located at (−1.5, 4.0) on the map grid while the injured person is located at (2.0, 2.8). How far is the search team from the missing person?



Use the distance formula to find the distance between (2.0, 2.8) and (−1.5, 4.0).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{(-1.5 - 2.0)^2 + (4.0 - 2.8)^2}$ *Let $(x_1, y_1) = (2.0, 2.8)$ and $(x_2, y_2) = (-1.5, 4.0)$.*

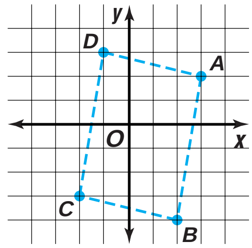
$d = \sqrt{13.69}$ or 3.7

The map distance is 3.7 units. Each unit equals 15 kilometers. So, the actual distance is about 3.7(15) or 55.5 kilometers.

You can use the Distance Formula and what you know about slope to investigate geometric figures on the coordinate plane.

Example 3 **Determine whether quadrilateral *ABCD* with vertices *A*(3, 2), *B*(2, −4), *C*(−2, −3), and *D*(−1, 3) is a parallelogram.**

Recall that a quadrilateral is a parallelogram if one pair of opposite sides are parallel and congruent.



First, graph the figure. $\overline{DA}$ and $\overline{CB}$ are one pair of opposite sides.

To determine if $\overline{DA} \parallel \overline{CB}$, find the slopes of $\overline{DA}$ and $\overline{CB}$.

Look Back
Refer to Lesson 1-3 to review the slope formula.

slope of $\overline{DA}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \textit{Slope formula}$$
$$= \frac{2 - 3}{3 - (-1)} \quad \textit{Let } (x_1, y_1) = (-1, -3) \textit{ and } (x_2, y_2) = (3, 2).$$
$$= -\frac{1}{4}$$

slope of $\overline{CB}$

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \textit{Slope formula}$$
$$= \frac{-4 - (-3)}{2 - (-2)} \quad \textit{Let } (x_1, y_1) = (-2, -3) \textit{ and } (x_2, y_2) = (2, -4).$$
$$= -\frac{1}{4}$$

Their slopes are equal. Therefore, $\overline{DA} \parallel \overline{CB}$.

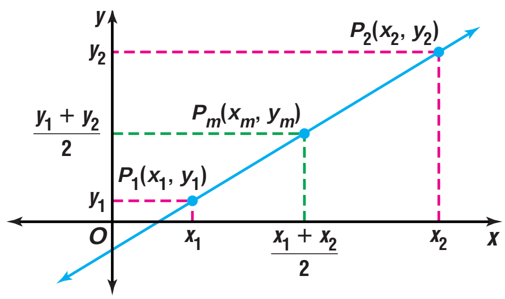
To determine if $\overline{DA} \cong \overline{CB}$, use the distance formula to find *DA* and *CB*.

$$DA = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad CB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[3 - (-1)]^2 + (2 - 3)^2} \qquad = \sqrt{[2 - (-2)]^2 + [-4 - (-3)]^2}$$
$$= \sqrt{17} \qquad = \sqrt{17}$$

The measures of $\overline{DA}$ and $\overline{CB}$ are equal. Therefore, $\overline{DA} \cong \overline{CB}$.

Since $\overline{DA} \parallel \overline{CB}$ and $\overline{DA} \cong \overline{CB}$, quadrilateral *ABCD* is a parallelogram. *You can also check your work by showing $\overline{DC} \parallel \overline{AB}$ and $\overline{DC} \cong \overline{AB}$.*

In addition to finding the distance between two points, you can use the coordinates of two points to find the midpoint of the segment between the points. In the figure at the right, the midpoint of $\overline{P_1P_2}$ is P_m. Notice that the *x*-coordinate of P_m is the average of the *x*-coordinates of P_1 and P_2. The *y*-coordinate of P_m is the average of the *y*-coordinates of P_1 and P_2.



Midpoint of a Line Segment	If the coordinates of P_1 and P_2 are (x_1, y_1) and (x_2, y_2), respectively, then the midpoint of $\overline{P_1P_2}$ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example 4 **Find the coordinates of the midpoint of the segment that has endpoints at (−2, 4) and (6, −5).**

Let (−2, 4) be (x_1, y_1) and (6, −5) be (x_2, y_2). Use the Midpoint Formula.

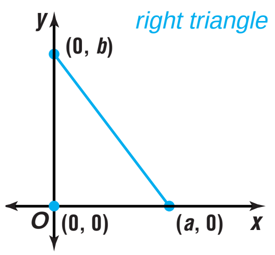
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-2 + 6}{2}, \frac{4 + (-5)}{2}\right)$$
$$= \left(2, -\frac{1}{2}\right)$$

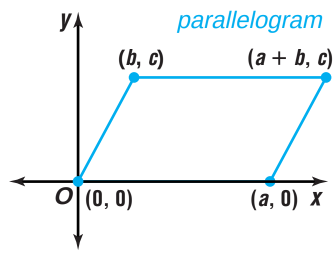
The midpoint of the segment is at $\left(2, -\frac{1}{2}\right)$.

Many theorems from plane geometry can be more easily proven by analytic methods. That is, they can be proven by placing the figure in a coordinate plane and using algebra to express and draw conclusions about the geometric relationships. The study of coordinate geometry from an algebraic perspective is called **analytic geometry.**

When using analytic methods to prove theorems from geometry, the position of the figure in the coordinate plane can be arbitrarily selected as long as size and shape are preserved. This means that the figure may be translated, rotated, or reflected from its original position. For polygons, one vertex is usually located at the origin, and one side coincides with the *x*-axis, as shown below.

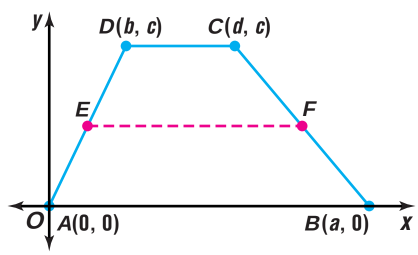
In a right triangle, the legs are on the axes.





Example 5 **Prove that the measure of the median of a trapezoid is equal to one half of the sum of the measures of the two bases.**

In trapezoid *ABCD*, choose two vertices as *A*(0, 0) and *B*(*a*, 0). Since $\overline{AB} \parallel \overline{DC}$, $\overline{DC}$ lies on a horizontal grid line of the coordinate plane. Therefore, *C* and *D* must have the same *y*-coordinate. Choose arbitrary letters to represent the *y*-coordinates, and the two *x*-coordinates; in this case, *D*(*b*, *c*) and *C*(*d*, *c*). Let *E* be the midpoint of $\overline{AD}$, and let *F* be the midpoint of $\overline{BC}$.



Now, find the coordinates of E and F by using the Midpoint Formula.

The coordinates of E are $\left(\frac{b+0}{2}, \frac{c+0}{2}\right)$ or $\left(\frac{b}{2}, \frac{c}{2}\right)$.

The coordinates of F are $\left(\frac{d+a}{2}, \frac{c+0}{2}\right)$ or $\left(\frac{d+a}{2}, \frac{c}{2}\right)$.

Then find the measures of each base and the median by using the Distance Formula.

$DC = \sqrt{(d-b)^2 + (c-c)^2}$ or $d - b$

$AB = \sqrt{(a-0)^2 + (0-0)^2}$ or a

$EF = \sqrt{\left(\frac{d+a}{2} - \frac{b}{2}\right)^2 + \left(\frac{c}{2} - \frac{c}{2}\right)^2}$ or $\frac{1}{2}(d - b + a)$

Calculate one half of the sum of the measures of the bases.

$\frac{1}{2}(DC + AB) = \frac{1}{2}(d - b + a)$ *$DC = d - b, AB = a$*

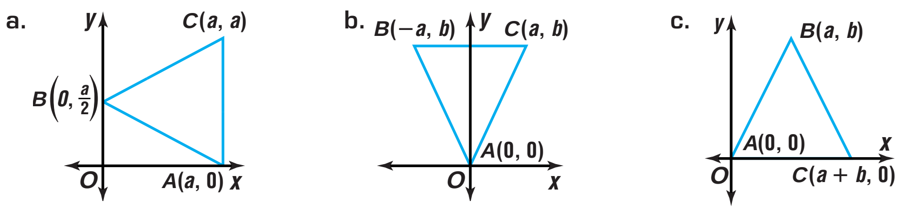
Since both $\frac{1}{2}(DC + AB)$ and EF equal $\frac{1}{2}(d - b + a)$, it follows that $\frac{1}{2}(DC + AB) = EF$. Therefore, the measure of the median of a trapezoid is one half of the sum of the measures of its bases.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why only the positive square root is considered when applying the distance formula.
2. **Describe** how can you show that a midpoint of a segment is equidistant from its endpoints given the coordinates of each point.
3. **Determine** whether each diagram represents an isosceles triangle. Explain your reasoning.



4. **Describe** four different ways of proving that a quadrilateral is a parallelogram if you are given the coordinates of its vertices.

Guided Practice

Find the distance between each pair of points with the given coordinates. Then, find the coordinates of the midpoint of the segment that has endpoints at the given coordinates.

5. (5, 1), (5, 11)
6. (0, 0), (−4, −3)
7. (−2, 2), (0, 4)

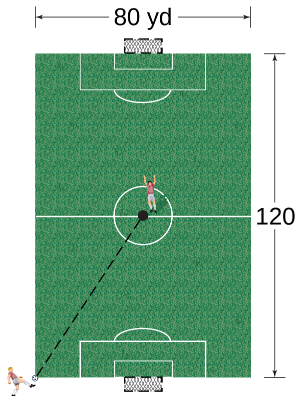
8. Determine whether the quadrilateral $ABCD$ with vertices $A(3, 4)$, $B(6, 2)$, $C(8, 7)$, and $D(5, 9)$ is a parallelogram. Justify your answer.

9. Determine whether the triangle XYZ with vertices $X(-3, 2)$, $Y(-1, -6)$, and $Z(5, 0)$ is isosceles. Justify your answer.

10. Consider rectangle $ABCD$.
   a. Draw and label rectangle $ABCD$ on the coordinate plane.
   b. Prove that $\overline{AC} \cong \overline{BD}$.
   c. Suppose the diagonals intersect at point E. Prove that $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$.
   d. What can you conclude about the diagonals of a rectangle? Explain.

Crew Stadium, Columbus, Ohio

11. **Sports** The dimensions of a soccer field are120 yards by 80 yards. A player kicks the ball from a corner to his teammate at the center of the playing field. Suppose the kicker is located at the origin.
   a. Find the ordered pair that represents the location of the kicker's teammate.
   b. Find the distance the ball travels.



EXERCISES

Practice

Find the distance between each pair of points with the given coordinates. Then, find the coordinates of the midpoint of the segment that has endpoints at the given coordinates.

12. $(-1, 1), (4, 13)$
13. $(1, 3), (-1, -3)$
14. $(8, 0), (0, 8)$
15. $(-1, -6), (5, -3)$
16. $\left(3\sqrt{2}, -5\right), \left(7\sqrt{2}, -1\right)$
17. $(a, 7), (a, -9)$
18. $(6 + r, s), (r - 2, s)$
19. $(c, d), (c + 2, d - 1)$
20. $(w - 2, w), (w, 4w)$

21. Find all values of a so that the distance between points at $(a, -9)$ and $(-2a, 7)$ is 20 units.

22. If $M\left(-3, \frac{5}{2}\right)$ is the midpoint of $\overline{CD}$ and C has coordinates $(4, -1)$, find the coordinates of D.

Determine whether the quadrilateral having vertices with the given coordinates is a parallelogram.

23. $(-2, 3), (-3, -2), (2, -3), (3, 2)$
24. $(4, 11), (8, 14), (4, 19), (0, 15)$

25. Collinear points lie on the same line. Find the value of k for which the points $(15, 1)$, $(-3, -8)$, and $(3, k)$ are collinear.

26. Determine whether the points $A(-3, 0)$, $B\left(-1, 2\sqrt{3}\right)$, and $C(1, 0)$ are the vertices of an equilateral triangle. Justify your answer.

27. Show that points $E(2, 5)$, $F(4, 4)$, $G(2, 0)$, and $H(0, 1)$ are the vertices of a rectangle.

interNET CONNECTION

Graphing Calculator Programs
To download a graphing calculator program that determines the distance and midpoint between two points, visit **www.amc.glencoe.com**

Prove using analytic methods. Be sure to include a coordinate diagram.

28. The measure of the line segment joining the midpoints of two sides of a triangle is equal to one-half the measure of the third side.

29. The diagonals of an isosceles trapezoid are congruent.

30. The medians to the congruent sides of an isosceles triangle are congruent. (*Hint*: A median of a triangle is a segment connecting a vertex to the midpoint of the side opposite the vertex.)

31. The diagonals of a parallelogram bisect each other.

32. The line segments joining the midpoints of consecutive sides of any quadrilateral form a parallelogram.

Applications and Problem Solving

Real World Application

33. Geometry The vertices of a rectangle are at $(-3, 1)$, $(-1, 3)$, $(3, -1)$, and $(1, -3)$. Find the area of the rectangle.

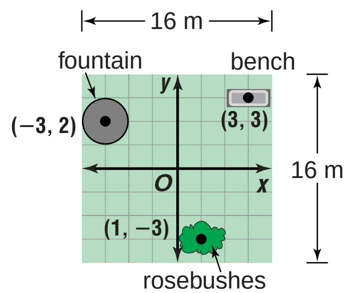
34. Web Page Design Many Internet Web pages are designed so that when the cursor is positioned over a specified area of an image, lines of text are displayed. The programming string "$x|y|$width$|$height" defines the location of the bottom left corner of the designated region using x and y coordinates and then defines the width and height of the region in pixels.

a. If the center of a Web page is located at the origin, graph the two regions defined by $-22|12|10|8$ and $31|-10|8|10$.

b. Suppose the two regions are to be no less than 40 pixels apart. Calculate the distance between the regions at their closest points to determine if this criteria is met.

35. Critical Thinking Prove analytically that the segments joining midpoints of consecutive sides of an isosceles trapezoid form a rhombus. Include a coordinate diagram with your proof.

36. Landscaping The diagram shows the plans made by a landscape artist for a homeowner's 16-meter by 16-meter backyard. The homeowner has requested that the rosebushes, fountain, and garden bench be placed so that they are no more than 14 meters apart.



a. Has the landscape artist met the homeowner's requirements? Explain.

b. The homeowner has purchased a sundial to be placed midway between the fountain and the rosebushes. Determine the coordinates indicating where the sundial should be placed.

37. Critical Thinking Consider point $M(t, 3t - 12)$.

a. Prove that for all values of t, M is equidistant from $A(0, 3)$ and $B(9, 0)$.

b. Describe the figure formed by the points M for all values of t. What is the relationship between the figure and points A and B?

Mixed Review

38. Find $(-5 + 12\boldsymbol{i})^2$. *(Lesson 9-8)*

Extra Practice See p. A44.

CONTENTS

39. **Physics** Suppose that during a storm, the force of the wind blowing against a skyscraper can be expressed by the vector (115, 2018, 0), where each measure in the ordered triple represents the force in Newtons. What is the magnitude of this force? *(Lesson 8-3)*

40. Verify that $2\sec^2 x = \frac{1}{1 + \sin x} + \frac{1}{1 - \sin x}$ is an identity. *(Lesson 7-2)*

41. A circle has a radius of 12 inches. Find the degree measure of the central angle subtended by an arc 11.5 inches long. *(Lesson 6-1)*

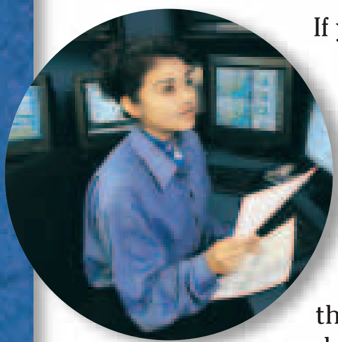
42. Find $\sin 390°$. *(Lesson 5-3)*

43. Solve $z^2 - 8z = -14$ by completing the square. *(Lesson 4-2)*

44. **SAT Practice** **Grid-In** If $x^2 = 16$ and $y^2 = 4$, what is the greatest possible value of $(x - y)^2$?

CAREER CHOICES

Meteorologist



If you find the weather intriguing, then you may want to investigate a career in meteorology. Meteorologists spend their time studying weather and forecasting changes in the weather. They analyze charts, weather maps, and other data to make predictions about future weather patterns. Meteorologists also research different types of weather, such as tornadoes and hurricanes, and may even teach at universities.

As a meteorologist, you may choose to specialize in one of several areas such as climatololgy, operational meteorology, or industrial meteorology. As a meteorologist, you might even be seen on television forecasting the weather for your area!

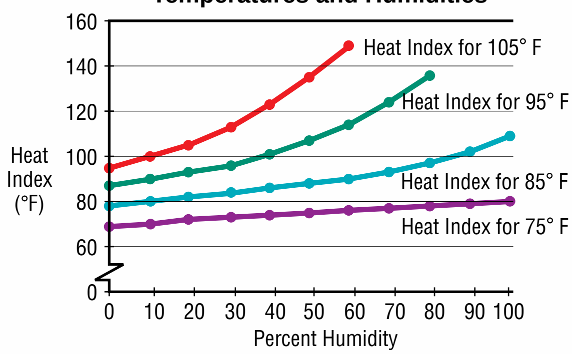
Career Overview

Degree Preferred:
Bachelor's degree in meteorology

Related Courses:
mathematics, geography, physics, computer science

Outlook:
slower than average job growth through the year 2006

Heat Index for Various Temperatures and Humidities



Source: *The World Almanac* 1999

interNET CONNECTION For more information on careers in meteorology, visit: **www.amc.glencoe.com**

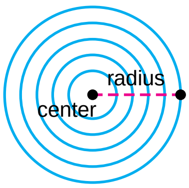
10-2 Circles

OBJECTIVES

- Use and determine the standard and general forms of the equation of a circle.
- Graph circles.

Real World Application

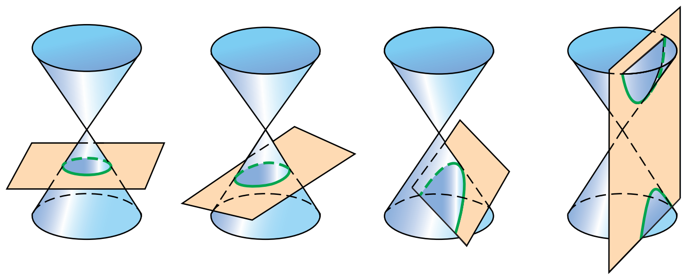
SEISMOLOGY Portable autonomous digital seismographs (PADSs) are used to investigate the strong ground motions produced by the aftershocks of large earthquakes. Suppose a PADS is deployed 2 miles west and 3.5 miles south of downtown Olympia, Washington, to record the aftershocks of a recent earthquake. While there, the PADS detects and records the seismic activity of another quake located 24 miles away. What are all the possible locations of this earthquake's epicenter? *This problem will be solved in Example 2.*



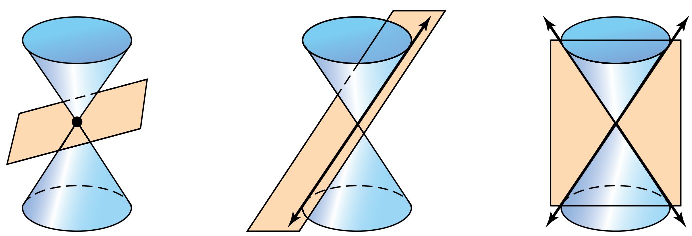
The pattern of the shock waves from an earthquake form **concentric** circles. A **circle** is the set of all points in the plane that are equidistant from a given point in the plane, called the **center.** The distance from the center to any point on the circle is called the **radius** of the circle. Concentric circles have the same center but not necessarily the same radius.

A circle is one type of **conic section.** Conic sections, which include circles, parabolas, ellipses and hyperbolas, were first studied in ancient Greece sometime between 600 and 300 B.C. The Greeks were largely concerned with the properties, not the applications, of conics. In the seventeenth century, applications of conics became prominent in the development of calculus.

Conic sections are used to describe all of the possible ways a plane and a double right cone can intersect. In forming the four basic conics, the plane does not pass through the vertex of the cone.

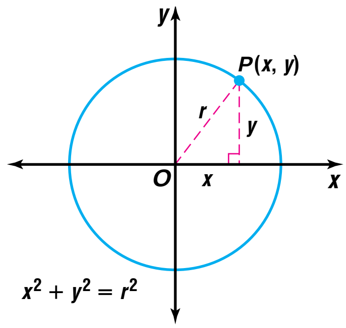


When the plane does pass through the vertex of a conical surface, as illustrated below, the resulting figure is called a **degenerate conic.** A degenerate conic may be a point, line, or two intersecting lines.



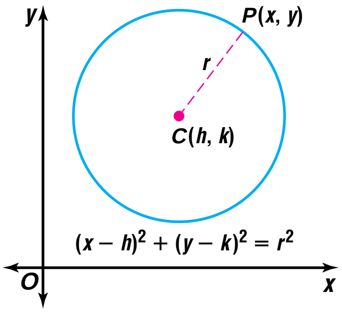
Radius can also refer to the line segment from the center to any point on the circle.

In the figure at the right, the center of the circle is at the origin. By drawing a perpendicular from any point $P(x, y)$ on the circle but not on an axis to the x-axis, you form a right triangle. The Pythagorean Theorem can be used to write an equation that describes every point on a circle whose center is located at the origin.



$x^2 + y^2 = r^2$ *Pythagorean Theorem*

This is the equation for the parent graph of all circles.



Suppose the center of this circle is translated from the origin to $C(h, k)$. You can use the distance formula to write the equation for this translated circle.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ *Distance formula*

$r = \sqrt{(x - h)^2 + (y - k)^2}$ *$d = r$, $(x_2, y_2) = (x, y)$, and $(x_1, y_1) = (h, k)$*

$r^2 = (x - h)^2 + (y - k)^2$ *Square each side.*

This equation is the standard form of the equation of a circle.

Standard Form of the Equation of a Circle	The standard form of the equation of a circle with radius r and center at (h, k) is $(x - h)^2 + (y - k)^2 = r^2$.

Examples

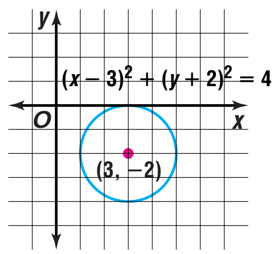
1 **Write the standard form of the equation of the circle that is tangent to the x-axis and has its center at $(3, -2)$. Then graph the equation.**

Since the circle is tangent to the x-axis, the distance from the center to the x-axis is the radius. The center is 2 units below the x-axis. Therefore, the radius is 2.

$(x - h)^2 + (y - k)^2 = r^2$ *Standard form*

$(x - 3)^2 + [y - (-2)]^2 = 2^2$ *$h = 3$, $k = -2$, $r = 2$*

$(x - 3)^2 + (y + 2)^2 = 4$



The standard form of the equation for this circle is $(x - 3)^2 + (y + 2)^2 = 4$.

2 **SEISMOLOGY** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Write an equation for the set of points representing all possible locations of the earthquake's epicenter. Let downtown Olympia, Washington, be located at the origin.

b. Graph the equation found in part a.

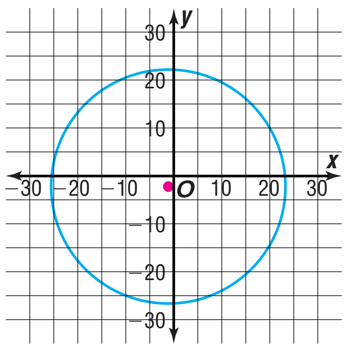
a. The location of the PADS, 2 miles west and 3.5 miles south of downtown Olympia, Washington, can be expressed as the ordered pair $(-2, -3.5)$. Since any point 24 miles from the seismograph could be the epicenter, the radius of the circle is 24.

$$(x - h)^2 + (y - k)^2 = r^2 \quad \textit{Standard form}$$

$$[x - (-2)]^2 + [y - (-3.5)]^2 = 24^2 \quad (h, k) = (-2, -3.5) \text{ and } r = 24$$

$$(x + 2)^2 + (y + 3.5)^2 = 576$$

b. The location of the epicenter lies on the circle with equation $(x + 2)^2 + (y + 3.5)^2 = 576$.



$(x + 2)^2 + (y + 3.5)^2 = 576$

Graphing Calculator Tip

To graph a circle on a graphing calculator, first solve for y. Then graph the two resulting equations on the same screen by inserting $\{1, -1\}$ in front of the equation for the symbol $\pm$. Use **5:ZSquare** in the **ZOOM** menu to make the graph look like a circle

The standard form of the equation of a circle can be expanded to obtain a general form of the equation.

$$(x - h)^2 + (y - k)^2 = r^2 \quad \textit{Standard form}$$

$$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = r^2 \quad \text{Expand } (x - h)^2 \text{ and } (y - k)^2.$$

$$x^2 + y^2 + (-2h)x + (-2k)y + (h^2 + k^2) - r^2 = 0$$

Since h, k, and r are constants, let D, E, and F equal $-2h$, $-2k$ and $(h^2 + k^2) - r^2$, respectively.

$$x^2 + y^2 + Dx + Ey + F = 0$$

This equation is called the general form of the equation of a circle.

General Form of the Equation of a Circle

The general form of the equation of a circle is

$$x^2 + y^2 + Dx + Ey + F = 0,$$

where D, E, and F are constants.

Notice that the coefficients of x^2 and y^2 in the general form must be 1. If those coefficients are not 1, division can be used to transform the equation so that they are 1. Also notice that there is no term containing the product of the variables, xy.

When the equation of a circle is given in general form, it can be rewritten in standard form by completing the square for the terms in x and the terms in y.

Example 3 **The equation of a circle is $2x^2 + 2y^2 - 4x + 12y - 18 = 0$.**

a. Write the standard form of the equation.

b. Find the radius and the coordinates of the center.

c. Graph the equation.

a.

$$2x^2 + 2y^2 - 4x + 12y - 18 = 0$$

$$x^2 + y^2 - 2x + 6y - 9 = 0 \quad \textit{Divide each side by 2.}$$

$$(x^2 - 2x + ?) + (y^2 + 6y + ?) = 9 \quad \textit{Group to form perfect square trinomials.}$$

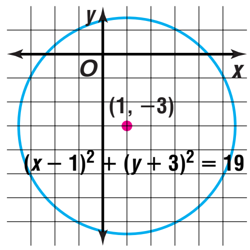
$$(x^2 - 2x + 1) + (y^2 + 6y + 9) = 9 + 1 + 9 \quad \textit{Complete the square.}$$

$$(x - 1)^2 + (y + 3)^2 = 19 \quad \textit{Factor the trinomials.}$$

$$(x - 1)^2 + (y + 3)^2 = \left(\sqrt{19}\right)^2 \quad \textit{Express 19 as } \left(\sqrt{19}\right)^2 \textit{ to show that } r = \sqrt{19}.$$

b. The center of the circle is located at $(1, -3)$, and the radius is $\sqrt{19}$.

c. Plot the center at $(-1, 3)$. The radius of $\sqrt{19}$ is approximately equal to 4.4.



From geometry, you know that any two points in the coordinate plane determine a unique line. It is also true that any three noncollinear points in the coordinate plane determine a unique circle. The equation of this circle can be found by substituting the coordinates of the three points into the general form of the equation of a circle and solving the resulting system of three equations.

Example 4 **Write the standard form of the equation of the circle that passes through the points at (5, 3), (−2, 2), and (−1, −5). Then identify the center and radius of the circle.**

Substitute each ordered pair for (x, y) in $x^2 + y^2 + Dx + Ey + F = 0$, to create a system of equations.

$$(5)^2 + (3)^2 + D(5) + E(3) + F = 0 \quad (x, y) = (5, 3)$$

$$(-2)^2 + (2)^2 + D(-2) + E(2) + F = 0 \quad (x, y) = (-2, 2)$$

$$(-1)^2 + (-5)^2 + D(-1) + E(-5) + F = 0 \quad (x, y) = (-1, -5)$$

Simplify the system of equations.

$$5D + 3E + F + 34 = 0$$
$$-2D + 2E + F + 8 = 0$$
$$-D - 5E + F + 26 = 0$$

The solution to the system is $D = -4$, $E = 2$, and $F = -20$.

The general form of the equation of the circle is $x^2 + y^2 - 4x + 2y - 20 = 0$. After completing the square, the standard form is $(x - 2)^2 + (y + 1)^2 = 25$. The center of the circle is at $(2, -1)$, and its radius is 5.

Look Back
Refer to Lesson 2-2 to review solving systems of three equations.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how to convert the general form of the equation of a circle to the standard form of the equation of a circle.
2. **Write** the equations of five concentric circles with different radii whose centers are at $(-4, 9)$.
3. **Describe** how you might determine the equation of a circle if you are given the endpoints of the circle's diameter.
4. **Find a counterexample** to this statement: The graph of any equation of the form $x^2 + y^2 + Dx + Ey + F = 0$ is a circle.
5. **You Decide** Kiyo says that you can take the square root of each side of an equation. Therefore, he decides that $(x - 3)^2 + (y - 1)^2 = 49$ and $(x - 3) + (y - 1) = 7$ are equivalent equations. Ramon says that the equations are not equivalent. Who is correct? Explain.

Guided Practice

Write the standard form of the equation of each circle described. Then graph the equation.

6. center at $(0, 0)$, radius 9
7. center at $(-1, 4)$ and tangent to $x = 3$

Write the standard form of each equation. Then graph the equation.

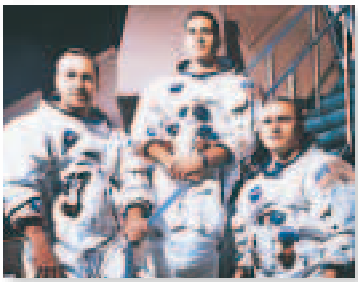
8. $x^2 + y^2 - 4x + 14y - 47 = 0$
9. $2x^2 + 2y^2 - 20x + 8y + 34 = 0$

Write the standard form of the equation of the circle that passes through points with the given coordinates. Then identify the center and radius.

10. $(0, 0)$, $(4, 0)$, $(0, 4)$
11. $(1, 3)$, $(5, 5)$, $(5, 3)$

Write the equation of the circle that satisfies each set of conditions.

12. The circle passes through the point at $(1, 5)$ and has its center at $(-2, 1)$.
13. The endpoints of a diameter are at $(-2, 6)$ and at $(10, -10)$.



14. **Space Science** Apollo 8 was the first manned spacecraft to orbit the moon at an average altitude of 185 kilometers above the moon's surface. Determine an equation to model the orbit of the Apollo 8 command module if the radius of the moon is 1740 kilometers. Let the center of the moon be at the origin.

← Apollo 8 crew: (from left) James A. Lovell, Jr., William A. Anders, Frank Borman

EXERCISES

Practice

Write the standard form of the equation of each circle described. Then graph the equation.

15. center at $(0, 0)$, radius 5
16. center at $(-4, 7)$, radius $\sqrt{3}$
17. center at $(-1, -3)$, radius $\frac{\sqrt{2}}{2}$
18. center at $(-5, 0)$, radius $\frac{9}{2}$
19. center at $(6, 1)$, tangent to the y-axis
20. center at $(3, -2)$, tangent to $y = 2$

Write the standard form of each equation. Then graph the equation.

21. $36 - x^2 = y^2$

22. $x^2 + y^2 + y = \frac{3}{4}$

23. $x^2 + y^2 - 4x + 12y + 30 = 0$

24. $2x^2 + 2y^2 + 2x - 4y = -1$

25. $6x^2 - 12x + 6y^2 + 36y = 36$

26. $16x^2 + 16y^2 - 8x - 32y = 127$

27. Write $x^2 + y^2 + 14x + 24y + 157 = 0$ in standard form. Then graph the equation.

interNET CONNECTION

Graphing Calculator Programs
For a graphing calculator program that determines the radius and the coordinates of the center of a circle from an equation written in general form, visit **www.amc.glencoe.com**

Write the standard form of the equation of the circle that passes through the points with the given coordinates. Then identify the center and radius.

28. $(0, -1), (-3, -2), (-6, -1)$

29. $(7, -1), (11, -5), (3, -5)$

30. $(-2, 7), (-9, 0), (-10, -5)$

31. $(-2, 3), (6, -5), (0, 7)$

32. $(4, 5), (-2, 3), (-4, -3)$

33. $(1, 4), (2, -1), (-3, 0)$

34. Write the standard form of the equation of the circle that passes through the origin and points at $(2.8, 0)$ and $(5, 2)$.

Write the equation of the circle that satisfies each set of conditions.

35. The circle passes through the origin and has its center at $(-4, 3)$.

36. The circle passes through the point $(5, 6)$ and has its center at $(2, 3)$.

37. The endpoints of a diameter are at $(2, 3)$ and at $(-6, -5)$.

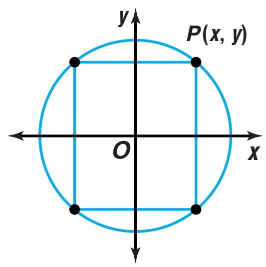
38. The points at $(-3, 4)$ and $(2, 1)$ are the endpoints of a diameter.

39. The circle is tangent to the line with equation $x + 3y = -2$ and has its center at $(5, 1)$.

40. The center of the circle is on the x-axis, its radius is 1, and it passes through the point at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Graphing Calculator

41. A rectangle is inscribed in a circle centered at the origin with diameter 12.



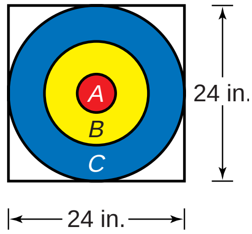
a. Write the equation of the circle that meets these conditions.
b. Write the dimensions of the rectangle in terms of x.
c. Write a function $A(x)$ that represents the area of the rectangle.
d. Use a graphing calculator to graph the function $y = A(x)$.
e. Find the value of x, to the nearest tenth, that maximizes the area of the rectangle. What is the maximum area of the rectangle?

42. Select the standard viewing window and then select **ZSquare** from the ZOOM menu.
a. Select **9:Circle(** from the **DRAW** menu and then enter 2 , 3 , 4) . Then press ENTER.
b. Describe what appears on the viewing screen.
c. Write the equation for this graph.
d. Use what you have learned to write the command to graph the equation $(x + 4)^2 + (y - 2) = 36$. Then graph the equation.

Applications and Problem Solving

Real World Application

43. Sports Cindy is taking an archery class and decides to practice her skills at home. She attaches the target shown at right to a bale of hay. The circles on the target are concentric and equally spaced apart.



a. If the common center of the circles is located at the origin, write an equation that models the largest circle.

b. If the smallest circle is modeled by the equation $x^2 + y^2 = 6.25$, find the area of the region marked B.

44. Geometry Write the equation of a circle that circumscribes the triangle whose sides are the graphs of $4x - 7y = 27$, $x - 5y + 3 = 0$, and $2x + 3y - 7 = 0$.

45. Critical Thinking Consider a family of circles in which $h = k$ and the radius is 2. Let k be any real number.

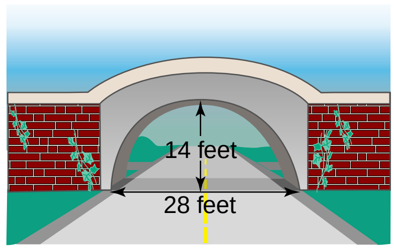
a. Write the equation of the family of circles.

b. Graph three members of this family on the same set of axes.

c. Write a description of all members of this family of circles.

Look Back
Refer to Lesson 3-2 to review families of graphs.

46. Transportation A moving truck 7 feet wide and 13 feet high is approaching a semi-circular brick archway at an apartment complex. The base of the archway is 28 feet wide. The road under the archway is divided, allowing for two-way traffic.

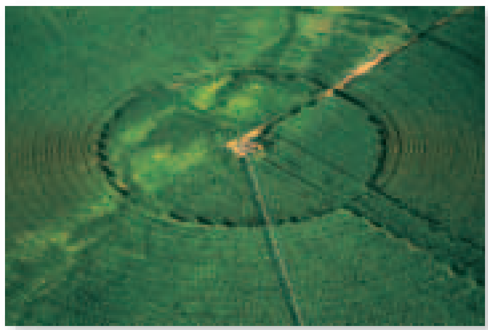


a. Write an equation, centered at the origin, of the archway that models its shape.

b. If the truck remains just to the right of the median, will it be able to pass under the archway without damage? Explain.

47. Critical Thinking Find the radius and the coordinates of the center of a circle defined by the equation $x^2 + y^2 - 8x + 6y + 25 = 0$. Describe the graph of this circle.

48. Agriculture One method of irrigating crops is called the center pivot system. This system rotates a sprinkler pipe from the center of the field to be irrigated. Suppose a farmer places one of these units at the center of a square plot of land 2500 feet on each side. With the center of this plot at the origin, the irrigator sends out water far enough to reach a point located at (475, 1140).



a. Find an equation representing the farthest points the water can reach.

b. Find the area of the land that receives water directly.

c. About what percent of the farmer's plot does not receive water directly?

49. Critical Thinking Consider points $A(3, 4)$, $B(-3, -4)$, and $P(x, y)$.

a. Write an equation for all x and y for which $\overline{PA} \perp \overline{PB}$.

b. What is the relationship between points A, B, and P if $\overline{PA} \perp \overline{PB}$?

Mixed Review

50. Find the distance between points at $(4, -3)$ and $(-2, 6)$. *(Lesson 10-1)*

51. Simplify $(2 + \boldsymbol{i})(3 - 4\boldsymbol{i})(1 + 2\boldsymbol{i})$. *(Lesson 9-5)*

52. Sports Patrick kicked a football with an initial velocity of 60 ft/s at an angle of 60° to the horizontal. After 0.5 seconds, how far has the ball traveled horizontally and vertically. *(Lesson 8-7)*

53. Toys A toy boat floats on the water bobbing up and down. The distance between its highest and lowest point is 5 centimeters. It moves from its highest point down to its lowest point and back up to its highest point every 20 seconds. Write a cosine function that models the movement of the boat in relationship to the equilibrium point. *(Lesson 6-6)*

54. Find the area to the nearest square unit of $\triangle ABC$ if $a = 15$, $b = 25$, and $c = 35$. *(Lesson 5-8)*

55. Amusement Parks The velocity of a roller coaster as it moves down a hill is $\sqrt{v_0^2 + 64h}$. The designer of a coaster wants the coaster to have a velocity of 95 feet per second when it reaches the bottom of the hill. If the initial velocity of the coaster at the top of the hill is 15 feet per second, how high should the designer make the hill? *(Lesson 4-7)*

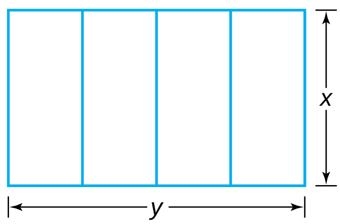
56. Determine whether the graph of $y = 6x^4 - 3x^2 + 1$ is symmetric with respect to the x-axis, the y-axis, the line $y = x$, the line $y = -x$, or the origin. *(Lesson 3-1)*

57. Business A pharmaceutical company manufactures two drugs. Each case of drug A requires 3 hours of processing time and 1 hour of curing time per week. Each case of drug B requires 5 hours of processing time and 5 hours of curing time per week. The schedule allows 55 hours of processing time and 45 hours of curing time weekly. The company must produce no more than 10 cases of drug A and no more than 9 cases of drug B. *(Lesson 2-7)*

a. If the company makes a profit of \$320 on each case of drug A and \$500 on each case of drug B, how many cases of each drug should be produced in order to maximize profit?

b. Find the maximum profit.

58. SAT/ACT Practice An owner plans to divide and enclose a rectangular property with dimensions x and y into rectangular regions as shown at the right. In terms of x and y, what is the total length of fence needed if every line segment represents a section of fence and there is no overlap between sections of fence.



A $5x + 2y$

B $5x + 8y$

C $4x + 2y$

D $4xy$

E xy

Extra Practice See p. A44.

CONTENTS

10-3

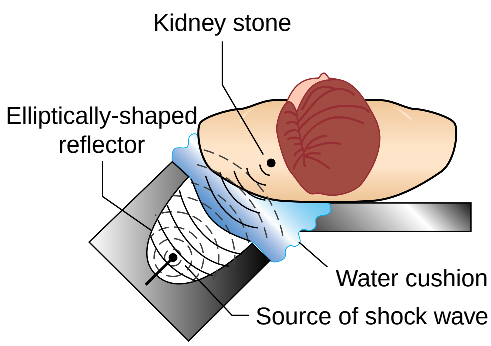
Ellipses

OBJECTIVES

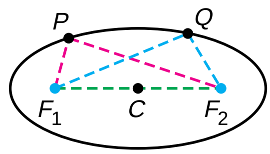
- Use and determine the standard and general forms of the equation of an ellipse.
- Graph ellipses.

Real World Application

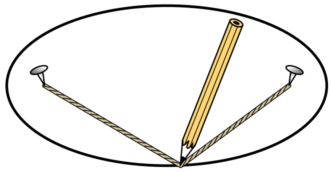
MEDICAL TECHNOLOGY To eliminate kidney stones, doctors sometimes use a medical tool called a *lithotripter,* (*LITH-uh-trip-tor*) which means "stone crusher." A lithotripter is a device that uses ultra-high-frequency shock waves moving through water to break up the stone. After x-raying a patient's kidney to precisely locate and measure the stone, the lithotripter is positioned so that the shock waves reflect off the inner surface of the elliptically-shaped tub and break up the stone. *A problem related to this will be solved in Example 1.*



An **ellipse** is the set of all points in the plane, the sum of whose distances from two fixed points, called **foci,** is constant. In the figure at the right, F_1 and F_2 are the foci, and the midpoint C of the line segment joining the foci is called the **center** of the ellipse. P and Q are any two points on the ellipse. By definition, $PF_1 + PF_2 = QF_1 + QF_2$.

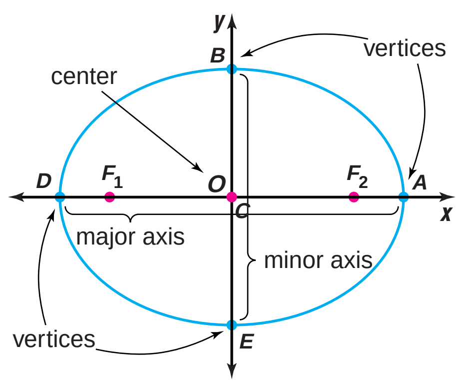


Foci (FOH sigh) is the plural of focus.

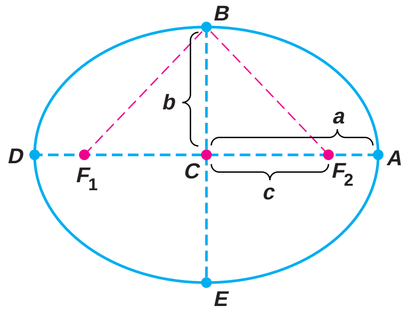


To help visualize this definition, imagine tacking two ends of a string at the foci and using a pencil to trace a curve as it is held tight against the string. The curve which results will be an ellipse since the sum of the distances to the foci, the total length of the string, remains constant.

The parent graph of an ellipse, shown at the right, is centered at the origin. An ellipse has two axes of symmetry, in this case, the x-axis and the y-axis. Notice that the ellipse intersects each axis of symmetry two times. The longer line segment, $\overline{AD}$, which contains the foci, is called the **major axis.** The shorter segment, $\overline{BE}$, is called the **minor axis.** The endpoints of each axis are the **vertices** of the ellipse.



The center separates each axis into two congruent segments. Suppose we let b represent the length of the **semi-minor axis** $\overline{BC}$ and a represent the length of the **semi-major axis** $\overline{CA}$. The foci are located along the major axis, c units from the center. There is a special relationship among the values a, b, and c.



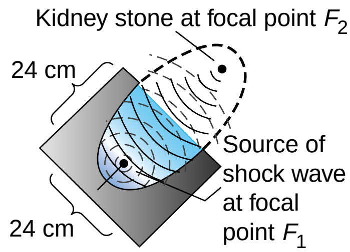
Suppose you draw $\overline{BF_1}$ and $\overline{BF_2}$. The lengths of these segments are equal because $\triangle BF_1C \cong \triangle BF_2C$. Since B and A are two points on the ellipse, you can use the definition of an ellipse to find BF_2.

$BF_1 + BF_2 = AF_1 + AF_2$	*Definition of ellipse*
$BF_1 + BF_2 = AF_1 + DF_1$	$AF_2 = DF_1$
$BF_1 + BF_2 = AD$	*Segment addition:* $AF_1 + DF_1 = AD$
$BF_1 + BF_2 = 2a$	*Substitution:* $AD = 2a$
$2(BF_2) = 2a$	$BF_1 = BF_2$
$BF_2 = a$	

Since $BF_2 = a$ and $\triangle BCF_2$ is a right triangle, $b^2 + c^2 = a^2$ by the Pythagorean Theorem.

Example 1 Real World Application

MEDICAL TECHNOLOGY **Refer to the application at the beginning of this lesson. Suppose the reflector of a mobile lithotripter is 24 centimeters wide and 24 centimeters deep. How far, to the nearest hundredth of a centimeter, should the shock wave emitter be placed from a patient's kidney stone?**



For the lithotripter to break up the stone, the shock wave emitter must be positioned at one focal point of the ellipse and the kidney stone at the other. To determine the distance between the emitter and the kidney stone, we must first find the lengths of the semi-major and semi-minor axes.

The semi-major axis in this ellipse is equal to the depth of the reflector, 24 centimeters. So, $a = 24$.

The semi-minor axis in this ellipse is half the width of the reflector, 12 centimeters. So, $b = 12$.

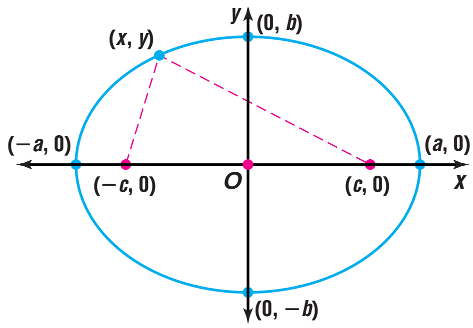
To find the focal length of the ellipse, use the formula $c^2 = a^2 - b^2$.

$c^2 = a^2 - b^2$	
$c^2 = (24)^2 - (12)^2$	$a = 24$ *and* $b = 12$
$c^2 = 432$	*Simplify.*
$c = \sqrt{432}$ or approximately 20.8	*Take the square root of each side.*

The distance between the two foci of the ellipse is $2c$ or 41.6.

Therefore, the emitter should be placed approximately 41.6 centimeters away from the patient's kidney along the ellipse's major axis.

The standard form of the equation of an ellipse can be derived from the definition and the distance formula. Consider the special case when the center is at the origin.



Suppose the foci are at $F_1(-c, 0)$ and $F_2(c, 0)$, and (x, y) is any point on the ellipse. By definition, the sum of the distances from a point at (x, y) to the foci is constant. To find a value for this constant, let (x, y) be one of the vertices on the x-axis, for example, $(a, 0)$. Let d_1 and d_2 be the distances from the point at $(a, 0)$ to F_1 and F_2, respectively. Use the Distance Formula.

$$\begin{aligned} d_1 &= \sqrt{[a - (-c)]^2 + (0 - 0)^2} \\ &= \sqrt{(a + c)^2} \\ &= a + c \end{aligned}$$

$$\begin{aligned} d_2 &= \sqrt{(a - c)^2 + (0 - 0)^2} \\ &= \sqrt{(a - c)^2} \\ &= a - c \end{aligned}$$

$d_1 + d_2 = a + c + a - c$ or $2a$ Therefore, one value for the constant is $2a$.

Using the Distance Formula and this value for the constant, a general equation for any point at (x, y) is $2a = \sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2}$.

$$2a = \sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2}$$

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2} \quad \textit{Isolate a radical.}$$

$$(x + c)^2 + y^2 = 4a^2 - 4a\sqrt{(x - c)^2 + y^2} + (x - c)^2 + y^2 \quad \textit{Square each side.}$$

$$a^2 - xc = a\sqrt{(x - c)^2 + y^2} \quad \textit{Simplify.}$$

$$a^4 - 2a^2xc + x^2c^2 = a^2[(x - c)^2 + y^2] \quad \textit{Square each side again.}$$

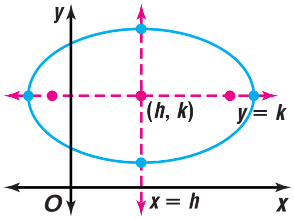
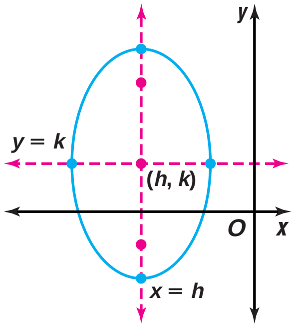
$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2) \quad \textit{Simplify.}$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad \textit{Divide each side by } a^2(a^2 - c^2).$$

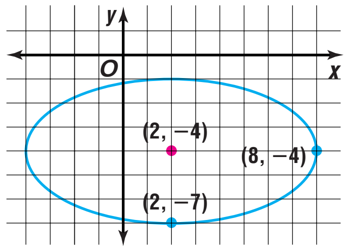
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad b^2 = a^2 - c^2$$

The resulting equation, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is the equation of an ellipse whose center is the origin and whose foci are on the x-axis. When the foci are on the y-axis, the equation is of the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$.

The standard form of the equation of an ellipse with a center other than the origin is a translation of the parent graph to a center at (h, k). The table on the next page gives the standard form, graph, and general description of the equation of each type of ellipse.

Standard Form of the Equation of an Ellipse	Orientation	Description
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, where $c^2 = a^2 - b^2$		Center: (h, k) Foci: $(h \pm c, k)$ Major axis: $y = k$ Major axis vertices: $(h \pm a, k)$ Minor axis: $x = h$ Minor axis vertices: $(h, k \pm b)$
$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$, where $c^2 = a^2 - b^2$		Center: (h, k) Foci: $(h, k \pm c)$ Major axis: $x = h$ Major axis vertices: $(h, k \pm a)$ Minor axis: $y = k$ Minor axis vertices: $(h \pm b, k)$

Example 2 **Consider the ellipse graphed at the right.**



a. Write the equation of the ellipse in standard form.

b. Find the coordinates of the foci.

a. The center of the graph is at $(2, -4)$. Therefore, $h = 2$ and $k = -4$.

Since the ellipse's horizontal axis is longer than its vertical axis, a is the distance between points at $(2, -4)$ and $(8, -4)$ or 6. The value of b is the distance between points at $(2, -4)$ and $(2, -7)$ or 3.

Therefore, the standard form of the equation of this ellipse is

$$\frac{(x-2)^2}{6^2} + \frac{(y+4)^2}{3^2} = 1 \text{ or } \frac{(x-2)^2}{36} + \frac{(y+4)^2}{9} = 1.$$

b. Using the equation $c = \sqrt{a^2 - b^2}$, we find that $c = 5$. The foci are located on the horizontal axis, 5 units from the center of the ellipse. Therefore, the foci have coordinates $(7, -4)$ and $(-3, -4)$.

In all ellipses, $a^2 \geq b^2$. You can use this information to determine the orientation of the major axis from the equation. If a^2 is the denominator of the x^2 term, the major axis is parallel to the x-axis. If a^2 is the denominator of the y^2 term, the major axis is parallel to the y-axis.

Example 3 For the equation $\frac{(y-3)^2}{25} + \frac{(x+4)^2}{9} = 1$, find the coordinates of the center, foci, and vertices of the ellipse. Then graph the equation.

Determine the values of a, b, c, h, and k.

Since $a^2 > b^2$, $a^2 = 25$ and $b^2 = 9$.

$a^2 = 25$ $\quad$ $b^2 = 9$ $\quad$ $c = \sqrt{a^2 - b^2}$

$a = \sqrt{25}$ or 5 $\quad$ $b = \sqrt{9}$ or 3 $\quad$ $c = \sqrt{25 - 9}$ or 4

$x - h = x + 4$ $\quad$ $y - k = y - 3$

$h = -4$ $\quad$ $k = 3$

Since a^2 is the denominator of the y term, the major axis is parallel to the y-axis.

Graphing Calculator Tip

You can graph an ellipse by first solving for y. Then graph the two resulting equations on the same screen.

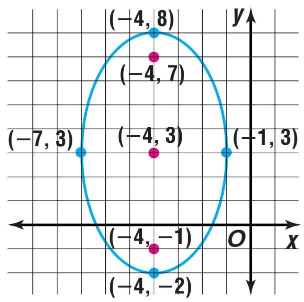
center: $(-4, 3)$ $\quad$ *(h, k)*

foci: $(-4, 7)$ and $(-4, -1)$ $\quad$ *$(h, k \pm c)$*

major axis vertices: $(-4, 8)$ and $(-4, -2)$ $\quad$ *$(h, k \pm a)$*

minor axis vertices: $(-1, 3)$ and $(-7, 3)$ $\quad$ *$(h \pm b, k)$*

Graph these ordered pairs. Other points on the ellipse can be found by substituting values for x and y. Complete the ellipse.



As with circles, the standard form of the equation of an ellipse can be expanded to obtain the general form. The result is a second-degree equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A \neq 0$ and $C \neq 0$, and A and C have the same sign. An equation in general form can be rewritten in standard form to determine the center at (h, k), the measure of the semi-major axis, a, and the measure of the semi-minor axis, b.

Example 4 Find the coordinates of the center, the foci, and the vertices of the ellipse with the equation $4x^2 + 9y^2 - 40x + 36y + 100 = 0$. Then graph the equation.

First write the equation in standard form.

$$4x^2 + 9y^2 - 40x + 36y + 100 = 0$$

$$4(x^2 - 10x + ?) + 9(y^2 + 4y + ?) = -100 + ? + ?$$ *The GCF of the x terms is 4. The GCF of the y terms is 9.*

$$4(x^2 - 10x + 25) + 9(y^2 + 4y + 4) = -100 + 4(25) + 9(4)$$ *Complete the square.*

$$4(x-5)^2 + 9(y+2)^2 = 36$$ *Factor.*

$$\frac{(x-5)^2}{9} + \frac{(y+2)^2}{4} = 1$$ *Divide each side by 36.*

(continued on the next page)

Since $a^2 > b^2$, $a^2 = 9$ and $b^2 = 4$. Thus, $a = 3$ and $b = 2$.

Since $c^2 = a^2 - b^2$, $c = \sqrt{5}$.

Since a^2 is the denominator of the x term, the major axis is parallel to the x-axis.

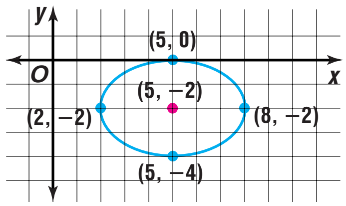
center: $(5, -2)$ *(h, k)*

foci: $\left(5 \pm \sqrt{5}, -2\right)$ *$(h \pm c, k)$*

major axis vertices: $(8, -2)$ and $(2, -2)$ *$(h \pm a, k)$*

minor axis vertices: $(5, 0)$ and $(5, -4)$ *$(h, k \pm b)$*

Sketch the ellipse.



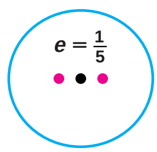
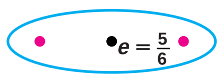
The **eccentricity** of an ellipse, denoted by e, is a measure that describes the shape of an ellipse. It is defined as $e = \frac{c}{a}$. Since $0 < c < a$, you can divide by a to show that $0 < e < 1$.

$0 < c < a$

$0 < \frac{c}{a} < 1$ *Divide by a.*

$0 < e < 1$ *Replace $\frac{c}{a}$ with e.*

The table shows the relationship between the value of e, the location of the foci, and the shape of the ellipse.

Value of e	Location of Foci	Graph
close to 0	near center of ellipse	
close to 1	far from center of ellipse	

Sometimes, you may need to find the value of b when you know the values of a and e. In any ellipse, $b^2 = a^2 - c^2$ and $\frac{c}{a} = e$. By using the two equations, it can be shown that $b^2 = a^2(1 - e^2)$. *You will derive this formula in Exercise 4.*

All of the planets in our solar system have elliptical orbits with the sun as one focus. These orbits are often described by their eccentricity.

Example 5 **ASTRONOMY** **Of the nine planetary orbits in our solar system, Pluto's has the greatest eccentricity, 0.248. Astronomers have determined that the orbit is about 29.646 AU (astronomical units) from the sun at its closest point to the sun (perihelion). The length of the semi-major axis is about 39.482 AU.**

Real World Application

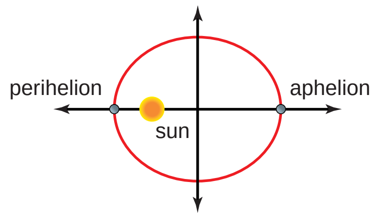
1 AU = the average distance between the sun and Earth, about 9.3×10^7 miles

a. Sketch the orbit of Pluto showing the sun in its position.

b. Find the length of the semi-minor axis of the orbit.

c. Find the distance of Pluto from the sun at its farthest point (aphelion).

a. The sketch at the right shows the sun as a focus for the elliptical orbit.



b. b is the length of the semi-minor axis.

$$b^2 = a^2(1 - e^2)$$
$$b = \sqrt{a^2(1 - e^2)}$$
$$b = \sqrt{(39.482)^2(1 - 0.248^2)} \quad \textit{a = 39.482, e = 0.248}$$
$$b \approx 38.249 \text{ AU}$$

c. *aphelion = length of major axis − distance from sun to perihelion*

$$d = 2(39.482) - 29.646$$
$$= 49.318$$

Pluto is about 49 AU from the sun at its aphelion.

CHECK FOR UNDERSTANDING

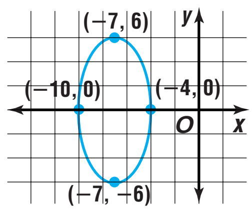
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** the equation of an ellipse centered at the origin, with $a = 8$, $b = 5$, and the major axis on the y-axis.
2. **Explain** how to determine whether the foci of an ellipse lie on the horizontal or vertical axis of an ellipse.
3. **Describe** the result when the foci and center of an ellipse coincide and give the eccentricity of such an ellipse.
4. **Derive** the equation $b^2 = a^2(1 - e^2)$.
5. **You Decide** Manuel says that the graph of $3y - 36 = 2x^2 - 18x$ is an ellipse with its major axis parallel to the y-axis. Shanice disagrees. Who is correct? Explain your answer.

Guided Practice

6. Write the equation of the ellipse graphed at the right in standard form. Then find the coordinates of its foci.



For the equation of each ellipse, find the coordinates of the center, foci, and vertices. Then graph the equation.

7. $\frac{x^2}{36} + \frac{y^2}{4} = 1$
8. $\frac{x^2}{81} + \frac{(y - 4)^2}{49} = 1$
9. $25x^2 + 9y^2 + 100x - 18y = 116$
10. $9x^2 + 4y^2 - 18x + 16y = 11$

Write the equation of the ellipse that meets each set of conditions.

11. The center is at $(-2, -3)$, the length of the vertical major axis is 8 units, and the length of the minor axis is 2 units.

12. The foci are located at $(-1, 0)$ and $(1, 0)$ and $a = 4$.

13. The center is at $(1, 2)$, the major axis is parallel to the x-axis, and the ellipse passes through points at $(1, 4)$ and $(5, 2)$.

14. The center is at $(3, 1)$, the vertical semi-major axis is 6 units long, and $e = \frac{1}{3}$.

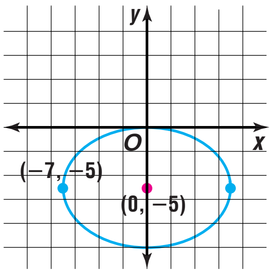
15. Astronomy The elliptical orbit of Mars has its foci at $(0.141732, 0)$ and $(-0.141732, 0)$, where 1 unit equals 1 AU. The length of the major axis is 3.048 AU. Determine the equation that models Mars' elliptical orbit.

EXERCISES

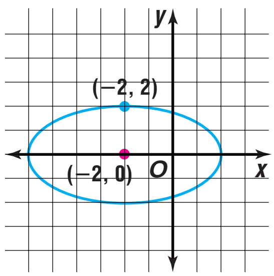
Practice

Write the equation of each ellipse in standard form. Then find the coordinates of its foci.

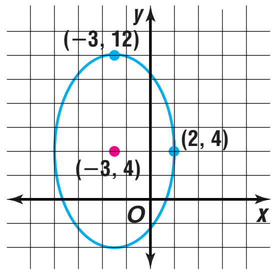
16.



17.



18.



For the equation of each ellipse, find the coordinates of the center, foci, and vertices. Then graph the equation.

19. $\frac{(x+2)^2}{1} + \frac{(y-1)^2}{4} = 1$

20. $\frac{(x-6)^2}{100} + \frac{(y-7)^2}{121} = 1$

21. $\frac{(x-4)^2}{16} + \frac{(y+6)^2}{9} = 1$

22. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

23. $4x^2 + y^2 - 8x + 6y + 9 = 0$

24. $16x^2 + 25y^2 - 96x - 200y = -144$

25. $3x^2 + y^2 + 18x - 2y + 4 = 0$

26. $6x^2 - 12x + 6y^2 + 36y = 36$

27. $18y^2 + 12x^2 - 144y - 48x = -120$

28. $4y^2 - 8y + 9x^2 - 54x + 49 = 0$

29. $49x^2 + 16y^2 + 160y - 384 = 0$

30. $9y^2 + 108y + 4x^2 - 56x = -484$

Write the equation of the ellipse that meets each set of conditions.

31. The center is at $(-3, -1)$, the length of the horizontal semi-major axis is 7 units, and the length of the semi-minor axis is 5 units.

32. The foci are at $(-2, 0)$ and $(2, 0)$, and $a = 7$.

33. The length of the semi-minor axis is $\frac{3}{4}$ the length of the horizontal semi-major axis, the center is at the origin, and $b = 6$.

34. The semi-major axis has length $2\sqrt{13}$ units, and the foci are at $(-1, 1)$ and $(-1, -5)$.

35. The endpoints of the major axis are at $(-11, 5)$ and $(7, 5)$. The endpoints of the minor axis are at $(-2, 9)$ and $(-2, 1)$.

36. The foci are at $(1, -1)$ and $(1, 5)$, and the ellipse passes through the point at $(4, 2)$.

37. The center is the origin, $\frac{1}{2} = \frac{c}{a}$, and the length of the horizontal semi-major axis is 10 units.

38. The ellipse is tangent to the x- and y-axes and has its center at $(4, -7)$.

39. The ellipse has its center at the origin, $a = 2$, and $e = \frac{3}{4}$.

40. The foci are at $(3, 5)$ and $(1, 5)$, and the ellipse has eccentricity 0.25.

41. The ellipse has a vertical major axis of 20 units, its center is at $(3, 0)$, and $e = \frac{7}{10}$.

42. The center is at $(1, -1)$, one focus is located at $\left(1, -1 + \sqrt{5}\right)$, and the ellipse has eccentricity $\frac{\sqrt{5}}{3}$.

Graphing Calculator

Graph each equation. Then use the TRACE function to approximate the coordinates of the vertices to the nearest integer.

43. $x^2 + 4y^2 - 6x + 24y = -41$

44. $4x^2 + y^2 - 8x - 2y = -1$

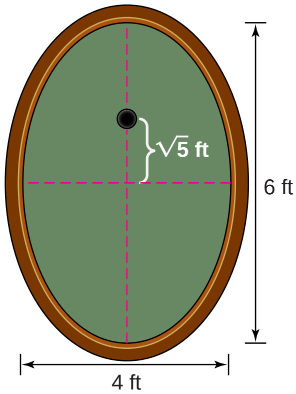
45. $4x^2 + 9y^2 - 16x + 18y = 11$

46. $25y^2 + 16x^2 - 150y + 32x = 159$

Applications and Problem Solving

Real World Application

47. **Entertainment** *Elliptipool* is a billiards game that use an elliptically-shaped pool table with only one pocket in the surface. A cue ball and a target ball are used in play. The object of the game is to strike the target ball with the cue ball so that the target ball rolls into the pocket after one bounce off the side. Suppose the cue ball and target ball can be placed anywhere on the half of the table opposite the pocket. The pool table shown at the right is 4 feet wide and 6 feet long. The pocket is located $\sqrt{5}$ feet from the center of the table along the ellipse's major axis. Assuming no spin is placed on either ball and the target ball is struck squarely, where should each be placed to have the best chance of hitting the target ball into the pocket? Explain your reasoning.



48. **Critical Thinking** As the foci of an ellipse move farther apart with the major axis fixed, what figure does the ellipse approach? Justify your answer.

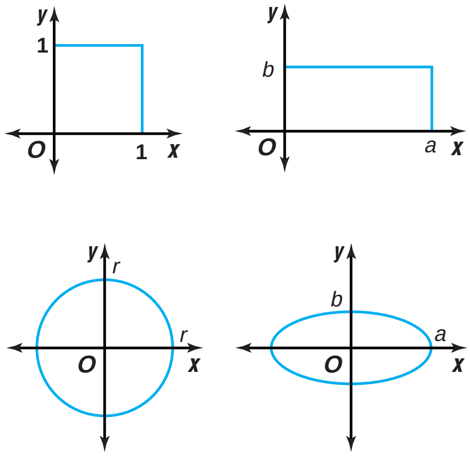
49. **History** A whispering gallery is designed using an elliptical ceiling. It operates on the principle that sound projected from one focus of an ellipse reflects off the ceiling and back to the other focus. The United States Capitol contains such an elliptical room. The room is 96 feet in length, 46 feet in width, and has a ceiling that is about 23 feet high.

a. Write an equation modeling the shape of the room. Assume that is centered at the origin and the major axis is horizontal.

b. John Quincy Adams is known to have overheard conversations being held at the opposing party leader's desk by standing in a certain spot in this room. Describe two possible places where Adams might have stood to overhear.

c. About how far did Adams stand from the desk?

50. Geometry The square has an area of 1 square unit. If the square is stretched horizontally by a factor of a and compressed vertically by a factor of b, the area of the rectangle formed is ab square units.

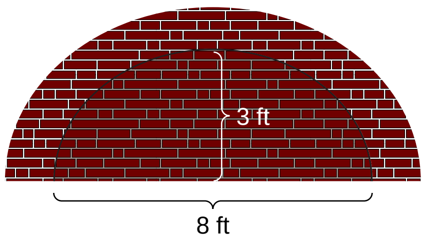


a. The area of a circle with equation $x^2 + y^2 = r^2$ is πr^2. Develop a formula for the area of an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

b. Use the formula found in part **a** to find the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

51. Critical Thinking Show that the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetric with respect to the origin.

52. Construction The arch of a fireplace is to be constructed in the shape of a semi-ellipse. The opening is to have a height of 3 feet at the center and a width of 8 feet along the base. To sketch the outline of the fireplace, the contractor uses an 8-foot string tied to two thumbtacks.



a. Where should the thumbtacks be placed?

b. Explain why this technique works.

53. Astronomy The satellites orbiting Earth follow elliptical paths with the center of Earth as one focus. The table below lists data on five satellites that have orbited or currently orbit Earth.

Satellites Orbiting Earth

Name	Launch Date	Time or expected time aloft	Semi-major axis a (km)	Eccentricity
Sputnik I	Oct. 1957	57.6 days	6955	0.052
Vanguard I	Mar. 1958	300 years	8872	0.208
Skylab 4	Nov. 1973	84.06 days	6808	0.001
GOES 4	Sept. 1980	10^6 years	42,166	0.0003
Intelsat 5	Dec. 1980	10^6 years	41,803	0.007

a. Which satellite has the most circular orbit? Explain your reasoning.

b. Soviet satellite Sputnik I was the first artificial satellite to orbit Earth. If the radius of Earth is approximately 6357 kilometers, find the greatest distance Sputnik I orbited from the surface of Earth to the nearest kilometer.

Mixed Review

54. Write the standard form of the equation of the circle that passes through points at $(0, -9)$, $(7, -2)$, and $(-5, -10)$. Identify the circle's center and radius. Then graph the equation. *(Lesson 10-2)*

55. Determine whether the quadrilateral with vertices at points $(-1, -2)$, $(5, -4)$, $(4, 1)$, and $(-5, 4)$ is a parallelogram. *(Lesson 10-1)*

56. If $\sin \theta = \frac{7}{8}$ and the terminal side of θ is in the first quadrant, find $\cos 2\theta$. *(Lesson 7-4)*

57. Write an equation of the cosine function with an amplitude of 4, a period of 180°, and a phase shift of 20°. *(Lesson 6-5)*

58. Solve $\triangle ABC$ if $C = 121°32'$, $B = 42°5'$, and $a = 4.1$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-6)*

59. Use the Remainder Theorem to find the remainder for the quotient $(x^4 - 4x^3 - 2x^2 - 1) \div (x - 5)$. Then, state whether the binomial is a factor of the polynomial. *(Lesson 4-3)*

60. Determine whether the point at $(-2, -16)$ is the location of a *minimum*, a *maximum*, or a *point of inflection* for the function $x^2 + 4x - 12$. *(Lesson 3-6)*

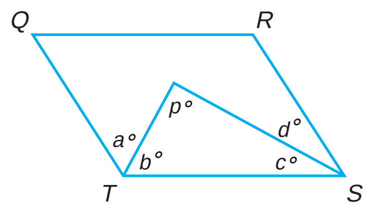
61. Sketch the graph of $g(x) = |x - 2|$. *(Lesson 3-2)*

62. **Motion** A ceiling fan has four evenly spaced blades that are each 2 feet long. Suppose the center of the fan is located at the origin and the blades of the fan lie in the x- and y-axis. Imagine a fly lands on the tip of the blade along the positive x-axis. Find the location of the fly where it landed and its location after a 90°, 180°, and 270° counterclockwise rotation of the fan. *(Lesson 2-4)*

63. **SAT Practice** In parallelogram $QRST$, $a = b$ and $c = d$. What is the value of p?

A 45 **B** 60 **C** 90 **D** 120

E Cannot be determined from the information given



GRAPHING CALCULATOR EXPLORATION

To graph an ellipse, such as the equation $\frac{(x-3)^2}{18} + \frac{(y+2)^2}{32} = 1$, on a graphing calculator, you must first solve for y.

$$\frac{(x-3)^2}{18} + \frac{(y+2)^2}{32} = 1$$

$$32(x-3)^2 + 18(y-2)^2 = 576$$

$$18(y-2)^2 = 576 - 32(x-3)^2$$

$$(y-2)^2 = \frac{576 - 32(x-3)^2}{18}$$

So, $y = \pm\sqrt{\frac{576 - 32(x-3)^2}{18}} + 2$.

The result is two equations. To graph both equations as **Y1**, replace $\pm$ with $\{1, -1\}$.

Like other graphs, there are families of ellipses. Changing certain values in the equation of an ellipse creates a new member of that family.

TRY THESE **For each situation, make a conjecture about the behavior of the graph. Then verify by graphing the original equation and the modified equation on the same screen using a square window.**

1. x is replaced by $(x - 4)$.
2. x is replaced by $(x + 4)$.
3. y is replaced by $(y - 4)$.
4. y is replaced by $(y + 4)$.
5. 32 is switched with 18.

WHAT DO YOU THINK?

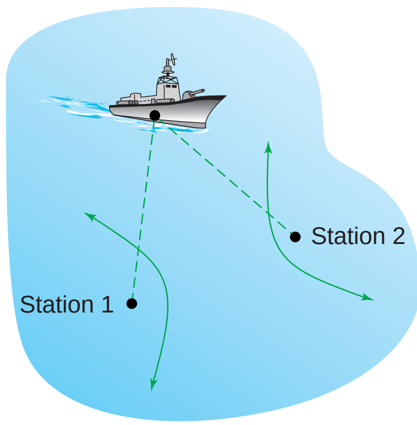
6. Describe the effects of replacing x in the equation of an ellipse with $(x \pm c)$ for $c > 0$.
7. Describe the effects of replacing y in the equation of an ellipse with $(y \pm c)$ for $c > 0$.
8. In the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, describe the effect of interchanging a and b.

10-4 Hyperbolas

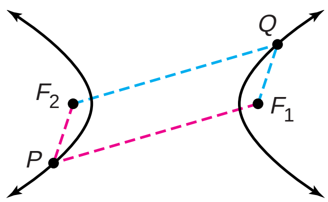
OBJECTIVES
- Use and determine the standard and general forms of the equation of a hyperbola.
- Graph hyperbolas.

Real World Application

NAVIGATION Since World War II, ships have used the LORAN (*LO*ng *RA*nge *N*avigation) system as a means of navigation independent of visibility conditions. Two stations, located a great distance apart, simultaneously transmit radio pulses to ships at sea. Since a ship is usually closer to one station than the other, the ship receives these pulses at slightly different times. By measuring the time differential and by knowing the speed of the radio waves, a ship can be located on a conic whose foci are the positions of the two stations. *A problem related to this will be solved in Example 4.*



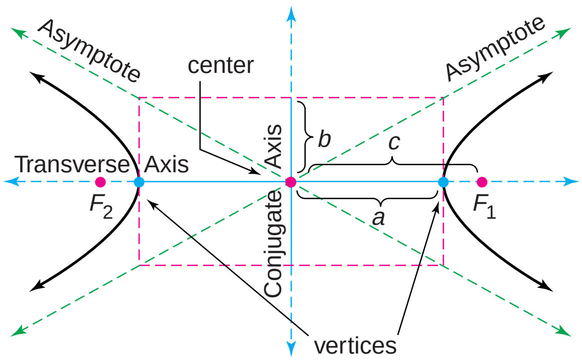
A **hyperbola** is the set of all points in the plane in which the difference of the distances from two distinct fixed points, called **foci,** is constant. That is, if F_1 and F_2 are the foci of a hyperbola and P and Q are any two points on the hyperbola, $|PF_1 - PF_2| = |QF_1 - QF_2|$.



The **center** of a hyperbola is the midpoint of the line segment whose endpoints are the foci. The point on each branch of the hyperbola that is nearest the center is called a **vertex.**

Look Back
Refer to Lesson 3-7 to review asymptotes.

The **asymptotes** of a hyperbola are lines that the curve approaches as it recedes from the center. As you move farther out along the branches, the distance between points on the hyperbola and the asymptotes approaches zero.

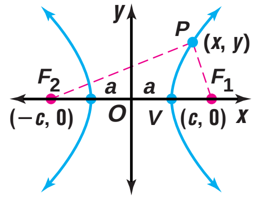


Note that $c > a$ for the hyperbola.

A hyperbola has two axes of symmetry. The line segment connecting the vertices is called the **transverse axis** and has a length of $2a$ units. The segment perpendicular to the transverse axis through the center is called the **conjugate axis** and has length $2b$ units.

For a hyperbola, the relationship among a, b, and c is represented by $a^2 + b^2 = c^2$. The asymptotes contain the diagonals of the rectangle guide, which is $2a$ units by $2b$ units. The point at which the diagonals meet coincides with the center of the hyperbola.

The standard form of the equation of a hyperbola with its origin as its center can be derived from the definition and the Distance Formula. Suppose the foci are on the x-axis at $(c, 0)$ and $(-c, 0)$ and the coordinates of any point on the hyperbola are (x, y).



$$|PF_2 - PF_1| = |VF_2 - VF_1| \quad \textit{Definition of hyperbola}$$

$$\left|\sqrt{(x + c)^2 + y^2} - \sqrt{(x + c)^2 + y^2}\right| = |c + a - (c - a)| \quad \textit{Distance Formula}$$

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x + c)^2 + y^2} = 2a \quad \textit{Simplify.}$$

$$\sqrt{(x - c)^2 + y^2} = 2a + \sqrt{(x + c)^2 + y^2} \quad \textit{Isolate a radical.}$$

$$(x - c)^2 + y^2 = 4a^2 + 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2 \quad \textit{Square each side.}$$

$$-4xc - 4a^2 = 4a\sqrt{(x + c)^2 + y^2} \quad \textit{Simplify.}$$

$$xc + a^2 = -a\sqrt{(x + c)^2 + y^2} \quad \textit{Divide each side by } -4.$$

$$x^2c^2 + 2a^2xc + a^4 = a^2x^2 + 2a^2xc + a^2c^2 + a^2y^2 \quad \textit{Square each side.}$$

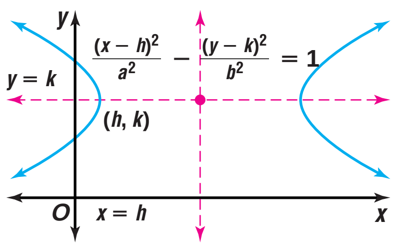
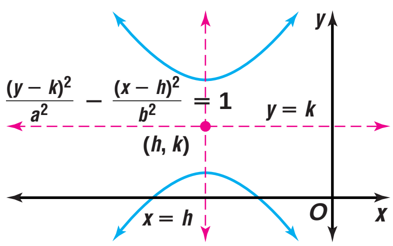
$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \quad \textit{Simplify.}$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1 \quad \textit{Divide by } a^2(c^2 - a^2).$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \textit{By the Pythagorean Theorem, } c^2 - a^2 = b^2.$$

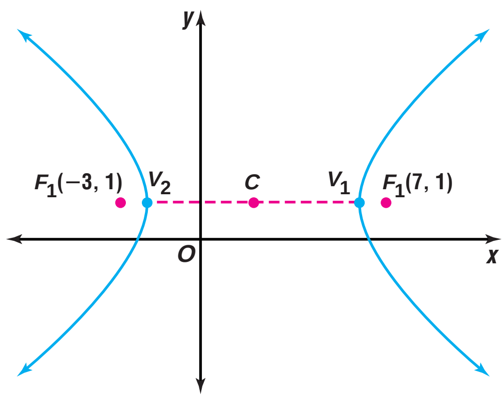
If the foci are on the y-axis, the equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

As with the other graphs we have studied in this chapter, the standard form of the equation of a hyperbola with center other than the origin is a translation of the parent graph to a center at (h, k).

<table>
<tr><th>Standard Form of the Equation of a Hyperbola</th><th>Orientation</th><th>Description</th></tr>
<tr><td>$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$,
where $b^2 = c^2 - a^2$</td><td>
</td><td>center: (h, k)
foci: $(h \pm c, k)$
vertices: $(h \pm a, k)$
equation of
transverse axis: $y = k$
(parallel to x-axis)</td></tr>
<tr><td>$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$,
where $b^2 = c^2 - a^2$</td><td>
</td><td>center: (h, k)
foci: $(h, k \pm c)$
vertices: $(h, k \pm a)$
equation of
transverse axis: $x = h$
(parallel to y-axis)</td></tr>
</table>

Example 1 **Find the equation of the hyperbola with foci at (7, 1) and (−3, 1) whose transverse axis is 8 units long.**

A sketch of the graph is helpful. Let F_1 and F_2 be the foci, let V_1 and V_2 be the vertices, and let C be the center.



To locate the center, find the midpoint of $\overline{F_1F_2}$.

$$\left(\frac{7 + (-3)}{2}, \frac{1 + 1}{2}\right) \text{ or } (2, 1)$$

Thus, $h = 2$ and $k = 1$ since (2, 1) is the center.

The transverse axis is 8 units long. Thus, $2a = 8$ or $a = 4$. So $a^2 = 16$.

Use the equation $b^2 = c^2 - a^2$ to find b^2.

$b^2 = c^2 - a^2$

$b^2 = 25 - 16$ *$c = 5$, $a = 4$*

$b^2 = 9$

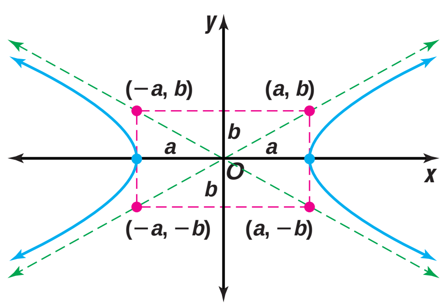
Recall that c is the distance from the center to a focus. Here $c = CF_1$ or 5.

Use the standard form when the transverse axis is parallel to the x-axis.

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \Rightarrow \frac{(x-2)^2}{16} - \frac{(y-1)^2}{9} = 1$$

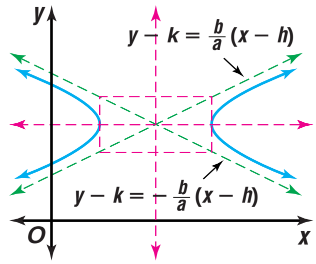
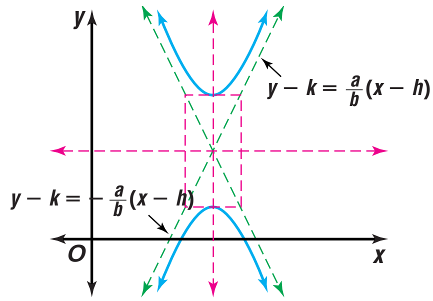
Before graphing a hyperbola, it is often helpful to graph the asymptotes. As noted in the beginning of the lesson, the asymptotes contain the diagonals of the rectangle guide defined by the transverse and conjugate axes. While not part of the graph, a sketch of this $2a$ by $2b$ rectangle provides an easy way to graph the asymptotes of the hyperbola. Suppose the center of a hyperbola is the origin and the transverse axis lies along the x-axis.

From the figure at the right, we can see that the asymptotes have slopes equal to $\pm\frac{b}{a}$. Since both lines have a y-intercept of 0, the equations for the asymptotes are $y = \pm\frac{b}{a}x$.



If the hyperbola were oriented so that the traverse axis was parallel to the y-axis, the slopes of the asymptotes would be $\pm\frac{a}{b}$. Thus, the equations of the asymptotes would be $y = \pm\frac{a}{b}x$.

The equations of the asymptotes of any hyperbola can be determined by a translation of the graph to a center at (h, k).

Equations of the Asymptotes of a Hyperbola	$y - k = \pm\frac{b}{a}(x - h)$, for a hyperbola with standard form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	
	$y - k = \pm\frac{a}{b}(x - h)$, for a hyperbola with standard form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$	

Example 2 **Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $\frac{(y+4)^2}{36} - \frac{(x-2)^2}{25} = 1$. Then graph the equation.**

Since the y terms are in the first expression, the hyperbola has a vertical transverse axis.

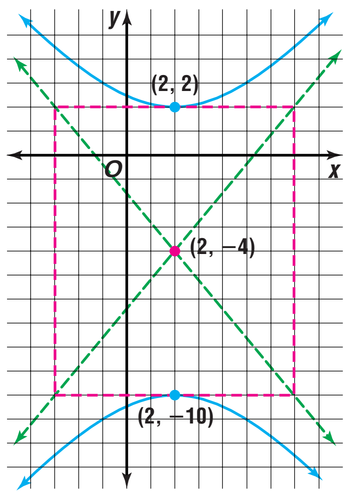
From the equation, $h = 2$, $k = -4$, $a = 6$, and $b = 5$. The center is at $(2, -4)$.

The equations of the asymptotes are $y + 4 = \pm\frac{6}{5}(x - 2)$.

The vertices are at $(h, k \pm a)$ or $(2, 2)$ and $(2, -10)$.

Since $c^2 = a^2 + b^2$, $c = \sqrt{61}$. Thus, the foci are at $\left(2, -4 + \sqrt{61}\right)$ and $\left(2, -4 - \sqrt{61}\right)$.

Graph the center, vertices, and the rectangle guide. Next graph the asymptotes. Then sketch the hyperbola.



Graphing Calculator Tip

You can graph a hyperbola on a graphing calculator by first solving for y and then graphing the two resulting equations on the same screen.

By expanding the standard form for a hyperbola, you can determine the general form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ where $A \neq 0$, $C \neq 0$, and A and C have different signs.

As with other general forms we have studied, the general form of a hyperbola can be rewritten in standard form. While it is important to be able to recognize the equation of a hyperbola in general form, the standard form provides important information about the hyperbola that makes it easier to graph.

Example 3 **Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $9x^2 - 4y^2 - 54x - 40y - 55 = 0$. Then graph the equation.**

Write the equation in standard form. Use the same process you used with ellipses.

$$9x^2 - 4y^2 - 54x - 40y - 55 = 0$$

$$9x^2 - 54x - 4y^2 - 40y = 55 \quad \textit{Rearrange terms.}$$

$$9(x^2 - 6x + ?) - 4(y^2 + 10y + ?) = 55 + ? + ? \quad \textit{Factor GCF for each variable.}$$

$$9(x^2 - 6x + 9) - 4(y^2 + 10y + 25) = 55 + 9(9) - 4(25) \quad \textit{Complete the square.}$$

$$9(x - 3)^2 - 4(y + 5)^2 = 36 \quad \textit{Factor.}$$

$$\frac{(x-3)^2}{4} - \frac{(y+5)^2}{9} = 1 \quad \textit{Divide each side by 36.}$$

The center is at $(3, -5)$. Since the x terms are in the first expression, the transverse axis is horizontal.

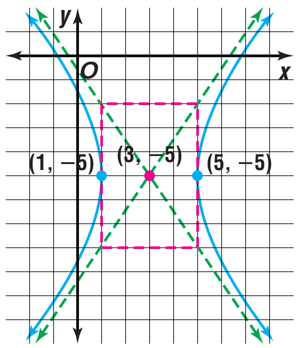
$a = 2, b = 3, c = \sqrt{13}$

The foci are at $\left(3 - \sqrt{13}, -5\right)$ and $\left(3 + \sqrt{13}, -5\right)$.

The vertices are at $(1, -5)$ and $(5, -5)$.

The asymptotes have equations $y + 5 = \pm\frac{3}{2}(x - 3)$.

Graph the vertices and the rectangle guide. Next graph the asymptotes. Then sketch the hyperbola.



One interesting application of hyperbolas is in navigation.

Example 4 Real World Application

NAVIGATION **Refer to the application at the beginning of the lesson. Suppose LORAN stations A and B are located 400 miles apart along a straight shore, with A due west of B. A ship approaching the shore receives radio pulses from the stations and is able to determine that it is 100 miles farther from station A than it is from station B.**

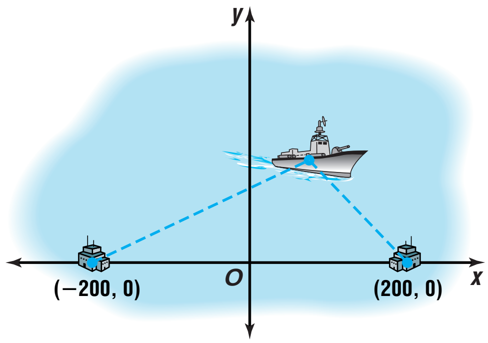
a. Find the equation of the hyperbola on which the ship is located.

b. Find the exact coordinates of the ship if it is 60 miles from shore.

a. First set up a rectangular coordinate system with the origin located midway between station A and station B.

The stations are located at the foci of the hyperbola, so $c = 200$.

The difference of the distances from the ship to each station is 100 miles. By definition of a hyperbola, this difference equals $2a$, so $a = 50$. The vertices of the hyperbola are located on the same axis as the foci, so the vertices of the hyperbola the ship is on are at $(-50, 0)$ and $(50, 0)$.



Since the hyperbola's transverse axis is the x-axis, the form of the equation of this hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Using the equation $b^2 = c^2 - a^2$, we can find the value for b^2.

$b^2 = c^2 - a^2$

$b^2 = 200^2 - 50^2$ *$a = 50$, $c = 200$*

$b^2 = 37{,}500$

Thus, the equation of the hyperbola is $\frac{x^2}{2500} - \frac{y^2}{37{,}500} = 1$

b. If the ship is 60 miles from shore, let $y = 60$ in the equation of the hyperbola and solve for x.

$$\frac{x^2}{2500} - \frac{60^2}{37{,}500} = 1$$

$$\frac{x^2}{2500} = 1 + \frac{60^2}{37{,}500}$$

$$\frac{x^2}{2500} = 1.096$$

$$x^2 = 2500(1.096)$$

$$x = \pm\sqrt{2740} \approx \pm 52.3$$

Since the ship is closer to station B than station A, we use the positive value of x to locate the ship at coordinates $(52.3, 60)$.

In the standard form of the equation of a hyperbola, if $a = b$, the graph is an **equilateral hyperbola.** Replacing a with b in the equations of the asymptotes of a hyperbola with a horizontal transverse axis reveals a property of equilateral hyperbolas.

$y - k = \frac{b}{a}(x - h)$		$y - k = -\frac{b}{a}(x - h)$
$y - k = \frac{b}{b}(x - h)$	*Let $a = b$*	$y - k = -\frac{b}{b}(x - h)$
$y - k = (x - h)$		$y - k = -(x - h)$

The slopes of the equations of the two asymptotes are negative reciprocals, 1 and −1. Thus, the asymptotes of an equilateral hyperbola are perpendicular.

Remember that $xy = c$ is the general equation that models inverse variation.

A special case of the equilateral hyperbola is a **rectangular hyperbola,** where the coordinate axes are the asymptotes. The general equation of a rectangular hyperbola is $xy = c$, where c is a nonzero constant. The sign of the constant c determines the location of the branches of the hyperbola.

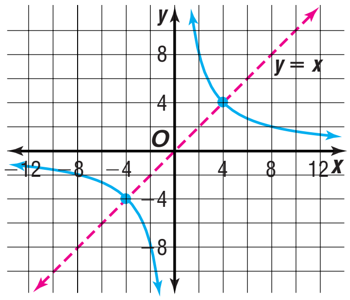
Rectangular Hyperbola: $xy = c$	
Value of c	**Location of branches of hyperbola**
Positive	Quadrants I and III
Negative	Quadrants II and IV

Example 5 **Graph $xy = 16$.**

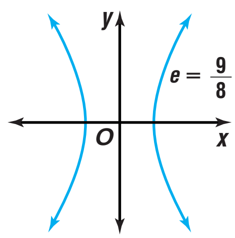
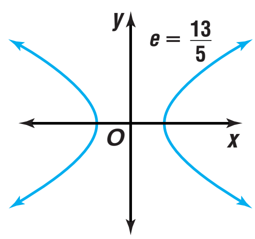
Since c is positive, the hyperbola lies in the first and third quadrants.

The transverse axis is along the graph of $y = x$.

The coordinates of the vertices must satisfy the equation of the hyperbola and also their graph must be points on the traverse axis. Thus, the vertices are at (4, 4) and (−4, −4).



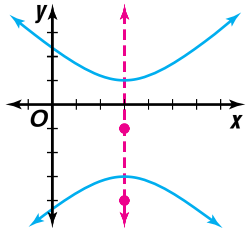
Like an ellipse, the shape of a hyperbola is determined by its eccentricity, which is again defined as $e = \frac{c}{a}$. However, in a hyperbola, $0 < a < c$. So, $0 < 1 < e$ or $e > 1$. The table below shows the relationship between the value of the e and the shape of the hyperbola.

Value of e	Graph
close to 1	
not close to 1	

Since $c^2 = a^2 + b^2$ in hyperbolas, it can be shown that $b^2 = a^2(e^2 - 1)$. *You will derive this formula in Exercise 3.*

Example 6 **Write the equation of the hyperbola with center at (3, −1), a focus at (3, −4), and eccentricity $\frac{3}{2}$.**

Sketch the graph using the points given. Since the center and focus have the same x-coordinate, the transverse axis is vertical. Use the form $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.



The focus is 3 units below the center, so $c = 3$. Now use the eccentricity to find the values of a^2 and b^2.

$e = \frac{c}{a}$

$\frac{3}{2} = \frac{3}{a}$ *$c = 3$ and $e = \frac{3}{2}$*

$2 = a$

$4 = a^2$

$b^2 = a^2(e^2 - 1)$

$b^2 = 4\left(\frac{9}{4} - \frac{4}{4}\right)$ *$a^2 = 4$ and $e = \frac{3}{2}$*

$b^2 = 5$

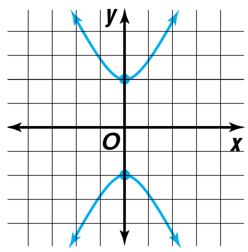
The equation is $\frac{(y+1)^2}{4} - \frac{(x-3)^2}{5} = 1$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** the standard forms of the equations of hyperbolas and ellipses.
2. **Determine** which of the following equations matches the graph of the hyperbola at right.
   a. $\frac{x^2}{4} - y^2 = 1$ b. $\frac{y^2}{4} - x^2 = 1$ c. $x^2 - \frac{y^2}{4} = 1$



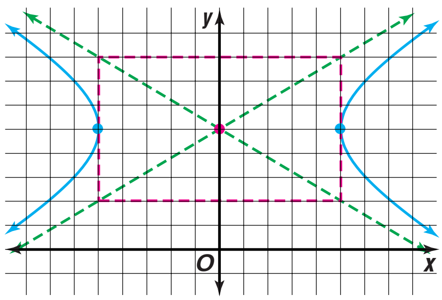
3. **Derive** the equation $b^2 = a^2(e^2 - 1)$ for a hyperbola.
4. *Math Journal* **Write** an explanation of how to determine whether the transverse axis of a hyperbola is horizontal or vertical.

Guided Practice

For the equation of each hyperbola, find the coordinates of the center, the foci, and the vertices and the equations of the asymptotes of its graph. Then graph the equation.

5. $\frac{x^2}{25} - \frac{y^2}{4} = 1$ 6. $\frac{(y-3)^2}{16} - \frac{(x-2)^2}{4} = 1$ 7. $y^2 - 5x^2 + 20x = 50$

8. Write the equation of the hyperbola graphed at the right.



9. Graph $xy = -9$.

Write an equation of the hyperbola that meets each set of conditions.

10. The center is at $(1, -4)$, $a = 5$, $b = 2$, and it has a horizontal transverse axis.

11. The length of the conjugate axis is 6 units, and the vertices are at $(3, 4)$ and $(3, 0)$.

12. The hyperbola is equilateral and has foci at $(0, 6)$ and $(0, -6)$.

13. The eccentricity of the hyperbola is $\frac{5}{3}$, and the foci are at $(10, 0)$ and $(-10, 0)$.

14. Aviation Airplanes are equipped with signal devices to alert rescuers to their position. Suppose a downed plane sends out radio pulses that are detected by two receiving stations, A and B. The stations are located 130 miles apart along a stretch of I-40, with A due west of B. The two stations are able to determine that the plane is 50 miles farther from station B than from station A.

a. Determine the equation of the hyperbola centered at the origin on which the plane is located.

b. Graph the equation, indicating on which branch of the hyperbola the plane is located.

c. If the pilot estimates that the plane is 6 miles from I-40, find the exact coordinates of its position.

EXERCISES

Practice

For the equation of each hyperbola, find the coordinates of the center, the foci, and the vertices and the equations of the asymptotes of its graph. Then graph the equation.

15. $\frac{x^2}{100} - \frac{y^2}{16} = 1$

16. $\frac{x^2}{9} - \frac{(y-5)^2}{81} = 1$

17. $\frac{x^2}{4} - \frac{y^2}{49} = 1$

18. $\frac{(y-7)^2}{64} - \frac{(x+1)^2}{4} = 1$

19. $x^2 - 4y^2 + 6x - 8y = 11$

20. $-4x^2 + 9y^2 - 24x - 90y + 153 = 0$

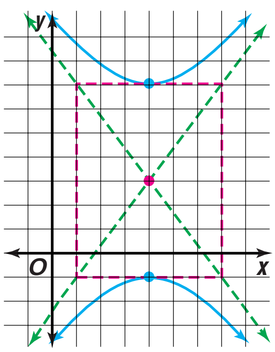
21. $16y^2 - 25x^2 - 96y + 100x - 356 = 0$

22. $36x^2 - 49y^2 - 72x - 294y = 2169$

23. Graph the equation $25y^2 - 9x^2 - 100y - 72x - 269 = 0$. Label the center, foci and the equations of the asymptotes.

Write the equation of each hyperbola.

24.



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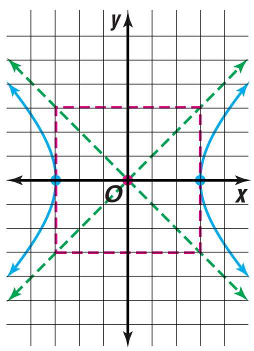
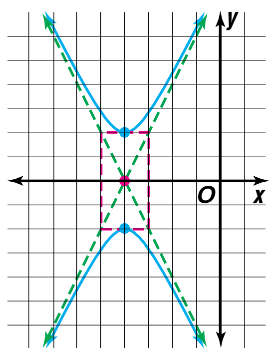


26.



Graph each equation.

27. $xy = 49$

28. $xy = -36$

29. $4xy = -25$

30. $9xy = 16$

Write an equation of the hyperbola that meets each set of conditions.

31. The center is at $(4, -2)$, $a = 2$, $b = 3$, and it has a vertical transverse axis.

32. The vertices are at $(0, 3)$ and $(0, -3)$, and a focus is at $(0, -9)$.

33. The length of the transverse axis is 6 units, and the foci are at $(5, 2)$ and $(-5, 2)$.

34. The length of the conjugate axis is 8 units, and the vertices are at $(-3, 9)$ and $(-3, -5)$.

35. The hyperbola is equilateral and has foci at $(8, 0)$ and $(-8, 0)$.

36. The center is at $(-3, 1)$, one focus is at $(2, 1)$, and the eccentricity is $\frac{5}{4}$.

37. A vertex is at $(4, 5)$, the center is at $(4, 2)$, and an equation of one asymptote is $4y + 4 = 3x$.

38. The equation of one asymptote is $3x - 11 = 2y$. The hyperbola has its center at $(3, -1)$ and a vertex at $(5, -1)$.

39. The hyperbola has foci at $(0, 8)$ and $(0, -8)$ and eccentricity $\frac{4}{3}$.

40. The hyperbola has eccentricity $\frac{6}{5}$ and foci at $(10, -3)$ and $(-2, -3)$.

41. The hyperbola is equilateral and has foci at $(9, 0)$ and $(-9, 0)$.

42. The slopes of the asymptotes are ± 2, and the foci are at $(1, 5)$ and $(1, -3)$.

Applications and Problem Solving

43. Chemistry According to Boyle's Law, the pressure P (in kilopascals) exerted by a gas varies inversely as the volume V (in cubic decimeters) of a gas if the temperature remains constant. That is, $PV = c$. Suppose the constant for oxygen at 25°C is 505.

a. Graph the function $PV = c$ for $c = 505$.

b. Determine the volume of oxygen if the pressure is 101 kilopascals.

c. Determine the volume of oxygen if the pressure is 50.5 kilopascals.

d. Study your results for parts **b** and **c**. If the pressure is halved, make a conjecture about the effect on the volume of gas.

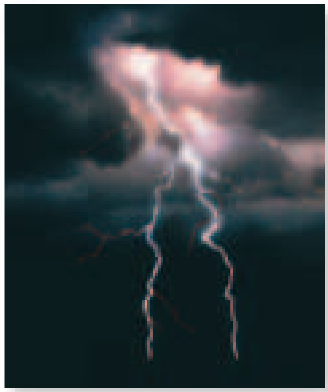
44. Critical Thinking Prove that the eccentricity of all equilateral hyperbolas is $\sqrt{2}$.

45. Nuclear Power A nuclear cooling tower is a *hyperboloid,* that is, a hyperbola rotated around its conjugate axis. Suppose the hyperbola used to generate the hyperboloid modeling the shape of the cooling tower has an eccentricity of $\frac{5}{3}$.

a. If the cooling tower is 150 feet wide at its narrowest point, determine an equation of the hyperbola used to generate the hyperboloid.

b. If the tower is 450 feet tall, the top is 100 feet above the center of the hyperbola, and the base is 350 feet below the center, what is the radius of the top and the base of the tower?

46. **Forestry** Two ranger stations located 4 miles apart observe a lightning strike. A ranger at station A reports hearing the sound of thunder 2 seconds prior to a ranger at station B. If sound travels at 1100 feet per second, determine the equation of the hyperbola on which the lightning strike was located. Place the two ranger stations on the x-axis with the midpoint between the two stations at the origin. The transverse axis is horizontal.



47. **Critical Thinking** A hyperbola has foci $F_1(-6, 0)$ and $F_1(6, 0)$. For any point $P(x, y)$ on the hyperbola, $|PF_1 - PF_2| = 10$. Write the equation of the hyperbola in standard form.

48. **Analytic Geometry** Two hyperbolas in which the transverse axis of one is the conjugate axis of the other are called *conjugate hyperbolas*. In equations of conjugate hyperbolas, the x^2 and y^2 terms are reversed. For example, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{16} = 1$ are equations of conjugate hyperbolas.
   a. Graph $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and $\frac{y^2}{9} - \frac{x^2}{16} = 1$ on the same coordinate plane.
   b. What is true of the asymptotes of conjugate hyperbolas?
   c. Write the equation of the conjugate hyperbola for $\frac{(x-3)^2}{16} - \frac{(y-2)^2}{25} = 1$.
   d. Graph the conjugate hyperbolas in part **c**.

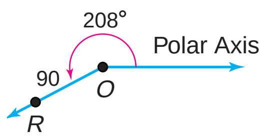
Mixed Review

49. Write the equation of the ellipse that has a semi-major axis length of 4 units and foci at (2, 3) and (2, −3). *(Lesson 10-3)*

50. Write $x^2 + y^2 - 4x + 14y - 28 = 0$ in standard form. Then graph the equation. *(Lesson 10-2)*

51. Show that the points with coordinates (−1, 3), (3, 6), (6, 2), and (2, −1) are the vertices of a square. *(Lesson 10-1)*

52. Name three different pairs of polar coordinates that represent point R. Assume $-360° \le \theta \le 360°$. *(Lesson 9-1)*



53. Find the inner product of vectors ⟨4, −1, 8⟩ and ⟨−5, 2, 2⟩. Are the vectors perpendicular? Explain. *(Lesson 8-4)*

54. Write the standard from of the equation of the line that has a normal 3 units long and makes an angle of 60° with the positive x-axis. *(Lesson 7-6)*

55. **Aviation** An airplane flying at an altitude of 9000 meters passes directly overhead. Fifteen seconds later, the angle of elevation to the plane is 60°. How fast is the airplane flying? *(Lesson 5-4)*

56. Approximate the real zeros of the function $f(x) = 4x^4 + 5x^3 - x^2 + 1$ to the nearest tenth. *(Lesson 4-5)*

57. **SAT/ACT Practice** If r and s are integers and $r + s = 0$, which of the following must be true?
   I. $r^3 > s^3$ II. $r^3 = s^3$ III. $r^4 = s^4$
   A I only **B** II only **C** III only
   D I and II only **E** I and III only

Extra Practice See p. A45.

10-5

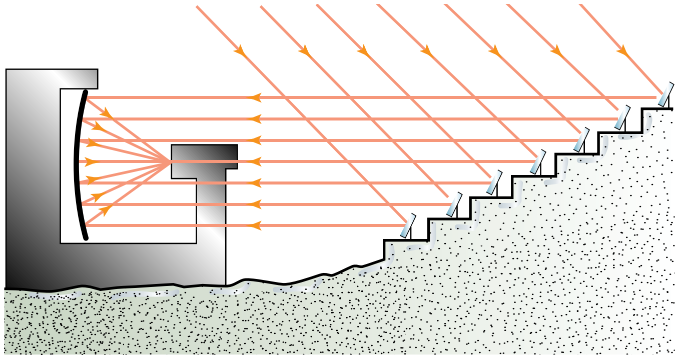
Parabolas

OBJECTIVES

- Use and determine the standard and general forms of the equation of a parabola.
- Graph parabolas.

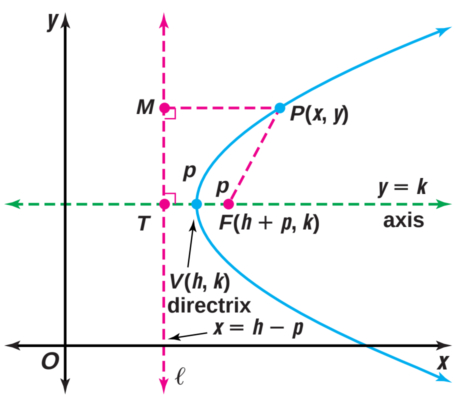
Real World Application

ENERGY The Odeillo Solar Furnace, located in southern France, uses a series of 63 flat mirrors, arranged on terraces on a hillside, to reflect the sun's rays on to a large parabolic mirror. These computer-controlled mirrors tilt to track the sun and ensure that its rays are always reflected to the central parabolic mirror.



This mirror in turn reflects the sun's rays to the focal point where a furnace is mounted on a tower. The concentrated energy generates temperatures of up to 6870°F. If the width of the Odeillo parabolic mirror is 138 feet and the furnace is located 58 feet from the center of the mirror, how deep is the mirror? *This problem will be solved in Example 2.*

In Chapter 4, you learned that the graphs of quadratic equations like $x = y^2$ or $y = x^2$ are called *parabolas.* A parabola is defined as the set of all points in a plane that are the same distance from a given point, called the **focus,** and a given line, called the **directrix.** *Remember that the distance from a point to a line is the length of the segment from the point perpendicular to the line.*



In the figure at the left, F is the focus of the parabola and ℓ is the directrix. This parabola is symmetric with respect to the line $y = k$, which passes through the focus. This line is called the **axis of symmetry,** or, more simply, the *axis* of the parabola. The point at which the axis intersects the parabola is called the **vertex.**

Suppose the vertex V has coordinates (h, k). Let p be the distance from the focus to the vertex, FV. By the definition of a parabola, the distance from any point on the parabola to the focus must equal the distance from that point to the directrix. So, if $FV = p$, then $VT = p$. The coordinates of F are $(h + p, k)$, and the equation of the directrix is $x = h - p$.

Now suppose that $P(x, y)$ is any point on the parabola other than the vertex. From the definition of a parabola, you know that $PF = PM$. Since M lies on the directrix, the coordinates of M are $(h - p, y)$.

For PF, let $F(h + p, k)$ be (x_1, y_1) and $P(x, y)$ be (x_2, y_2). Then for PM, let M be (x_1, y_1). You can use the Distance Formula to determine the equation for the parabola.

$$PF = PM$$

$$\sqrt{[x - (h + p)]^2 + (y - k)^2} = \sqrt{[x - (h - p)]^2 + (y - y)^2}$$

$$[x - (h + p)]^2 + (y - k)^2 = [x - (h - p)]^2 \quad \textit{Square each side.}$$

$$x^2 - 2x(h + p) + (h + p)^2 + (y - k)^2 = x^2 - 2x(h - p) + (h - p)^2$$

This equation can be simplified to obtain the equation

$$(y - k)^2 = 4p(x - h).$$

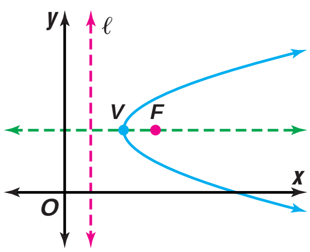
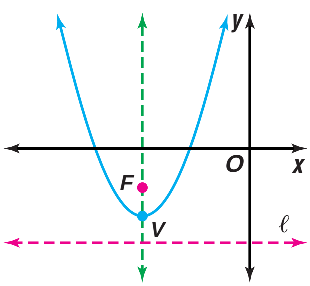
When p is positive, the parabola opens to the right.
When p is negative, the parabola opens to the left.

Unlike the equations of other conic sections, the equation of a parabola has only one squared term.

This is the equation of a parabola whose directrix is parallel to the y-axis. The equation of a parabola whose directrix is parallel to the x-axis can be obtained by switching the terms in the parentheses of the previous equation.

$$(x - h)^2 = 4p(y - k)$$

When p is positive, the parabola opens upward.
When p is negative, the parabola opens downward.

Standard Form of the Equation of a Parabola	Orientation when $p > 0$	Description
$(y - k)^2 = 4p(x - h)$		vertex: (h, k) focus: $(h + p, k)$ axis of symmetry: $y = k$ directrix: $x = h - p$ opening: right if $p > 0$ left if $p < 0$
$(x - h)^2 = 4p(y - k)$		vertex: (h, k) focus: $(h, k + p)$ axis of symmetry: $x = h$ directrix: $y = k - p$ opening: upward if $p > 0$ downward if $p < 0$

Example 1 **Consider the equation $y^2 = 8x + 48$.**

a. Find the coordinates of the focus and the vertex and the equations of the directrix and the axis of symmetry.

b. Graph the equation of the parabola.

a. First, write the equation in the form $(y - k)^2 = 4p(x - h)$.

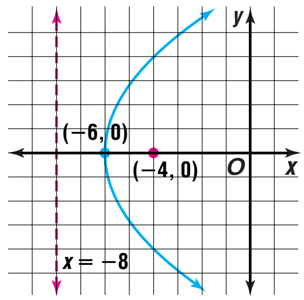
$$y^2 = 8x + 48$$
$$y^2 = 8(x + 6) \quad \textit{Factor.}$$
$$(y - 0)^2 = 4(2)(x + 6) \quad \textit{4p = 8, so p = 2}$$

In this form, we can see that $h = -6$, $k = 0$, and $p = 2$. We can use this to find the desired information.

Vertex: $(-6, 0)$ — *(h, k)*
Directrix: $x = -8$ — *$x = h - p$*
Focus: $(-4, 0)$ — *$(h + p, k)$*
Axis of Symmetry: $y = 0$ — *$y = k$*

The axis of symmetry is the x-axis. Since p is positive, the parabola opens to the right.

b. Graph the directrix, the vertex, and the focus. To determine the shape of the parabola, graph several other ordered pairs that satisfy the equation and connect them with a smooth curve.



Graphing Calculator Tip

You can graph a parabola that opens to the right or to left by first solving for y and then graphing the two resulting equations on the same screen.

One useful property of parabolic mirrors is that all light rays traveling parallel to the mirror's axis of symmetry will be reflected by the parabola to the focus.

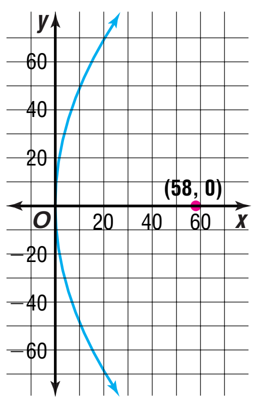
Example 2 Real World Application

ENERGY **Refer to the application at the beginning of the lesson.**

a. Find and graph the equation of a parabola that models the shape of the Odeillo mirror.

b. Find the depth of the parabolic mirror.

a. The shape of the mirror can be modeled by a parabola with vertex at the origin and opening to the right. The general equation of such a parabola is $y^2 = 4px$, where p is the focal length. Given a focal length of 58 feet, we can derive the model equation.



$$y^2 = 4px$$
$$y^2 = 4(58)x \quad \textit{p = 58}$$
$$y^2 = 232x$$

b. With the mirror's vertex at the origin, the distance from the vertex to one edge of the mirror is half the overall width of the mirror, $\frac{1}{2}$(138 feet) or 69 feet.

Use the model equation to find the depth x of the mirror when the distance from the center is 69 feet.

$$\begin{aligned} y^2 &= 232x \\ (69)^2 &= 232x \quad \textit{y = 69} \\ 4761 &= 232x \\ x &= \frac{4761}{232} \text{ or about } 20.5 \end{aligned}$$

The mirror is about 20.5 feet deep.

You can use the same process you used with circles to rewrite the standard form of the equation of a parabola in general form. *You will derive the general form in Exercise 37.*

General Form for the Equation of a Parabola

The general form of the equation of a parabola is $y^2 + Dx + Ey + F = 0$, when the directrix is parallel to the y-axis, or $x^2 + Dx + Ey + F = 0$, when the directrix is parallel to the x-axis.

It is necessary to convert an equation in general form to standard form to determine the coordinates of the vertex (h, k) and the distance from the vertex to the focus p.

Example 3 **Consider the equation $2x^2 - 8x + y + 6 = 0$.**

a. Write the equation in standard form.

b. Find the coordinates of the vertex and focus and the equations for the directrix and the axis of symmetry.

c. Graph the equation of the parabola.

a. Since x is squared, the directrix of this parabola is parallel to the x-axis.

$$\begin{aligned} 2x^2 - 8x + y + 6 &= 0 \\ 2x^2 - 8x &= -y - 6 && \textit{Isolate the x terms and the y terms.} \\ 2(x^2 - 4x + ?) &= -y - 6 + ? && \textit{The GCF of the x terms is 2.} \\ 2(x^2 - 4x + 4) &= -y - 6 + 2(4) && \textit{Complete the square.} \\ 2(x - 2)^2 &= -(y - 2) && \textit{Simplify and factor.} \\ (x - 2)^2 &= -\frac{1}{2}(y - 2) && \textit{Divide each side by 2.} \end{aligned}$$

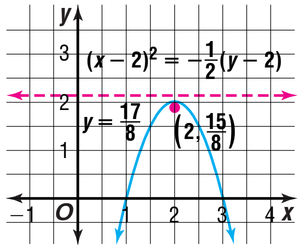
The standard form of the equation is $(x - 2)^2 = -\frac{1}{2}(y - 2)$.

b. Since $4p = -\frac{1}{2}$, $p = -\frac{1}{8}$.

vertex: $(2, 2)$ *(h, k)* focus: $\left(2, \frac{15}{8}\right)$ *(h, k + p)*

directrix: $y = \frac{17}{8}$ *y = k − p* axis of symmetry: $x = 2$ *x = h*

c. Now sketch the graph of the parabola using the information found in part b.



Parabolas are often used to demonstrate maximum or minimum points in real-world situations.

Example 4 **AERONAUTICS** **NASA's KC-135A aircraft flies in parabolic arcs to simulate the weightlessness experienced by astronauts in space. The aircraft starts its ascent at 24,000 feet. During the ascent, all on board experience 2g's or twice the pull of Earth's gravity. As the aircraft approaches its maximum height, the engines are stopped, and the aircraft is allowed to free fall at a precisely determined angle. Zero gravity is achieved for 25 seconds as the plane reaches the top of the parabola and begins its descent. After this 25-second period, the engines are throttled to bring the aircraft out of the dive. If the height of the aircraft in feet (y) versus time in seconds (x) is modeled by the equation $x^2 - 65x + 0.11y - 2683.75 = 0$, what is the maximum height achieved by the aircraft during its parabolic flight?**

Real World Application

First, write the equation in standard form.

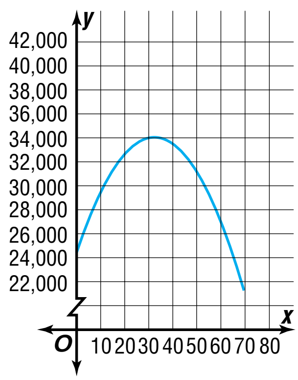
$$x^2 - 65x + 0.11y - 2683.75 = 0$$

$$x^2 - 65x = -0.11y + 2683.75 \quad \textit{Isolate the x terms and y terms.}$$

$$x^2 - 65x + 1056.25 = -0.11y + 2683.75 + 1056.25 \quad \textit{Complete the square.}$$

$$(x - 32.5)^2 = -0.11y + 3740$$

$$(x - 32.5)^2 = -0.11(y - 34{,}000)$$

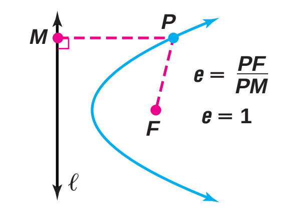
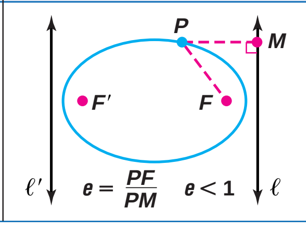
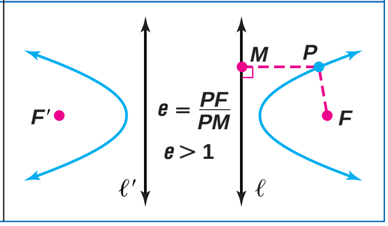


The vertex of the parabola is at (32.5, 34,000).

Remember that the vertex is the maximum or minimum point of a parabola. Since the parabola opens downward, the vertex is the maximum.

The x-coordinate of the vertex, 32.5, represents 32.5 seconds after the aircraft began the parabolic maneuver. The y-coordinate, 34,000, represents a maximum height of 34,000 feet.

All conics can be defined using the focus-directrix definition presented in this lesson. A conic section is defined to be the **locus** of points such that, for any point P in the locus, the ratio of the distance between that point and a fixed point F to the distance between that point and a fixed line ℓ, is constant. As we have seen, the point F is called the focus, and the line ℓ is the directrix. That ratio is the eccentricity of the curve, and its value can be used to determine the conic's classification. In the case of a parabola, $e = 1$. As shown previously, if $0 < e < 1$, the conic is an ellipse. If $e = 0$, the conic is a circle, and if $e > 1$, the conic is a hyperbola.

parabola	ellipse	hyperbola
$e = 1$	$e < 1, e \neq 0$	$e > 1$
		

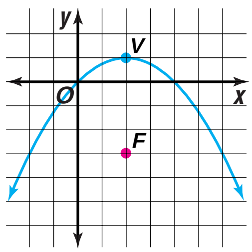
For those conics having more than one focus and directrix, F′ and ℓ′ represent alternates that define the same conic.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** a way in which you might distinguish the equation of a parabola from the equation of a hyperbola.
2. **Write** the equation of the graph shown at the right.
3. **Describe** the relationships among the vertex, focus, directrix and axis of symmetry of a parabola.
4. **Write** the equation in standard form of a parabola with vertex at $(-4, 5)$, opening to the left, and with a focus 5 units from its vertex.
5. **Identify** each of the following conic sections given their eccentricities.



a. $e = \frac{1}{2}$ **b.** $e = 1$ **c.** $e = 1.25$ **d.** $e = 0$

Guided Practice

For the equation of each parabola, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

6. $x^2 = 12(y - 1)$
7. $y^2 - 4x + 2y + 5 = 0$
8. $x^2 + 8x + 4y + 8 = 0$

Write the equation of the parabola that meets each set of conditions. Then graph the equation.

9. The vertex is at the origin, and the focus is at $(0, -4)$.
10. The parabola passes through the point at $(2, -1)$, has its vertex at $(-7, 5)$, and opens to the right.
11. The parabola passes through the point at $(5, 2)$, has a vertical axis, and has a minimum at $(4, -3)$.

12. **Sports** In 1998, Sammy Sosa of the Chicago Cubs was in a homerun race with Mark McGwire of the St. Louis Cardinals. One day, Mr. Sosa popped a baseball straight up at an initial velocity v_0 of 56 feet per second. Its distance s above the ground after t seconds is described by $s = v_0t - 16t^2 + 3$.

a. Graph the function $s = v_0t - 16t^2 + 3$ for the given initial velocity.

b. Find the maximum height achieved by the ball.

c. If the ball is allowed to fall to the ground, how many seconds, to the nearest tenth, is it in the air?

EXERCISES

Practice

For the equation of each parabola, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

13. $y^2 = 8x$

14. $x^2 = -4(y - 3)$

15. $(y - 6)^2 = 4x$

16. $y^2 + 12x = 2y - 13$

17. $y - 2 = x^2 - 4x$

18. $x^2 + 10x + 25 = -8y + 24$

19. $y^2 - 2x + 14y = -41$

20. $y^2 - 2y - 12x + 13 = 0$

21. $2x^2 - 12y - 16x + 20 = 0$

22. $3x^2 - 30y - 18x + 87 = 0$

23. Consider the equation $2y^2 + 16y + 16x + 64 = 0$. Identify the coordinates of the vertex and focus and the equations of the directrix and axis of symmetry. Then graph the equation.

Write the equation of the parabola that meets each set of conditions. Then graph the equation.

24. The vertex is at (−5, 1), and the focus is at (2, 1).

25. The equation of the axis is $y = 6$, the focus is at (0, 6), and $p = -3$.

26. The focus is at (4, −1), and the equation of the directrix is $y = -5$.

27. The parabola passes through the point at (5, 2), has a vertical axis, and has a maximum at (4, 3).

28. The parabola passes through the point at (−3, 1), has its vertex at (−2, −3), and opens to the left.

29. The focus is at (−1, 7), the length from the focus to the vertex is 2 units, and the function has a minimum.

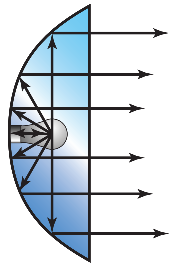
30. The parabola has a vertical axis and passes through points at (1, −7), (5, −3), and (7, −4).

31. The parabola has a horizontal axis and passes through the origin and points at (−1, 2) and (3, −2).

32. The parabola's directrix is parallel to the x-axis, and the parabola passes through points at (1, 1), (0, 9), and (2, 1).

Applications and Problem Solving

33. **Automotive** Automobile headlights contain parabolic reflectors, which work on the principle that light placed at the focus of a parabola will reflect off the mirror-like surface in lines parallel to the axis of symmetry. Suppose a bulb is placed at the focus of a headlight's reflector, which is 2 inches from the vertex.

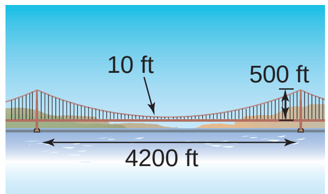


   a. If the depth of the headlight is to be 4 inches, what should the diameter of the headlight be at its opening?
   b. Find the diameter of the headlight at its opening if the depth is increased by 25%.

34. **Business** An airline has been charging \$140 per seat for a one-way flight. This flight has been averaging 110 passengers but can transport up to 180 passengers. The airline is considering a decrease in the price for a one-way ticket during the winter months. The airline estimates that for each \$10 decrease in the ticket price, they will gain approximately 20 passengers per flight.
   a. Based on these estimates, what ticket price should the airline charge to achieve the greatest income on an average flight?
   b. New estimates reveal that the increase in passengers per flight is closer to 10 for each \$10 decrease in the original ticket price. To maximize income, what should the new ticket price be?

35. **Critical Thinking** Consider the standard form of the equation of a parabola in which the vertex is known but the value of p is not known.
   a. As $|p|$ becomes greater, what happens to the shape of the parabola?
   b. As $|p|$ becomes smaller, what happens to the shape of the parabola?

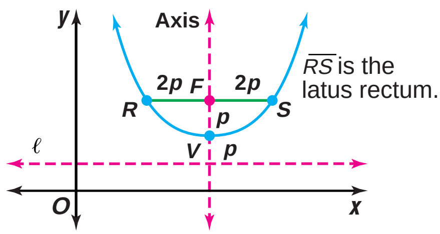
36. **Construction** The Golden Gate Bridge in San Francisco, California, is a catenary suspension bridge, which is very similar in appearance to a parabola. The main span cables are suspended between two towers that are 4200 feet apart and 500 feet above the roadway. The cable extends 10 feet above the roadway midway between the two towers.



   a. Find an equation that models the shape of the cable.
   b. How far from the roadway is the cable 720 feet from the bridge's center?

37. **Critical Thinking** Using the standard form of the equation of a parabola, derive the general form of the equation of a parabola.

38. **Critical Thinking** The *latus rectum* of a parabola is the line segment through the focus that is perpendicular to the axis and has endpoints on the parabola. The length of the latus rectum is $|4p|$ units, where p is the distance from the vertex to the focus.



   a. Write the equation of a parabola with vertex at $(-3, 2)$, axis $y = 2$, and latus rectum 8 units long.
   b. The latus rectum of the parabola with equation $(x - 1)^2 = -16(y - 4)$ coincides with the diameter of a circle. Write the equation of the circle.

Mixed Review

39. Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $\frac{(y-3)^2}{25} - \frac{(x-2)^2}{16} = 1$. Then graph the equation. *(Lesson 10-4)*

40. Find the coordinates of the center, foci, and vertices of the ellipse whose equation is $4x^2 + 25y^2 + 250y + 525 = 0$. Then graph the ellipse. *(Lesson 10-3)*

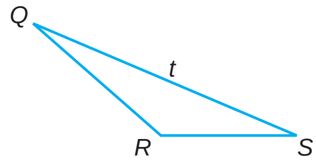
41. Graph $r = 12 \cos 2\theta$. *(Lesson 9-2)*

42. Find the values of θ for which $\cos \theta = 1$ is true. *(Lesson 6-3)*

43. **Geometry** A regular hexagon is inscribed in a circle with a radius 6.4 centimeters long. Find the apothem; that is, the distance from the center of the circle to the midpoint of a side. *(Lesson 5-4)*

44. Describe the end behavior of $g(x) = \frac{4}{x^2 + 1}$. *(Lesson 3-5)*

45. **SAT/ACT Practice** Triangle QRS has sides of lengths 14, 19, and t, where t is the length of the longest side. If t is the cube of an integer, what is the perimeter of the triangle?



A 41 **B** 58 **C** 60 **D** 69 **E** 76

MID-CHAPTER QUIZ

1. Given: $A(3, 3)$, $B(6, 9)$, and $C(9, 3)$ (Lesson 10-1)

a. Show that these points form an isosceles triangle.

b. Determine the perimeter of the triangle to the nearest hundredth.

2. Determine the midpoint of the diagonals of the rectangle with vertices $A(-4, 9)$, $B(5, 9)$, $C(5, 5)$, and $D(-4, 5)$. (Lesson 10-1)

3. Find the coordinates of the center and radius of the circle with equation $x^2 + y^2 - 6y - 8x = -16$. Then graph the circle. (Lesson 10-2)

4. Write the equation of the circle with center at $(-5, 2)$ and radius $\sqrt{7}$. (Lesson 10-2)

5. **Astronomy** A satellite orbiting Earth follows an elliptical path with the center of Earth as one focus. The eccentricity of the orbit is 0.16, and the major axis is 10,440 miles long. (Lesson 10-3)

a. If the mean diameter of Earth is 7920 miles, find the greatest and least distance of the satellite from the surface of Earth.

b. Assuming that the center of the ellipse is the origin and the foci lie on the x-axis, write the equation of the orbit of the satellite.

6. Identify the center, vertices, and foci of the ellipse with equation $9x^2 + 25y^2 - 72x + 250y + 544 = 0$. Then graph the equation. (Lesson 10-3)

7. Identify the center, vertices, foci, and equations of the asymptotes of the graph of the hyperbola with equation $3y^2 + 24y - x^2 - 2x + 41 = 0$. Then graph the equation. (Lesson 10-4)

8. Write the equation of a hyperbola that passes through the point at $(4, 2)$ and has asymptotes with equations $y = 2x$ and $y = -2x + 4$. (Lesson 10-4)

9. Identify the vertex, focus, and equations of the axis of symmetry and directrix for the parabola with equation $y^2 - 4x + 2y + 5 = 0$. Then graph the equation. (Lesson 10-5)

10. Write the equation of the parabola that passes through the point at $(9, -2)$, has its vertex at $(5, -1)$, and opens downward. (Lesson 10-5)

Extra Practice See p. A45.

CONTENTS

10-6

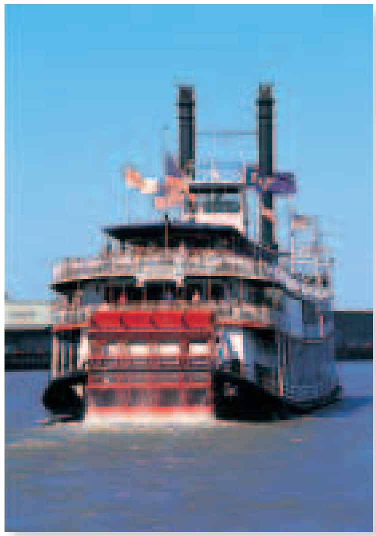
Rectangular and Parametric Forms of Conic Sections

OBJECTIVES

- Recognize conic sections in their rectangular form by their equations.
- Find a rectangular equation for a curve defined parametrically and vice versa.

Real World Application

TRANSPORTATION The first self-propelled boats on the western rivers of the United States were the paddlewheels. This boat used a steam engine to turn one or more circular wheels that had a paddle attached to the end of each spoke. In 1811, Robert Fulton and Nicholas Roosevelt built the first paddlewheel large enough for commercial use on the Ohio and Mississippi Rivers. By the end of the 19th century, paddlewheel boats had fought wars and carried people and cargo on nearly every river in the United States. Despite advancements in technology, paddlewheels are still in use today, though mainly for sentimental reasons. *You will solve a problem related to this in Exercise 40.*



We have determined a general equation for each conic section we have studied. All of these equations are forms of the general equation for conic sections.

General Equation for Conic Sections

The equation of a conic section can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not all zero.

The graph of a second-degree equation in two variables always represents a conic or degenerate case, unless the equation has no graph at all in the real number plane. Most of the conic sections that we have studied have axes that are parallel to the coordinate axes. The general equations of these conics have no xy term; thus, $B = 0$. The one conic section we have discussed whose axes are not parallel to the coordinate axes is the hyperbola whose equation is $xy = k$. In its equation, $B \neq 0$.

To identify the conic section represented by a given equation, it is helpful to write the equation in standard form. However, when $B = 0$, you can also identify the conic section by how the equation compares to the general equation. The table on the next page summarizes the standard forms and differences among the general forms.

General Form: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$		
Conic Section	**Standard Form of Equation**	**Variation of General Form of Conic Equations**
circle	$(x - h)^2 + (y - k)^2 = r^2$	$A = C$
parabola	$(y - k)^2 = 4p(x - h)$ or $(x - h)^2 = 4p(y - k)$	Either A or C is zero.
ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$	A and C have the same sign and $A \neq C$.
hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$	A and C have opposite signs.
	$xy = k$	$A = C = D = E = 0$

The circle is actually a special form of the ellipse, where $a^2 = b^2 = r^2$.

Remember that graphs can also be degenerate cases.

Example 1 **Identify the conic section represented by each equation.**

a. $6y^2 + 3x - 4y - 12 = 0$

$A = 0$ and $C = 6$. Since $A = 0$, the conic is a parabola.

b. $3y^2 - 2x^2 + 5y - x - 15 = 0$

$A = -2$ and $C = 3$. Since A and C have different signs, the conic is a hyperbola.

c. $9x^2 + 27y^2 - 6x - 108y + 82 = 0$

$A = 9$ and $C = 27$. Since A and C have the same signs and are not equal, the conic is an ellipse.

d. $4x^2 + 4y^2 + 5x + 2y - 150 = 0$

$A = 4$ and $C = 4$. Since $A = C$, the conic is a circle.

So far we have discussed equations of conic sections in their rectangular form. Some conic sections can also be described parametrically.

Look Back You can refer to Lesson 8-6 to review writing and graphing parametric equations.

The general form for a set of parametric equations is

$$x = f(t) \text{ and } y = g(t), \text{ where } t \text{ is in some interval } I.$$

As t varies over I in some order, a curve containing points (x, y) is traced out in a certain direction.

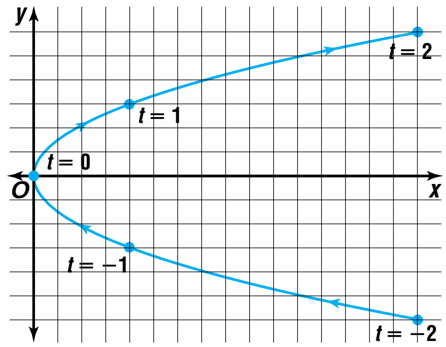
A parametric equation can be transformed into its more familiar rectangular form by eliminating the parameter t from the parametric equations.

Example 2 **Graph the curve defined by the parametric equations $x = 4t^2$ and $y = 3t$, where $-2 \leq t \leq 2$. Then identify the curve by finding the corresponding rectangular equation.**

Make a table of values assigning values for t and evaluating each expression to find values for x and y.

t	x	y	(x, y)
−2	16	−6	(16, −6)
−1	4	−3	(4, −3)
0	0	0	(0, 0)
1	4	3	(4, 3)
2	16	6	(16, 6)

Then graph the curve.



Notice the arrows indicating the direction in which the curve is traced for increasing values of t.

The graph appears to be part of a parabola. To identify the curve accurately, find the corresponding rectangular equation by eliminating t from the given parametric equations.

First, solve the equation $y = 3t$ for t.

$$y = 3t$$
$$\frac{y}{3} = t \quad \text{Solve for } t.$$

Then substitute $\frac{y}{3}$ for t in the equation $x = 4t^2$.

$$x = 4t^2$$
$$x = 4\left(\frac{y}{3}\right)^2 \quad t = \frac{y}{3}$$
$$x = \frac{4y^2}{9}$$

The equation $x = \frac{4y^2}{9}$ is the equation of a parabola with vertex at (0, 0) and its axis of symmetry along the x-axis. Notice that the domain of the rectangular equation is $x \geq 0$, which is greater than that of its parametric representation. By restricting the domain to $0 \leq x \leq 16$, our rectangular representation matches our parametric representation for the graph.

Some parametric equations require the use of trigonometric identities to eliminate the parameter t.

Example 3 **Find the rectangular equation of the curve whose parametric equations are $x = 2\cos t$ and $y = 2\sin t$, where $0 \le t \le 2\pi$. Then graph the equation using arrows to indicate how the graph is traced.**

Solve the first equation for $\cos t$ and the second equation for $\sin t$.

$$\cos t = \frac{x}{2} \text{ and } \sin t = \frac{y}{2}$$

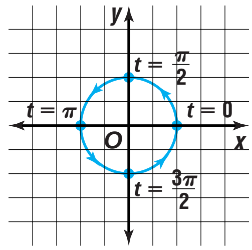
Use the trigonometric identity $\cos^2 t + \sin^2 t = 1$ to rewrite the equation to eliminate t.

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \textit{Substitution}$$

$$\frac{x^2}{4} + \frac{y^2}{4} = 1$$

$$x^2 + y^2 = 4 \quad \textit{Multiply each side by 4.}$$



This is the equation of a circle with center at (0, 0) and radius 2. As t increases from $t = 0$ to $t = 2\pi$, we see that the curve is traced in a counterclockwise motion.

Graphing Calculator Appendix

For keystroke instructions on how to graph parametric equations, see page A21.

You can also use substitution to find the parametric equations for a given conic section. If the conic section is defined as a function, $y = f(x)$, one way of finding the parametric equations is by letting $x = t$ and $y = f(t)$, where t is in the domain of f.

Example 4 **Find parametric equations for the equation $y = x^2 + 3$.**

Let $x = t$. Then $y = t^2 + 3$. Since the domain of the function $f(t)$ is all real numbers, the parametric equations are $x = t$ and $y = t^2 + 3$, where $-\infty < t < \infty$.

GRAPHING CALCULATOR EXPLORATION

The graph of the parametric equations $x = \cos t$ and $y = \sin t$, where $0 \le t \le 2\pi$ is the unit circle. Interchanging the trigonometric functions or changing the coefficients can alter the graph's size and shape as well as its starting point and the direction in which it is traced. Watch while the graph is being drawn to see the effects.

TRY THESE

1. Graph the parametric equations $x = -\cos t$ and $y = \sin t$, where $0 \le t \le 2\pi$.
   a. Where does the graph start?
   b. In which direction is the graph traced?
2. Graph the parametric equations $x = \sin t$ and $y = \cos t$, where $0 \le t \le 2\pi$.
   a. Where does the graph start?
   b. In which direction is the graph traced?
3. Graph $x = 2\cos t$, and $y = 3\sin t$, where $0 \le t \le 2\pi$. What is the shape of the graph?

WHAT DO YOU THINK?

4. What is the significance of the number a in the equations $x = a\cos t$ and $y = a\sin t$, where $0 \le t \le 2\pi$?
5. What is the result of changing the interval to $0 \le t \le 4\pi$ in Exercises 1-3?

Parametric equations are particularly useful in describing the motion of an object along a curved path.

Example 5 Real World Application

ASTRONOMY **The orbit of Saturn around the sun is modeled by the equation $\frac{x^2}{(9.50)^2} + \frac{y^2}{(9.48)^2} = 1$. It takes Saturn approximately 30 Earth years to complete one revolution of its orbit.**

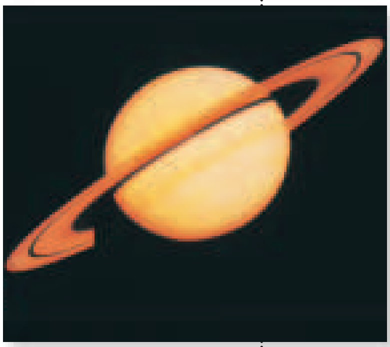
a. Find parametric equations that model the motion of Saturn beginning at (9.50, 0) and moving counterclockwise around the sun.

b. Use the parametric equations to determine Saturn's position after 18 years.

a. From the given equation, you can determine that the orbital path of Saturn is an ellipse with a major axis of 9.50 AU and a minor axis of 9.48 AU.

Like a circle, the parametric representation for an ellipse involves the use of sines and cosines. The parametric representation for the given equation is an ellipse with $x = 9.5$ and $y = 0$ when $t = 0$, so the following equations are true.

$$\frac{x}{9.50} = \cos \omega t \qquad \text{and} \qquad \frac{y}{9.48} = \sin \omega t$$



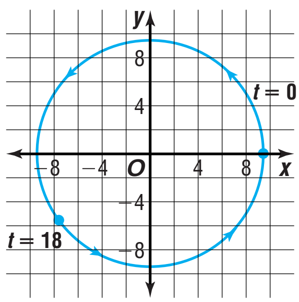
You can verify that by using the equations $x = 9.50$ and $y = 0$ when $t = 0$ and $\cos^2 \omega t + \sin^2 \omega t = 1$.

To move counterclockwise, the motion will have to begin with the value of x decreasing and y increasing, so $\omega > 0$. Since Saturn completes an orbit in 30 Earth years, the sine and cosine functions have a period $\frac{2\pi}{\omega} = 30$, so $\omega = \frac{\pi}{15}$.

Thus, the parametric equations corresponding to the rectangular equation $\frac{x^2}{(9.50)^2} + \frac{y^2}{(9.48)^2} = 1$ are $x = 9.50 \cos \frac{\pi}{15}t$ and $y = 9.48 \sin \frac{\pi}{15}t$, where $0 \le t \le 30$.

You can verify the equations above using a graphing calculator to trace the ellipse.

b. The position of Saturn after 18 years is found by letting $t = 18$ in both parametric equations.



$x = 9.50 \cos\left(\frac{\pi}{15}t\right)$

$x = 9.50 \cos\left[\frac{\pi}{15}(18)\right]$ $\quad t = 18$

$x \approx -7.69$

$y = 9.48 \sin\left(\frac{\pi}{15}t\right)$

$y = 9.48 \sin\left[\frac{\pi}{15}(18)\right]$ $\quad t = 18$

$y \approx -5.57$

Eighteen years later, Saturn is located at (−7.69, −5.57).

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** the general form of the equations of the four conic sections we have studied.
2. **Give** new restrictions on the parameter t in Example 2 so that the domains of the rectangular and parametric equations are the same.
3. **Write** the rectangular equation of a parabola with vertex at the origin and opening to the left. Then write the parametric equations that correspond to that parabola.

Guided Practice

Identify the conic section represented by each equation. Then write the equation in standard form and graph the equation.

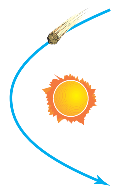
4. $x^2 + 9y^2 + 2x - 18y + 1 = 0$
5. $y^2 - 8x = -8$
6. $x^2 - 4x - y^2 - 5 - 4y = 0$
7. $x^2 - 6x + y^2 - 12y + 41 = 0$

Find the rectangular equation of the curve whose parametric equations are given. Then graph the equation using arrows to indicate orientation.

8. $x = t, y = -t^2 - 6t + 2; -\infty < t < \infty$
9. $x = 2\cos t, y = 3\sin t; 0 \leq t \leq 2\pi$

Find parametric equations for each rectangular equation.

10. $y = 2x^2 - 5x$
11. $x^2 + y^2 = 36$



12. **Astronomy** Some comets traveling at great speeds follow parabolic paths with the sun as their focus. Suppose the motion of a certain comet is modeled by the parametric equations $x = \frac{t^2}{80}$, and $y = t$ for $-\infty < t < \infty$. Find the rectangular equation that models the comet's path.

EXERCISES

Practice

Identify the conic section represented by each equation. Then write the equation in standard form and graph the equation.

13. $x^2 - 4y - 6x + 9 = 0$
14. $x^2 - 8x + y^2 + 6y + 24 = 0$
15. $x^2 - 3y^2 + 2x - 24y - 41 = 0$
16. $9x^2 + 25y^2 - 54x - 50y - 119 = 0$
17. $x^2 = y + 8x - 16$
18. $2xy = 3$
19. $5x^2 + 2y^2 - 40x - 20y + 110 = 0$
20. $x^2 - 8x + 11 = -y^2$
21. $8y^2 - 9x^2 - 16y + 36x - 100 = 0$
22. $4y^2 + 4y + 8x = 15$
23. Identify the conic section represented by $-4y^2 + 10x = 16y - x^2 - 5$. Write the equation in standard form and then graph the equation.
24. In the general equation of a conic, $A = C = 2, B = 0, D = -8, E = 12$, and $F = 6$. Write the equation in standard form. Then graph the equation.

Find the rectangular equation of the curve whose parametric equations are given. Then graph the equation using arrows to indicate orientation.

25. $x = t, y = 2t^2 - 4t + 1; -\infty < t < \infty$
26. $x = \cos 2t, y = \sin 2t; 0 \leq t \leq 2\pi$
27. $x = -\cos t, y = \sin t; 0 \leq t \leq 2\pi$
28. $x = 3\sin t, y = 2\cos t; 0 \leq t \leq 2\pi$
29. $x = -\sin 2t, y = 2\cos 2t; 0 \leq t \leq \pi$
30. $x = 2t - 1, y = \sqrt{t}; 0 \leq t \leq 4$

31. Find a rectangular equation for the curve whose parametric equations are $x = -3 \cos 2t$ and $y = 3 \sin 2t$, $0 \leq t \leq 2\pi$.

Find parametric equations for each rectangular equation.

32. $x^2 + y^2 = 25$

33. $x^2 + y^2 - 16 = 0$

34. $\frac{x^2}{4} + \frac{y^2}{25} = 1$

35. $\frac{y^2}{16} + x^2 = 1$

36. $y = x^2 - 4x + 7$

37. $x = y^2 + 2y - 1$

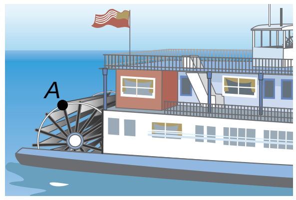
38. Find parametric equations for the rectangular equation $(y + 3)^2 = 4(x - 2)$.

Graphing Calculator

39. Consider the rectangular equation $x = \sqrt{y}$.
   a. By using different choices for t, find two different parametric representations of this equation.
   b. Graph the rectangular equation by hand. Then use a graphing calculator to sketch the graphs of each set of parametric equations.
   c. Are your graphs from part **b** the same?
   d. What does this suggest about parametric representations of rectangular equations?

Applications and Problem Solving

40. **Transportation** A riverboat's paddlewheel has a diameter of 12 feet and at full speed, makes one clockwise revolution in 2 seconds.

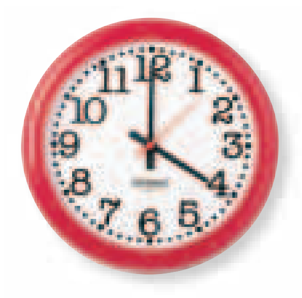


   a. Write a rectangular equation to model the shape of the paddlewheel.
   b. Write parametric equations describing the position of a point A on the paddlewheel for any given time t. Assume that at $t = 0$, A is at the very top of the wheel.
   c. How far will point A, which is a fixed point on the wheel, move in 1 minute?

41. **Critical Thinking** Identify the graph of each equation using the method described in this lesson. Then identify the graph of each equation after first rewriting the equation in standard form and solving for y. Explain the discrepancies, if any, in your answers.
   a. $2x^2 + 5y^2 = 0$
   b. $x^2 + y^2 - 4x - 6y + 13 = 0$
   c. $y^2 - 9x^2 = 0$

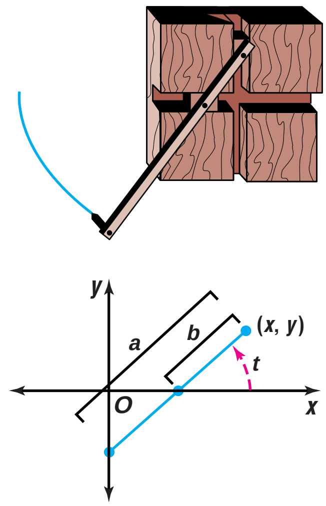
42. **Critical Thinking** Explain why a substitution of $x = t^2$ is not appropriate when trying to find a parametric representation of $y = x^2 - 5$?

43. **Timing** The path traced by the tip of the second-hand of a clock can be modeled by the equation of a circle in parametric form.



   a. If the radius of the clock is 6 inches, find an equation in rectangular form that models the shape of the clock.
   b. Find parametric equations that describe the motion of the tip as it moves from 12 o'clock noon to 12 o'clock noon of the next day.
   c. Simulate the motion described by graphing the equations on a graphing calculator.

44. Framing Portraits are often framed so that the opening through which the picture is seen is an ellipse. These oval mats must be custom cut using an oval cutter whose design relies upon the parametric equations of an ellipse. The elliptical compass at the right consists of a stick with a pencil attached to one end and two pivot holes at the other. Through these holes, the stick is anchored to two small blocks, one of which can slide horizontally and the other vertically in its groove. Use the diagram of the elliptical compass at right to verify that $x = a \cos t$ and $y = b \sin t$. (*Hint*: Draw an extra vertical and an extra horizontal line to create right triangles and then use trigonometry.)



Mixed Review

45. Find the coordinates of the vertex, focus, and the equations of the axis of symmetry and directrix of the parabola with equation $x^2 - 12y + 10x = -25$. Then graph the equation. *(Lesson 10-5).*

46. Graph $xy = -25$. *(Lesson 10-4)*

47. Write $3x^2 + 3y^2 - 18x + 12y = 9$ in standard form. Then graph the equation. *(Lesson 10-2)*

48. A 30-pound force is applied to an object at an angle of 60° with the horizontal. Find the magnitude of the horizontal and vertical components of the force. *(Lesson 8-1)*

49. Statistics The prediction equation $y = -0.13x + 37.8$ gives the fuel economy y for a car with horsepower x. Is the equation a better predictor for Car 1, which has a horsepower of 135 and average 19 miles per gallon, or for Car 2, which has a horsepower of 245 and averages 16 miles per gallon? Explain. *(Lesson 7-7)*

50. Find the value of $\sin\left(2 \text{ Sin}^{-1} \frac{1}{2}\right)$. *(Lesson 6-8)*

51. Find the area to the nearest square unit of $\triangle ABC$ if $a = 48$, $b = 32$, and $c = 44$. *(Lesson 5-8)*

52. Solve $\sqrt{2y - 3} - \sqrt{2y + 3} = -1$. *(Lesson 4-7)*

53. If y varies jointly as x and z and $y = 16$ when $x = 5$ and $z = 2$, find y when $x = 8$ and $z = 3$. *(Lesson 3-8)*

54. Find the determinant of $\begin{bmatrix} 5 & 9 \\ 7 & -3 \end{bmatrix}$. Then state whether an inverse exists for the matrix. *(Lesson 2-5)*

55. Write the point-slope form of the equation of the line through the points $(-6, 4)$ and $(3, 7)$. Then write the equation in slope-intercept form. *(Lesson 1-4)*

56. SAT/ACT Practice For all values where $x \neq y$, let $x \# y$ represent the lesser of the numbers x and y, and let $x @ y$ represent the greater of the number x and y. What is the value of $(1 \# 4) @ (2 \# 3)$?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Extra Practice See p. A45.

10-7

Transformations of Conics

OBJECTIVES

- Find the equations of conic sections that have been translated or rotated.
- Graph rotations and/or translations of conic equations.
- Identify the equations of conic sections using the discriminant.
- Find the angle of rotation for a given equation.

Real World Application

SPORTS At the 1996 Olympics in Atlanta, Georgia, U.S. high school student Kim Rhode won the gold medal in the women's double trap shooting event, being staged at the Olympics for the first time. The double trap event consists of firing double barrel shotguns at flying clay targets that are launched two at a time out of a house located 14.6 to 24.7 meters in front of the contestants. Targets are thrown out of the house in a random arc, but always at the same height. *A problem related to this will be solved in Example 1.*

Thus far, we have used a transformation called a translation to show how the parent graph of each of the conic sections is translated to a center other than the origin. For example, the equation of the circle $x^2 + y^2 = r^2$ becomes $(x - h)^2 + (y - k)^2 = r^2$ for a center of (h, k). A translation of a set of points with respect to (h, k) is often written as follows.

$$T_{(h,\,k)} \longleftrightarrow \text{translation with respect to } (h, k)$$

Example Real World Application

1 **SPORTS** **Refer to the application above. A video game simulating the sport of double trap allows a player to shoot at two elliptically-shaped targets released from a house at the bottom of the screen. With the house located at the origin, a target at its initial location is modeled by the equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Suppose a player misses one of the two targets released and the center of the target leaves the screen at the point (24, 30). Find an equation that models the shape and position of the target with its center translated to this point.**

To write the equation of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ for $T_{(24,\,30)}$, let $h = 24$ and $k = 30$. Then replace x with $x - h$ and y with $y - k$.

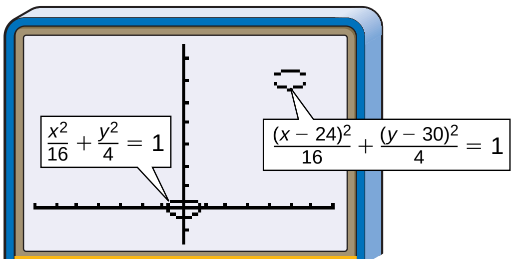
$x^2 \Rightarrow (x - 24)^2$

$y^2 \Rightarrow (y - 30)^2$

Thus, the translated equation is

$$\frac{(x - 24)^2}{16} + \frac{(y - 30)^2}{4} = 1$$

The graph shows the parent ellipse and its translation.



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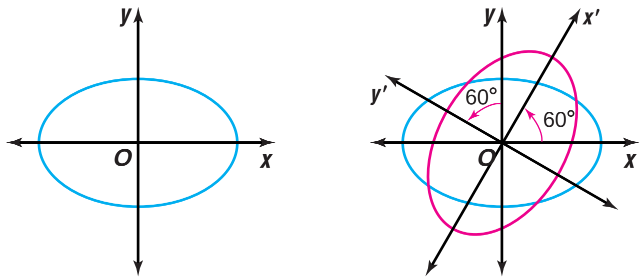
> **Look Back**
> You can refer to Lesson 2-4 to review rotation.

Another type of transformation you have studied is a rotation. Except for hyperbolas whose equations are of the form $xy = k$, all of the conic sections we have studied thus far have been oriented with their axes parallel to the coordinate axes. In the general form of these conics $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $B = 0$. Whenever $B \neq 0$, then the axes of the conic section are not parallel to the coordinate axes. That is, the graph is rotated.

The figures below show an ellipse whose center is the origin and its rotation. Notice that the angle of rotation has the same measure as the angles formed by the positive x-axis and the major axis and the positive y-axis and the minor axis.



The coordinates of the points of a rotated figure can be found by using a rotation matrix.

A positive value of θ indicates a counterclockwise rotation. A negative value of θ indicates a clockwise rotation.

A rotation of θ about the origin can be described by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Let $P(x, y)$ be a point on the graph of a conic section. Then let $P'(x', y')$ be the image of P after a counterclockwise rotation of θ. The values of x' and y' can be found by matrix multiplication.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

The inverse of the rotation matrix represents a rotation of $-\theta$. Multiply each side of the equation by the inverse rotation matrix to solve for x and y.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'\cos\theta + y'\sin\theta \\ -x'\sin\theta + y'\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x'\cos\theta + y'\sin\theta \\ -x'\sin\theta + y'\cos\theta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The result is two equations that can be used to determine the equation of a conic with respect to a rotation of θ.

Rotation Equations

To find the equation of a conic section with respect to a rotation of θ, replace

x with $x'\cos\theta + y'\sin\theta$
and y with $-x'\sin\theta + y'\cos\theta$.

Example 2 **Find the equation of the graph of $\frac{x^2}{16} - \frac{y^2}{9} = 1$ after it is rotated 45° about the origin. Then sketch the graph and its rotation.**

The graph of this equation is a hyperbola.

Find the expressions to replace x and y.

Replace x with $x' \cos 45° + y' \sin 45°$ or $\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$.

Replace y with $-x' \sin 45° + y' \cos 45°$ or $-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'$.

Computation is often easier if the equation is rewritten as an equation with denominators of 1.

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$9x^2 - 16y^2 = 144 \quad \textit{Multiply each side by 144.}$$

$$9\left(\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 - 16\left(-\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y'\right)^2 = 144 \quad \textit{Replace x and y.}$$

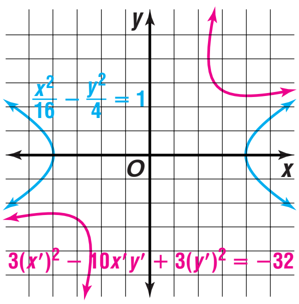
$$9\left[\frac{1}{2}(x')^2 + x'y' + \frac{1}{2}(y')^2\right] - 16\left[\frac{1}{2}(x')^2 - x'y' + \frac{1}{2}(y')^2\right] = 144 \quad \textit{Expand the binomial.}$$

$$-\frac{7}{2}(x')^2 + 25x'y' - \frac{7}{2}(y')^2 = 144 \quad \textit{Simplify.}$$

$$7(x')^2 - 50x'y' + 7(y')^2 = -288 \quad \textit{Multiply each side by } -2.$$

The equation of the hyperbola after the 45° rotation is $7(x')^2 - 50x'y' + 7(y')^2 = -288$.

The graph below shows the hyperbola and its rotation.



In Lesson 10-6, you learned to identify a conic from its general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where $B = 0$. When $B \neq 0$, the equation can be identified by examining the discriminant of the equation.

You may remember from the Quadratic Formula that the discriminant of a second-degree equation is defined as $B^2 - 4AC$ and will remain unchanged under any rotation. That is, $B^2 - 4AC = (B')^2 - 4A'C'$.

Identifying Conics By Using the Discriminant	For the general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, • if $B^2 - 4AC < 0$, the graph is a circle ($A = C$, $B = 0$) or an ellipse ($A \neq C$ or $B \neq 0$); • if $B^2 - 4AC > 0$, the graph is a hyperbola; • if $B^2 - 4AC = 0$, the graph is a parabola.

Remember that the graphs can also be degenerate cases.

Example 3 **Identify the graph of the equation $x^2 - 4xy + 4y^2 + 5\sqrt{5}y + 1 = 0$.**

Since the equation contains an *xy*-term, use the discriminant of the equation to identify the conic.

$$B^2 - 4AC = (-4)^2 - 4(1)(4) \quad A = 1, B = -4, C = 4$$
$$= 0$$

Since $B^2 - 4AC = 0$, the graph of the equation is a parabola.

You can also use values from the general form to find the angle of rotation about the origin.

Angle of Rotation About the Origin	For the general equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, the angle of rotation θ about the origin can be found by $\theta = \frac{\pi}{4}$, if $A = C$, or $\tan 2\theta = \frac{B}{A - C}$, if $A \neq C$.

Example 4 **Identify the graph of the equation $2x^2 + 9xy + 14y^2 - 5 = 0$. Then find θ and use a graphing calculator to draw the graph.**

$$B^2 - 4AC = (9)^2 - 4(2)(14) \quad A = 2, B = 9, \text{ and } C = 14$$
$$= -31$$

Since the discriminant is less than 0 and $A \neq C$, the graph is an ellipse.

Now find θ using $\tan 2\theta = \frac{B}{A - C}$, since $A \neq C$.

$$\tan 2\theta = \frac{B}{A - C}$$

$$\tan 2\theta = \frac{9}{2 - 14}$$

$$\tan 2\theta = -0.75$$

$$2\theta = -36.86989765 \quad \text{Take the inverse tangent of each side.}$$

$$\theta \approx -18° \quad \text{Round to the nearest degree.}$$

To graph the equation, you must solve for *y*. Rewrite the equation in quadratic form, $ay^2 + by + c = 0$.

$$\underset{a}{14y^2} + \underset{b}{(9x)y} + \underset{c}{(2x^2 - 5)} = 0$$

(continued on the next page)

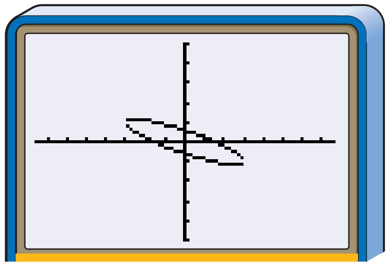
Now use the Quadratic Formula to solve for y.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-(9x) \pm \sqrt{(9x)^2 - 4(14)(2x^2 - 5)}}{2(14)}$$

$$y = \frac{-9x \pm \sqrt{-31x^2 + 280}}{28}$$

Enter the equations and graph.



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This method of solving for y to graph a rotated equation is useful when trying to identify the equation of a degenerate conic.

Example 5 **The graph of $xy - y^2 + 2x^2 = 0$ is a degenerate case. Identify the graph and then draw it.**

In this equation, $B^2 - 4AC > 0$. At first glance, this equation may appear to be the equation of a hyperbola. A closer inspection reveals that this is a degenerate case.

Solve the equation for y by first rewriting the equation in quadratic form.

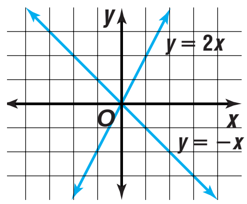
$$\underset{a}{(-1)}y^2 + \underset{b}{(x)}y + \underset{c}{2x^2} = 0$$

Now use the quadratic formula to solve for y.

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-x \pm \sqrt{x^2 - 4(-1)(2x^2)}}{2(-1)}$$

$$y = 2x \text{ or } y = -x$$



So the graph of $xy - y^2 + 2x^2$ is actually the graph of $y = 2x$ and $y = -x$, which are intersecting lines.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Sketch** a parabola with vertex at the origin and a horizontal line of symmetry. Then sketch the parabola for $T_{(-3, 3)}$. Label each graph with its equation in standard form.
2. **Write** the expressions needed to replace x and y for an equation rotated 30°.
3. **Indicate** the angle of rotation about the origin needed to transform the equation $\frac{x^2}{100} + \frac{y^2}{25} = 1$ into $\frac{x^2}{25} + \frac{y^2}{100} = 1$.

4. **You Decide** Ebony says that $7x^2 - 6\sqrt{3}xy + 13y^2 = 0$ is the equation of an ellipse. Teisha disagrees. She says the equation is a hyperbola. Who is correct? Justify your answer.

Guided Practice

Identify the graph of each equation. Write an equation of the translated or rotated graph in general form.

5. $x^2 + y^2 = 7$ for $T_{(3, 2)}$
6. $y = 2x^2 - 7x + 5$ for $T_{(-4, 5)}$
7. $x^2 - y^2 = 9, \theta = 60°$
8. $x^2 - 5x + y^2 = 3, \theta = \frac{\pi}{4}$

Identify the graph of each equation. Then find θ to the nearest degree.

9. $9x^2 + 4xy + 4y^2 + 2 = 0$
10. $8x^2 + 5xy - 4y^2 + 2 = 0$

11. The graph of $3(x - 1)^2 + 4(y + 4)^2 = 0$ is a degenerate case. Identify the graph and then draw it.

12. **Communications** A satellite dish tracks a satellite directly overhead. Suppose the equation $y = \frac{1}{6}x^2$ models the shape of the dish when it is oriented in this position. Later in the day the dish is observed to have rotated approximately 30°.
   a. Find an equation that models the new orientation of the dish.
   b. Sketch the graphs of both equations on the same set of axes using a graphing calculator.

EXERCISES

Practice

Identify the graph of each equation. Write an equation of the translated or rotated graph in general form.

13. $y = 3x^2 - 2x + 5$ for $T_{(2, -3)}$
14. $4x^2 + 5y^2 = 20$ for $T_{(5, -6)}$
15. $3x^2 + y^2 = 9$ for $T_{(-1, 3)}$
16. $4y^2 + 12x^2 = 24$ for $T_{(-1, 4)}$
17. $9x^2 - 25y^2 = 225$ for $T_{(0, 5)}$
18. $(x + 3)^2 = 4y$ for $T_{(-7, 2)}$
19. $x^2 - 8y = 0, \theta = 90°$
20. $2x^2 + 2y^2 = 8, \theta = 30°$
21. $y^2 + 8x = 0, \theta = \frac{\pi}{6}$
22. $xy = -8, \theta = \frac{\pi}{4}$
23. $x^2 - 5x + y^2 = 3, \theta = \frac{\pi}{3}$
24. $16x^2 - 4y^2 = 64, \theta = 60°$

25. Write the equation of the ellipse $6x^2 + 5y^2 = 30$ after a rotation of 30° about the origin.

Identify the graph of each equation. Then find θ to the nearest degree.

26. $9x^2 + 4xy + 5y^2 - 40 = 0$
27. $x^2 - xy - 4y^2 - x - y + 4 = 0$
28. $8x^2 + 8xy + 2y^2 = 0$
29. $2x^2 + 9xy + 14y^2 - 5 = 0$
30. $2x^2 + 4xy + 5y^2 + 3x - 4y - 20 = 0$
31. $2x^2 + 4\sqrt{3}xy + 6y^2 + \sqrt{3}x - y = 0$

32. Identify the equation $2x^2 + 4xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y + 12 = 0$ as a circle, ellipse, parabola, or hyperbola. Then find θ to the nearest degree.

The graph of each equation is a degenerate case. Identify the graph and then draw it.

33. $(x - 2)^2 - (x + 3)^2 = 5(y + 2)$

34. $2x^2 + 6y^2 + 8x - 12y + 14 = 0$

35. $y^2 - 9x^2 = 0$

36. $(x - 2)^2 + (y - 2)^2 + 4(x + y) = 8$

Graphing Calculator

Use the Quadratic Formula to solve each equation for *y*. Then use a graphing calculator to draw the graph.

37. $x^2 - 2xy + y^2 - 5x - 5y = 0$

38. $2x^2 + 9xy + 14y^2 = 5$

39. $8x^2 + 5xy - 4y^2 = -2$

40. $2x^2 + 4\sqrt{3}xy + 6y^2 + 3x = y$

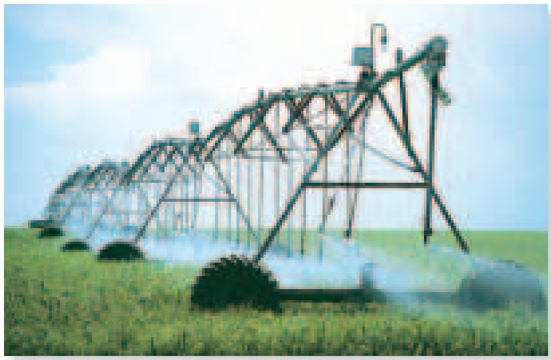
41. $2x^2 + 4xy + 2y^2 + 2\sqrt{2}x - 2\sqrt{2}y = -12$

42. $9x^2 + 4xy + 6y^2 = 20$

Applications and Problem Solving

Real World Application

43. Agriculture Many farmers in the Texas panhandle divide their land into smaller one square mile units. This square is then divided into four smaller units, or quadrants, of equal size. Each quadrant contains its own central pivot irrigation system. Suppose the center of the square mile of land is the origin.



a. Determine the translation in feet needed to place an irrigation system in Quadrant I.

b. Determine the equation for the path of the outer end of the irrigation system in Quadrant I.

44. Critical Thinking Identify the graph of each equation. Then determine the minimum angle of rotation needed to transform each equation so that the rotated graph coincides with its original graph.

a. $x^2 + 6x + 9 - y = 0$

b. $8x^2 + 6y^2 = 24$

c. $4xy = 25$

d. $15x^2 + 15y^2 = 60$

45. Critical Thinking Prove that a circle with equation $x^2 + y^2 = r^2$ remains unchanged under any rotation θ.

46. Astronomy Suppose the equation $31x^2 - 10\sqrt{3}xy + 21y^2 = 144$ models the shape of a reflecting mirror in a telescope.

a. Determine whether the reflector in the telescope is elliptical, parabolic, or hyperbolic.

b. Using a graphing calculator, sketch the graph of the equation.

c. Determine the angle through which the mirror has been rotated.

47. Critical Thinking Consider the equation $9x^2 - 2\sqrt{3}xy + 11y^2 - 24 = 0$.

a. Determine the minimum angle of rotation needed to transform the graph of this equation to a graph whose axes are on the x- and y-axes.

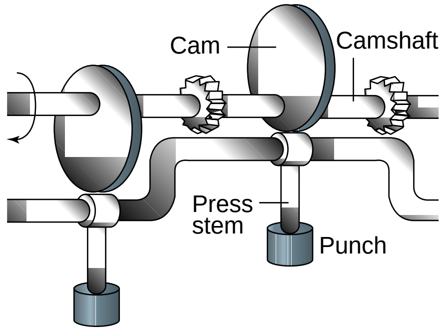
b. Use the angle of rotation in part **a** to find the new equation of the graph.

48. Manufacturing A cam in a punch press is shaped like an ellipse with the equation $\frac{x^2}{81} + \frac{y^2}{36} = 1$. The camshaft goes through the focus on the positive axis.

a. Graph a model of the cam.

b. Find an equation that translates the model so that the camshaft is at the origin.

c. Find the equation of the model in part b when the cam is rotated to an upright position.



Mixed Review

49. Identify the conic section represented by the equation $5y^2 - 3x^2 + 4x - 3y - 100 = 0$. *(Lesson 10-6)*

50. Write the equation of the ellipse that has its center at $(2, -3)$, $a = 1$, and $e = \frac{2\sqrt{6}}{5}$. *(Lesson 10-3)*

51. Graph $r = \frac{1}{\cos(\theta + 15°)}$. *(Lesson 9-4)*

52. Boating A boat heads due west across a lake at 8 m/s. If a current of 5 m/s moves due south, what is the boat's resultant velocity? *(Lesson 8-1)*

53. Which value is greater, cos 70° or cos 170°? *(Lesson 6-3)*

54. Change $\frac{5\pi}{16}$ radians to degree measure to the nearest minute. *(Lesson 5-1)*

55. Decompose $\frac{2y + 5}{y^2 + 3y + 2}$ into partial fractions. *(Lesson 4-6)*

56. If y varies inversely as x and $y = 4$ when $x = 12$, find y when $x = 5$. *(Lesson 3-8)*

57. Solve the system of equations algebraically. *(Lesson 2-2)*

$8m - 3n - 4p = 6$
$4m + 9n - 2p = -4$
$6m + 12n + 5p = -1$

58. Graph $h(x) = [\![x]\!] - 3$. *(Lesson 1-7)*

59. SAT Practice If $1 < b < 2$ and $2 < a < 3$, which statement is true about the expression $\frac{5a^8b^5}{180a^6b^2}$?

A The value of the expression is never greater than 1.

B The value of the expression is always between $\frac{1}{9}$ and 2.

C The value of the expression is always greater than 1.

D The value of the expression is always between $\frac{1}{36}$ and $\frac{3}{2}$.

E The value of the expression is always between 0 and $\frac{1}{2}$.

Extra Practice See p. A45.

CONTENTS

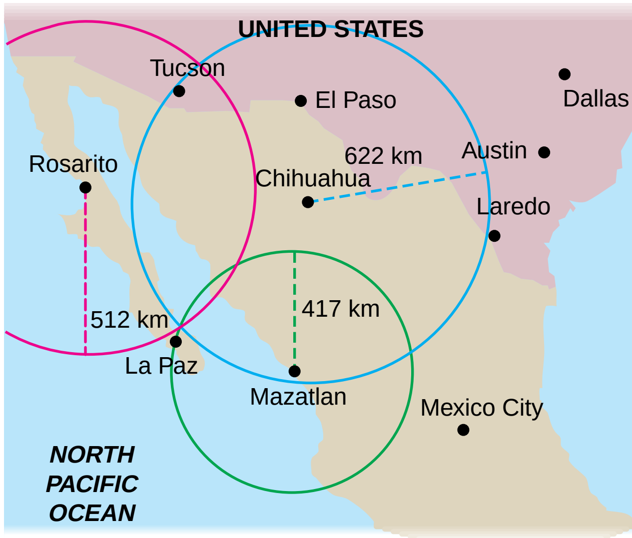
10-8 Systems of Second-Degree Equations and Inequalities

OBJECTIVE

- Graph and solve systems of second degree equations and inequalities.

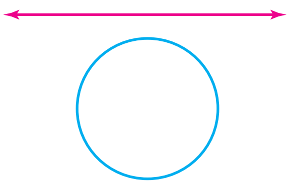
Real World Application

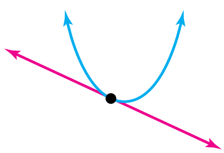
SEISMOLOGY The principal use of a seismograph network is to locate the epicenters of earthquakes. Seismograph stations in Chihuahua, Mazatlan, and Rosarito, Mexico, form one such network. Suppose this network detects an earthquake 622 kilometers from the Chihuahua station, 417 kilometers from the Mazatlan station, and 582 kilometers from the Rosarito station. Seismographic networks use complex software to approximate the location of the epicenter. The intersection of the three circles on the map shows the location of the epicenter to be near La Paz, Mexico. *You will solve a problem related to this in Exercise 40.*

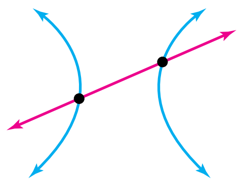


The equation of each circle in the application above is a second-degree equation. So the three circles represent a system of second-degree equations. The coordinates of the point that satisfies all three equations is the solution to the system, which is the location of the earthquake's epicenter. You can solve this system graphically by locating the point where all three circles intersect.

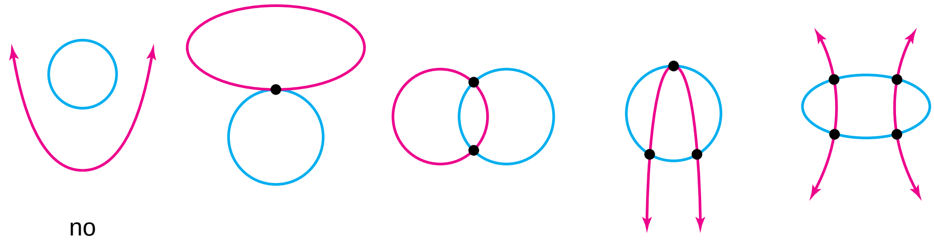
The number of solutions of a system of second-degree equations equals the number of times the graphs of the equations intersect. If the system is composed of a line and a conic, there may be 0, 1, or 2 solutions. *Remember that a line is a degenerate conic.*







If the system is composed of two conics, there may be 0, 1, 2, 3, or 4, solutions.



no solution | 1 solution | 2 solutions | 3 solutions | 4 solutions

While you can determine the number of solutions by graphing the equations of a system, the exact solution is not always apparent. To find the exact solution, you must use algebra.

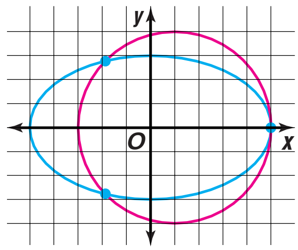
Example 1

a. Graph the system of equations. Use the graph to find approximate solutions.

b. Solve the system algebraically.

$$9x^2 + 25y^2 = 225$$
$$x^2 + y^2 - 2x = 15$$

a. The graph of the first equation is an ellipse. The graph of the second equation is a circle. When a system is composed of an ellipse and a circle, there may be 0, 1, 2, 3, or 4 possible solutions. Graph each equation. There appear to be 3 solutions close to (5, 0), (−2, −3), and (−2, 3).



Graphing Calculator Tip

You can use the CALC feature on a graphing calculator to approximate the intersection of any two functions graphed.

b. Since both equations contain a single term involving y, y^2, you can solve the system as follows.

First, multiply each side of the second equation by −25.

$$-25(x^2 + y^2 - 2x) = -25(15)$$
$$-25x^2 - 25y^2 + 50x = -375$$

Then, add the equations.

$$\begin{array}{rcl} 9x^2 + 25y^2 & = & 225 \\ -25x^2 - 25y^2 + 50x & = & -375 \\ \hline -16x^2 + 50x & = & -150 \end{array}$$

Using the quadratic formula, $x = 5$ or $-\frac{15}{8}$.

Now find y by substituting these values for x in one of the original equations.

$$x^2 + y^2 - 2x = 15$$
$$5^2 + y^2 - 2(5) = 15 \quad x = 5$$
$$25 + y^2 - 10 = 15$$
$$y^2 = 0$$
$$y = 0$$

$$x^2 + y^2 - 2x = 15$$
$$\left(-\frac{15}{8}\right)^2 + y^2 - 2\left(-\frac{15}{8}\right) = 15 \quad x = -\frac{15}{8}$$
$$\frac{225}{64} + y^2 + \frac{15}{4} = 15$$
$$y^2 = \frac{495}{64}$$
$$y = \pm\frac{3\sqrt{55}}{8}$$

The solutions are (5, 0), $\left(-\frac{15}{8}, \frac{3\sqrt{55}}{8}\right)$, and $\left(-\frac{15}{8}, -\frac{3\sqrt{55}}{8}\right)$. *How do the decimal approximations of these values compare to the approximations made in part **a**?*

Systems of second-degree equations are useful in solving real-world problems involving more than one parameter.

Example 2 Real World Application

SALES **During the month of January, Photo World collected \$2700 from the sale of a certain camera. After lowering the price by \$15, the store sold 30 more cameras and took in \$3375 from the sale of this camera the next month.**

a. Write a system of second-degree equations to model this situation.

b. Find the price of the camera during each month.

c. Use a graphing calculator to check your solution.

a. From the information in the problem, we can write two equations, each of which is the equation of a conic section.

Let x = number of cameras sold
Let y = price per camera in January.

Sales in January: $xy = 2700$
Sales in February: $(x + 30)(y - 15) = 3375$

These are equations of hyperbolas.

b. To solve the system algebraically, use substitution. You can rewrite the equation of the first hyperbola as $y = \frac{2700}{x}$. Before substituting, expand the left-hand side of the second equation and simplify the equation.

$$(x + 30)(y - 15) - 3375 = 0$$

$$xy - 15x + 30y - 450 - 3375 = 0 \quad \textit{Expand } (x - 30)(y + 15).$$

$$xy - 15x + 30y - 3825 = 0 \quad \textit{Simplify.}$$

$$2700 - 15x + 30\left(\frac{2700}{x}\right) - 3825 = 0 \quad xy = 2700,\ y = \frac{2700}{x}$$

$$-15x - 1125 + \frac{81{,}000}{x} = 0 \quad \textit{Simplify.}$$

$$-15x^2 - 1125x + 81{,}000 = 0 \quad \textit{Multiply each side by } x.$$

$$x^2 + 75x - 5400 = 0 \quad \textit{Divide each side by } -15.$$

$$(x - 45)(x + 120) = 0 \quad \textit{Factor.}$$

$x - 45 = 0$ or $x + 120 = 0$

$x = 45$ $\qquad$ $x = -120$

Since the number of cameras sold cannot be negative, the store sold 45 cameras during January.

The price of each camera sold during January was $\$\frac{2700}{45}$ or \$60, and the price per camera in February was \$60 − 15 or \$45.

c. Solve each equation for y. Then, graph the equations on the same screen.

$$y = \frac{2700}{x}$$

$$y = \frac{3375}{x + 30} + 15$$

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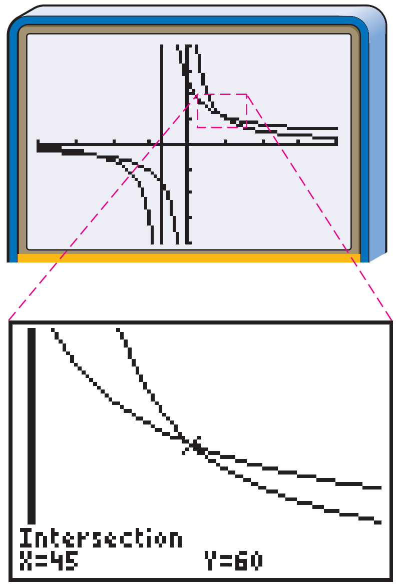


Use ZOOM to enlarge the section of the graph containing the intersection in the first quadrant. Use the **CALC: intersect** function to find the coordinates of the solution, (45, 60).

Look Back
You can refer to Lesson 2-6 to review solving systems of linear inequalities.

Previously you learned how to graph different types of inequalities by graphing the corresponding equation and then testing points in the regions of the graph to find solutions for the inequality. The same process is used when graphing systems of inequalities involving second-degree equations.

Example 3 **Graph the solutions for the system of inequalities.**

$$x^2 + 4y^2 \le 4$$
$$x^2 > y^2 + 1$$

First graph $x^2 + 4y^2 \le 4$. The ellipse should be a solid curve. Test a point either inside or outside the ellipse to see if its coordinates satisfy the inequality.

Test (0, 0):

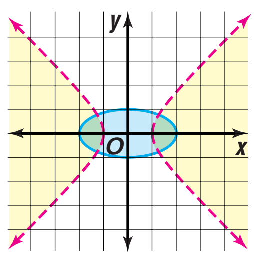
$$x^2 + y^2 \le 4$$
$$0^2 + 4(0)^2 \overset{?}{\le} 4 \quad \textit{(x, y) = (0, 0)}$$
$$0 \le 4 \quad ✓$$

Since (0, 0) satisfies the inequality, shade the interior of the ellipse. Then graph $x^2 \ge y^2 + 1$. The hyperbola should be dashed. Test a point inside the branches of the hyperbola or outside its branches. *Since a hyperbola is symmetric, you need not test points within both branches.*

Test (2, 0):

$$x^2 > y^2 + 1$$
$$2^2 \overset{?}{>} 0^2 + 1 \quad \textit{(x, y) = (2, 0)}$$
$$4 > 1 \quad ✓$$

Since (2, 0) satisfies the inequality, the regions inside the branches should be shaded. The intersection of the two graphs, which is shown in green, represents the solution of the system.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Draw** figures illustrating each of the possible numbers of solutions to a system involving the equations of a parabola and a hyperbola.
2. **Write** a system of equations involving two different conic sections that has exactly one solution, the origin.
3. **Describe** the graph of a system of second-degree equations having infinitely many solutions.
4. *Math Journal* **Write** a paragraph explaining how to solve a system of second-degree inequalities.

Guided Practice

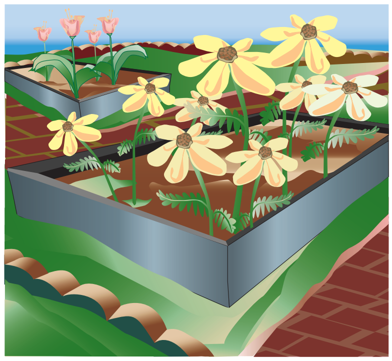
Solve each system of equations algebraically. Round to the nearest tenth. Check the solutions by graphing each system.

5. $\frac{(x-1)^2}{20} + \frac{(y-1)^2}{5} = 1$
   $x - y = 0$
6. $x^2 + y^2 = 16$
   $x + 2y = 10$
7. $9x^2 - 4y^2 = 36$
   $x^2 + y^2 = 4$
8. $x^2 = y$
   $xy = 1$

Graph each system of inequalities.

9. $x^2 + y^2 \geq 16$
   $x + y \leq 2$
10. $(x-5)^2 + 2y < 10$
    $y - 9 \geq -2x$
11. $x^2 + y^2 \leq 100$
    $x^2 + y^2 \geq 25$

12. **Gardening** A garden contains two square flowerbeds. The total area of the flowerbeds is 680 square feet, and the second bed has 288 more square feet than the first.
    a. Write a system of second-degree equations that models this situation.
    b. Graph the system found in part **a** and estimate the solution.
    c. Solve the system algebraically to find the length of each flowerbed within the garden.



EXERCISES

Practice

Solve each system of equations algebraically. Round to the nearest tenth. Check the solutions by graphing each system.

13. $x - 1 = 0$
    $y^2 = 49 - x^2$
14. $xy = 2$
    $x^2 = 3 + y^2$
15. $4x^2 + y^2 = 25$
    $-1 = 2x + y$
16. $x - y = 2$
    $x^2 = 100 - y^2$
17. $x - y = 0$
    $\frac{(x-1)^2}{9} - y^2 = 1$
18. $3x^2 = 9 - y^2$
    $x^2 + 2y^2 = 10$
19. $(y-1)^2 = 4 + x$
    $x + y = -1$
20. $x^2 + y^2 = 13$
    $xy + 6 = 0$

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CONTENTS

21. $x^2 + 4y^2 = 36$
$x^2 + y - 3 = 0$

22. $x^2 = 16 - y^2$
$2y - x + 3 = 0$

23. Find the coordinates of the point(s) of intersection for the graphs of $x^2 = 25 - 9y^2$ and $xy = -4$.

Graph each system of inequalities.

24. $x + y^2 \leq 9$
$y + x^2 \leq 0$

25. $x^2 + 4y^2 < 16$
$x^2 \leq y^2 + 4$

26. $x^2 + y^2 \leq 36$
$x + y^2 > 0$

27. $y^2 < 81 - 9x^2$
$16 \leq x^2 + y^2$

28. $y + 4 < (x - 3)^2$
$y^2 + x \geq 5$

29. $16x^2 + 49y^2 \leq 784$
$49x^2 + 16y^2 \geq 784$

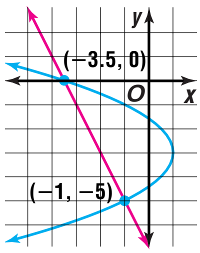
30. $x - (y - 1)^2 \leq 0$
$4y^2 \geq x^2 - 16$

31. $y - x^2 < 2$
$4x^2 + 9y^2 > 36$

32. $x > \frac{2}{y}$
$16x^2 - 25y^2 \geq 400$

33. Graph the solution to the system $(x + 3)^2 + (y + 2)^2 \geq 36$ and $x + 3 = 0$.

Write the system of equations or inequalities represented by each graph.

34.


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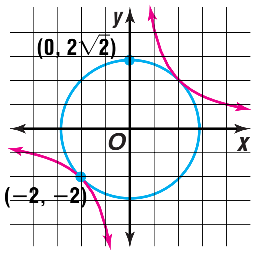

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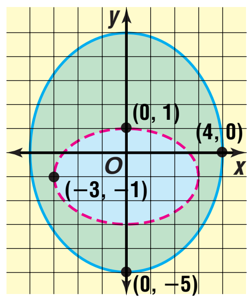


Applications and Problem Solving

37. **Construction** Carrie has 150 meters of fencing material to make a pen for her bird dog. She wants to form a rectangular pen with an area of 800 square meters. What will be the dimensions of her pen?
   a. Let x be the width of the field and y be its length. Write a system of equations that models this situation.
   b. How many solutions are possible for this type of system?
   c. Graph the system to estimate the dimensions of the pen.
   d. Solve this system algebraically, rounding the dimensions to the nearest tenth of a meter.

38. **Engineering** The Transport and Road Research Laboratory in Great Britain proposes the use of parabolic speed bumps 4 inches in height and 1 foot in width.
   a. Write a system of second-degree inequalities that models a cross-section of this speed bump. Locate the vertex of the speed bump at (0, 4).
   b. Graph the system found in part **a**.
   c. If the height of the speed bump is decreased to 3 inches, write a system of equations to model this new cross-section.

39. **Critical Thinking** Solve the system $x = -y + 1$, $xy = -12$, and $y^2 = 25 - x^2$ algebraically. Then graph the system to verify your solution(s).

40. Seismology Each of three stations in a seismograph network has detected an earthquake in their region. Seismograph readings indicate that the epicenter of the earthquake is 50 kilometers from the first station, 40 kilometers from the second station, and 13 kilometers from the third station. On a map in which each grid represents one square kilometer, the first station is located at the origin, the second station at (0, 30), and the third station at (35, 18).

a. Write a system of second-degree equations that models this situation.

b. Graph the system and use the graph to approximate the location of the epicenter.

c. Solve the system of equations algebraically to determine the location of the epicenter.

41. Critical Thinking Find the value of k so that the graphs of $x = 2y^2$ and $x + 3y = k$ are tangent to each other.

42. Entertaiment In a science fiction movie, astronomers track a large incoming asteroid and predict that it will strike Earth with disastrous results. Suppose a certain latitude of Earth's surface is modeled by $x^2 + y^2 = 40$ and the path of the asteroid is modeled by $x = 0.25y^2 + 5$.

a. Graph the two equations on the same axes.

b. Will the asteroid strike Earth? If so, what are the coordinates of the point of impact?

c. Describe this situation with parametric equations. Assume both the asteroid and Earth's surface are moving counterclockwise.

d. Graph the equations found in part **c** using a graphing calculator. Use a window that shows complete graphs of both Earth's surface and the asteroid's path.

Mixed Review

43. Find the equation of $\frac{x^2}{9} + y^2 = 1$ after a 30° rotation about the origin. *(Lesson 10-7)*

44. Write an equation in standard form of the line with the parametric equations $x = 4t + 1$ and $y = 5t - 7$. *(Lesson 8-6)*

45. Simplify $4 \csc \theta \cos \theta \tan \theta$. *(Lesson 7-1)*

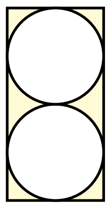
46. Mechanics A pulley of radius 10 centimeters turns at 5 revolutions per second. Find the linear velocity of the belt driving the pulley in meters per second. *(Lesson 6-2)*

47. Determine between which consecutive integers the real zeros of the function $f(x) = x^3 - 4$ are located. *(Lesson 4-5)*

48. Graph $y = (x + 2)^2 - 3$ and its inverse. *(Lesson 3-4)*

49. Is the relation {(4, 0), (3, 0), (5, −2), (4, −3), (0, −13)} a function? Explain. *(Lesson 1-1)*

50. SAT/ACT Practice In the figure at the right, two circles are tangent to each other and each is tangent to three sides of the rectangle. If the radius of each circle is 2, what is the area of the shaded region?



A $32 - 12\pi$ **B** $16 - 8\pi$ **C** $16 - 6\pi$

D $8 - 6\pi$ **E** $32 - 8\pi$

Extra Practice See p. A45.

10-8B Shading Areas on a Graph

An Extension of Lesson 10-8

OBJECTIVE

- Graph a system of second-degree inequalities using the **Shade(** command.

The **Shade(** command can be used to shade areas between the graphs of two equations. To shade an area on a graph, select **7:Shade(** from the **DRAW** menu. The instruction is pasted to the home screen. The argument, or restrictive information, for this command is as follows.

Shade(*lowerfunc,upperfunc,Xleft,Xright,pattern,patres*)

This command draws the lower function, *lowerfunc,* and the upper function, *upperfunc,* in terms of X on the current graph and shades the area that is specifically above *lowerfunc* and below *upperfunc.* This means that only the areas between the two functions defined are shaded.

Xleft and *Xright,* if included, specify left and right boundaries for the shading. *Xleft* and *Xright* must be numbers between **Xmin** and **Xmax**, which are the defaults.

The parameter *pattern* specifies one of four shading patterns.

pattern = **1** vertical lines (default)
pattern = **2** horizontal lines
pattern = **3** 45° lines with positive slope
pattern = **4** 45° lines with negative slope

The parameter *patres* specifies one of eight shading resolutions.

patres = **1** shades every pixel (default)	*patres* = **5** shades every fifth pixel
patres = **2** shades every second pixel	*patres* = **6** shades every sixth pixel
patres = **3** shades every third pixel	*patres* = **7** shades every seventh pixel
patres = **4** shades every fourth pixel	*patres* = **8** shades every eighth pixel

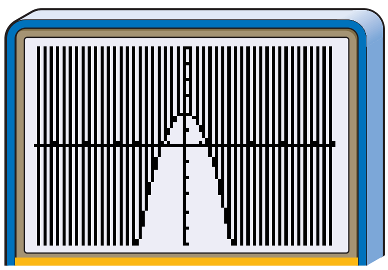
In a system of second-degree inequalities, this technique can be used to shade the interior region of a conic that is not a function.

Example **Graph the solutions for the system of inequalities below.**

$y \geq -x^2 + 2$

$x^2 + 9y^2 \leq 36$

The boundary equation of the first inequality, $y = -x^2 + 2$, is a function. This inequality is graphed by first entering the equation $y = -x^2 + 2$ into the **Y=** list. Since the test point (0, 3) satisfies the inequality, set the graph style to ◥ (shade above).



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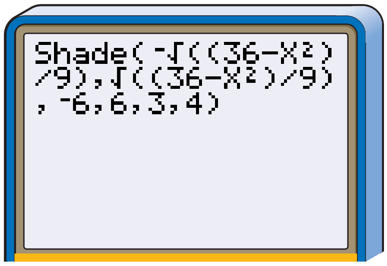
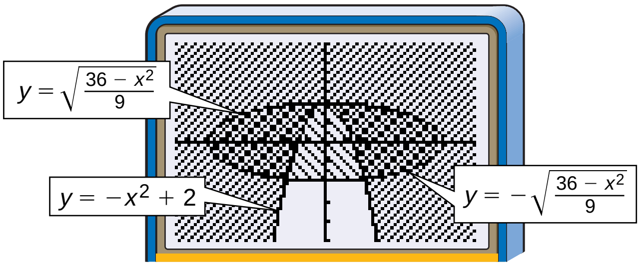
(continued on the next page)

The boundary equation for the second inequality is $x^2 + 9y^2 = 36$, which is defined using two functions, $y = -\sqrt{\frac{36 - x^2}{9}}$ and $y = \sqrt{\frac{36 - x^2}{9}}$. The lower function is $y = -\sqrt{\frac{36 - x^2}{9}}$, and the upper function is $y = \sqrt{\frac{36 - x^2}{9}}$. The two halves of the ellipse intersect at $x = -6$ and $x = 6$.

The expression to shade the area between the two halves of the ellipse is shown at the right.



Pressing ENTER will compute the graph as shown below.



[−9.1, 9.1] scl:1 by [−6, 6] scl:1

The solution set for this system of inequalities is the darker region in which the shadings for the two inequalities over lap.

To clear the any **Shade(** commands from the viewing window, select **1:ClrDraw** from the **DRAW** menu and then press ENTER. Remember to also clear any functions defined in the **Y=** list.

TRY THESE

Use the shade feature to graph each system of second-degree inequalities.

1. $y \le x^2 - 5$
$9y^2 - x^2 \le 36$

2. $x^2 + y^2 \le 16$
$x \ge y^2 - 4$

3. $16x^2 + 25y^2 \le 400$
$25x^2 - 16y^2 \ge 400$

4. $8x^2 + 32y^2 \le 256$
$32x^2 + 8y^2 \le 256$

WHAT DO YOU THINK?

5. Recall that the **Shade(** command can only shade the area between two functions.

a. To shade just the solution set for the system of inequalities in the example problem, how many regions would need a separate **Shade(** command?

b. How could you determine the approximate domain intervals for each region?

c. List and then execute three **Shade(** commands to shade the region representing the solution set for the example problem.

6. Use the **Shade(** command to create a "real-world" picture. Make a list of each command needed to create the picture, as well as a sketch of what the finished picture should look like.

VOCABULARY

analytic geometry (p. 618)
asymptotes (p. 642)
axis of symmetry (p. 653)
center (p. 623, 642)
circle (p. 623)
concentric (p. 623)
conic section (p. 623)
conjugate axis (p. 642)
degenerate conic (p. 623)
directrix (p. 653)
eccentricity (p. 636)
ellipse (p. 631)
equilateral hyperbola (p. 647)
focus (p. 631, 642, 653)
hyperbola (p. 642)
locus (p. 658)
major axis (p. 631)
minor axis (p. 631)
radius (p. 623)
rectangular hyperbola (p. 648)
semi-major axis (p. 632)
semi-minor axis (p. 632)
transverse axis (p. 642)
vertex (p. 631, 642, 653)

UNDERSTANDING AND USING THE VOCABULARY

State whether each statement is *true* or *false*. If false, replace the underlined word(s) to make a true statement.

1. Circles, ellipses, parabolas, and hyperbolas are all examples of conic sections.
2. Circles that have the same radius are concentric circles.
3. The line segment connecting the vertices of a hyperbola is called the conjugate axis.
4. The foci of an ellipse are located along the major axis of the ellipse.
5. In the general form of a circle, A and C have opposite signs.
6. A parabola is symmetric with respect to its vertex.
7. The shape of an ellipse is described by a measure called eccentricity.
8. A hyperbola is the set of all points in a plane that are the same distance from a given point and a given line.
9. The general equation of a rectangular hyperbola, where the coordinate axes are the asymptotes, is $xy = c$.
10. A point is the degenerate form of a parabola conic.

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 10-1 Find the distance and midpoint between two points on a coordinate plane.

- Find the distance between points at (3, 8) and (−5, 10).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-5 - 3)^2 + (10 - 8)^2}$$
$$= \sqrt{68} \text{ or } 2\sqrt{17}$$

REVIEW EXERCISES

Find the distance between each pair of points with the given coordinates. Then, find the midpoint of the segment that has endpoints at the given coordinates.

11. $(1, -6), (-3, -4)$

12. $(a, b), (a + 3, b + 4)$

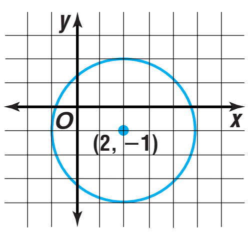
13. Determine whether the points $A(-5, -2)$, $B(3, 4)$, $C(10, 3)$, and $D(2, -3)$ are the vertices of a parallelogram. Justify your answer.

Lesson 10-2 Determine the standard form of the equation of a circle and graph it.

- Write $x^2 + y^2 - 4x + 2y - 4 = 0$ in standard form. Then graph the equation.

$$x^2 + y^2 - 4x + 2y - 4 = 0$$
$$x^2 - 4x + ? + y^2 + 2y + ? = 4$$
$$x^2 - 4x + 4 + y^2 + 2y + 1 = 4 + 4 + 1$$
$$(x - 2)^2 + (y + 1)^2 = 9$$

The center of the circle is located at (2, −1), and the radius is 3.



Write the standard form of the equation of each circle described. Then graph the equation.

14. center at (0, 0), radius $3\sqrt{3}$

15. center at (2, 1), tangent to the y-axis

Write the standard form of each equation. Then graph the equation.

16. $x^2 + y^2 = 6y$

17. $x^2 + 14x + y^2 + 6y = 23$

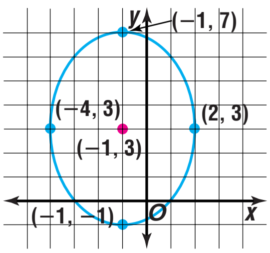
18. $3x^2 + 3y^2 + 6x + 12y - 60 = 0$

19. Write the standard form of the equation of the circle that passes through points at (1, 1), (−2, 2), and (−5, 1). Then identify the center and radius.

Lesson 10-3 Determine the standard form of the equation of an ellipse and graph it.

- For the equation, $\frac{(x + 1)^2}{9} + \frac{(y - 3)^2}{16} = 1$, find the coordinates of the center, foci, and vertices of the ellipse. Then graph the equation.

center: $(-1, 3)$
foci: $\left(-1, 3 + \sqrt{7}\right)$, $\left(-1, 3 - \sqrt{7}\right)$
vertices: $(2, 3)$, $(-4, 3)$, $(-1, -1)$, $(-1, 7)$



For the equation of each ellipse, find the coordinates of the center, foci, and vertices. Then graph the equation.

20. $\frac{(x - 5)^2}{16} + \frac{(y - 2)^2}{36} = 1$

21. $4x^2 + 25y^2 - 24x + 50y = 39$

22. $6x^2 + 4y^2 + 24x - 32y + 64 = 0$

23. $x^2 + 4y^2 + 124 = 8x + 48y$

24. Write the equation of an ellipse centered at (−4, 1) with a vertical semi-major axis 9 units long and a semi-minor axis 6 units long.

OBJECTIVES AND EXAMPLES

Lesson 10-4 Determine the standard and general forms of the equation of a hyperbola and graph it.

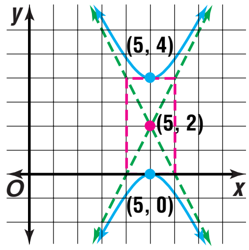
- Find the coordinates of the center, foci, and vertices, and the equations of the asymptotes of the graph of $\frac{(y-2)^2}{4} - (x-5)^2 = 1$. Then graph the equation.

center: $(5, 2)$

foci: $\left(5, 2 + \sqrt{5}\right)$, $\left(5, 2 - \sqrt{5}\right)$

vertices: $(5, 4)$, $(5, 0)$

asymptotes: $y - 2 = \pm 2(x - 5)$



REVIEW EXERCISES

For the equation of each hyperbola, find the coordinates of the center, the foci, and the vertices and the equations of the asymptotes of its graph. Then graph the equation.

25. $\frac{x^2}{25} - \frac{y^2}{16} = 1$

26. $\frac{(y+5)^2}{36} - \frac{(x-1)^2}{9} = 1$

27. $x^2 - 4y^2 - 16y = 20$

28. $9x^2 - 16y^2 - 36x - 96y + 36 = 0$

29. Graph $xy = 9$.

Write an equation of the hyperbola that meets each set of conditions.

30. The length of the conjugate axis is 10 units, and the vertices are at $(1, -1)$ and $(1, 5)$.

31. The vertices are at $(-2, -3)$ and $(6, -3)$, and a focus is at $(-4, -3)$.

Lesson 10-5 Determine the standard and general forms of the equation of a parabola and graph it.

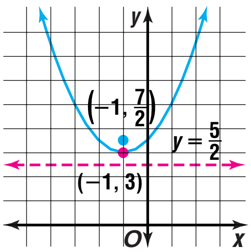
- For the equation $(x + 1)^2 = 2(y - 3)$, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

vertex: $(-1, 3)$

focus: $\left(-1, \frac{7}{2}\right)$

directrix: $y = \frac{5}{2}$

axis of symmetry: $x = -1$



For the equation of each parabola, find the coordinates of the vertex and focus, and the equations of the directrix and axis of symmetry. Then graph the equation.

32. $(x - 5)^2 = 8(y - 3)$

33. $(y + 2)^2 = -16(x - 1)$

34. $y^2 + 6y - 4x = -25$

35. $x^2 + 4x = y - 8$

Write an equation of the parabola that meets each set of conditions.

36. The parabola passes through the point at $(-3, 7)$, has its vertex at $(-1, 3)$, and opens to the left.

37. The focus is at $(5, 2)$, and the equation of the directrix is $y = -4$.

Lesson 10-6 Recognize conic sections in their rectangular form by their equations.

- Identify the conic section represented by the equation $2x^2 - 3x - y + 4 = 0$.

$A = 2$ and $C = 0$

Since $C = 0$, the conic is a parabola.

Identify the conic section represented by each equation.

38. $5x^2 - 7x + 2y^2 = 10$

39. $xy = 5$

40. $2x^2 + 4x + 2y^2 - 6y + 16 = 0$

41. $4y^2 + 6x - 5y = 20$

OBJECTIVES AND EXAMPLES

Lesson 10-6 Find a rectangular equation for a curve defined parametrically and vice versa.

- Find the rectangular equation of the curve whose parametric equations are $x = 3 \sin t$ and $y = \cos t$, where $0 \le t \le 2\pi$. Then graph the equation using arrows to indicate orientation.

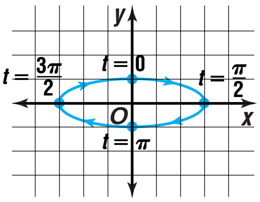
$x = 3\sin t \Rightarrow \sin t = \frac{x}{3}$

$y = \cos t \Rightarrow \cos t = y$

$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{3}\right)^2 + y^2 = 1$

$\frac{x^2}{9} + y^2 = 1$



REVIEW EXERCISES

Find the rectangular equation of the curve whose parametric equations are given. Then graph the equation using arrows to indicate orientation.

42. $x = t$, $y = -t^2 + 3$, $-\infty \le t \le \infty$

43. $x = \cos 4t$, $y = \sin 4t$, $0 \le t \le \frac{\pi}{2}$

44. $x = 2 \sin t$, $y = 3 \cos t$, $0 \le t \le 2\pi$

45. $x = \sqrt{t}$, $y = \frac{t}{2} - 1$, $0 \le t \le 9$

Find parametric equations for each rectangular equation.

46. $y = 2x^2 + 4$

47. $x^2 + y^2 = 49$

48. $\frac{x^2}{36} + \frac{y^2}{81} = 1$

49. $x = -y^2$

Lesson 10-7 Find the equations of conic sections that have been translated or rotated and find the angle of rotation for a given equation.

- To find the equation of a conic section with respect to a rotation of θ, replace

$$x \text{ with } x' \cos \theta + y' \sin \theta$$
$$\text{and } y \text{ with } -x' \sin \theta + y' \cos \theta.$$

Identify the graph of each equation. Write an equation of the translated or rotated graph in general form.

50. $4x^2 + 9y^2 = 36$, $\theta = \frac{\pi}{6}$

51. $y^2 - 4x = 0$, $\theta = 45°$

52. $4x^2 - 16(y - 1)^2 = 64$ for $T_{(1, -2)}$

Identify the graph of each equation. Then find θ to the nearest degree.

53. $6x^2 + 2\sqrt{3}xy + 8y^2 = 45$

54. $x^2 - 6xy + 9y^2 = 7$

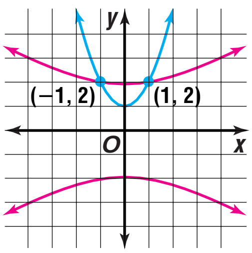
Lesson 10-8 Graph and solve systems of second degree equations and inequalities.

- Solve the system of equations $x^2 - y = -1$ and $x^2 - 3y^2 = -11$ algebraically.

$$\begin{array}{r} x^2 \quad - y + 1 = 0 \\ -x^2 + 3y^2 \quad - 11 = 0 \\ \hline 3y^2 - y - 10 = 0 \end{array}$$

$y = -\frac{5}{3}$ or $y = 2$

Substituting we find the solutions to be $(1, 2)$ and $(-1, 2)$. The graph shows these solutions to be true.



Solve each system of equations algebraically. Round to the nearest tenth. Check the solutions by graphing each system.

55. $(x - 1)^2 + 4(y - 1)^2 = 20$
$x = y$

56. $2x - y = 0$
$y^2 = 49 + x^2$

57. $x^2 - 4x - 4y = 4$
$(x - 2)^2 + 4y = 0$

58. $x^2 + y^2 = 12$
$xy = -4$

Graph each system of inequalities.

59. $x^2 + y \le 4$
$y^2 - x \le 0$

60. $xy \ge 9$
$x^2 + y^2 < 36$

61. $x^2 \le 16 - y^2$
$36y^2 > 324 - 9x^2$

62. $x^2 - 4y \ge 8$
$4y^2 - 25x^2 \ge 100$

APPLICATIONS AND PROBLEM SOLVING

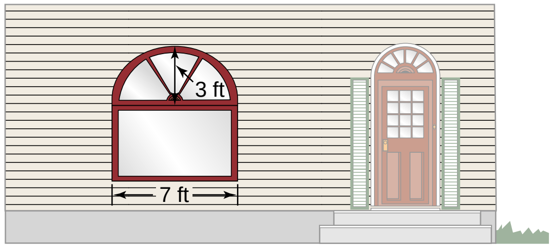
63. Gardening Migina bought a new sprinkler that covers part or all of a circular area. With the center of the sprinkler as the origin, the sprinkler sends out water far enough to reach a point located at (12, 16). *(Lesson 10-2)*

a. Find an equation representing the farthest points the sprinkler can reach.

b. Migina's backyard is 40 feet wide and 50 feet long. If Migina waters her backyard without moving the sprinkler, what percent of her backyard will not be watered directly?

64. Astronomy A satellite orbiting Earth follows an elliptical path with Earth at its center. The eccentricity of the orbit is 0.2, and the major axis is 12,000 miles long. Assuming that the center of the ellipse is the origin and the foci lie on the x-axis, write the equation of the orbit of the satellite. *(Lesson 10-6)*

65. Carpentry For a remodeling project, a carpenter is building a picture window that is topped with an arch in the shape of a semi-ellipse. The width of the window is to be 7 feet, and the height of the arch is to be 3 feet. To sketch the arch above the window, the carpenter uses a 7-foot string attached to two thumbtacks. Approximately where should the thumbtacks be placed? *(Lesson 10-3)*



ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. An ellipse has its center at the origin and an eccentricity of $\frac{1}{9}$. What is a possible equation for the ellipse?
2. A parabola has an axis of symmetry of $x = 2$ and a focus of (2, 5). What is a possible equation for the parabola in standard form?

PORTFOLIO

Choose one of the conic sections you studied in this chapter. Explain why it is a conic section and describe how you graph it.

Additional Assessment See page A65 for Chapter 10 practice test.

Unit 3 interNET Project

SPACE—THE FINAL FRONTIER

Out in Orbit!

- Search the internet for a satellite, space vehicle, or planet that travels in an orbit around a planet or star.
- Find data on the orbit of the satellite, space vehicle, or planet. This information should include the closest and farthest distance of that object from the planet it is orbiting.
- Make a scale drawing of the orbit of the satellite, space vehicle, or planet. Label important features and dimensions.
- Write a summary describing the orbit of the satellite, space vehicle, or planet. Be sure to discuss which conic section best models the orbit.

Ratio and Proportion Problems

Several problems on the SAT and ACT involve ratios or proportions. The ratio of x to y can be expressed in several ways.

$$\frac{x}{y} \qquad x{:}y \qquad x \text{ to } y$$

Think of a ratio as comparing parts of a whole. If the ratio of boys to girls in a class is 2:1, then one part is 2, one part is 1, and the whole is 3. The fraction of boys in the class is $\frac{2}{3}$.

Memorize the property of proportions.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

THE PRINCETON REVIEW

A ratio compares a part to a part. A fraction compares a part to a whole.

Ratios only tell you the *relative* sizes of quantities, not the actual quantities.

When setting up a proportion, label quantities to prevent careless errors.

ACT EXAMPLE

1. The ratio of boys to girls in a class is 4 to 5. If there are a total of 27 students in the class, how many boys are in the class?

A 4 **B** 9 **C** 12

D 14 **E** 17

HINT Notice what question is asked. Is it a ratio, a fraction, or a number?

Solution The ratio is 4 to 5, so the whole must be 9. The fraction of boys is $\frac{4}{9}$. The total number of students is 27, so the number of boys is $\frac{4}{9}$ of 27 or 12. The answer is choice **C**.

Alternate Solution Another method is to use a 'ratio box' to record the numbers and guide your calculations.

Boys	Girls	Whole
4	5	9
		27

In the Whole column, 9 must be multiplied by 3 to get the total of 27. So multiply the 4 by 3 to get the number of boys.

Boys	Girls	Whole
4	5	9
12		27

This is answer choice **C**.

SAT EXAMPLE

2. If 2 packages contain a total of 12 doughnuts, how many doughnuts are there in 5 packages?

A 12 **B** 24 **C** 30

D 36 **E** 60

HINT In a proportion, one ratio equals another ratio.

Solution Write a proportion.

packages → $\frac{2}{12} = \frac{5}{x}$ ← packages
doughnuts → ← doughnuts

Cross multiply.

$$\frac{2}{12} = \frac{5}{x}$$
$$2(x) = 5(12)$$
$$x = 30$$

The answer is choice **C**.

Alternate Solution You can also solve this problem without using a proportion. Since 2 packages contain 12 doughnuts, each package must contain 12 ÷ 2 or 6 doughnuts. Then five packages will contain 5 × 6 or 30 doughnuts.

This is answer choice **C**.

SAT AND ACT PRACTICE

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

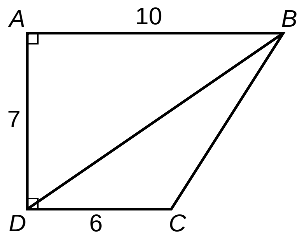
1. In a jar of red and green jelly beans, the ratio of green jelly beans to red jelly beans is 5:3. If the jar contains a total of 160 jelly beans, how many of them are red?

   A 30 **B** 53 **C** 60
   D 100 **E** 160

2. If $a^2b = 12^2$ and b is an odd integer, then a could be divisible by all of the following EXCEPT

   A 3 **B** 4 **C** 6 **D** 9 **E** 12

3. In the figure below, $\angle A$ and $\angle ADC$ are right angles, the length of $\overline{AD}$ is 7 units, the length of $\overline{AB}$ is 10 units, and the length of $\overline{DC}$ is 6 units. What is the area, in square units, of $\triangle DCB$?

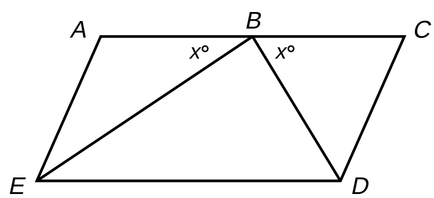


   A 21
   B 24
   C $3\sqrt{149}$
   D 142
   E 210

4. A science class has a ratio of girls to boys of 4 to 3. If the class has a total of 35 students, how many more girls are there than boys?

   A 20 **B** 15 **C** 7 **D** 5 **E** 1

5. In the figure below, $\overline{AC} \parallel \overline{ED}$. If the length of $BD = 3$, what is the length of BE?



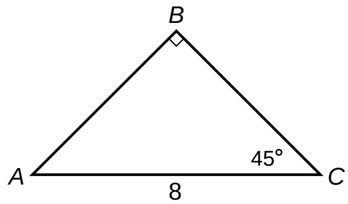
   Note: Figure not drawn to scale.

   A 3 **B** 4 **C** 5 **D** $3\sqrt{3}$
   E It cannot be determined from the information given.

6. What is the slope of the line that contains points at (6, 4) and (13, 5)?

   A $\frac{1}{8}$ **B** $-\frac{1}{9}$ **C** $\frac{1}{7}$ **D** 1 **E** 7

7. In $\triangle ABC$ below, if AC is equal to 8, then BC is equal to



   A $8\sqrt{2}$ **B** 8 **C** 6
   D $4\sqrt{2}$ **E** $3\sqrt{2}$

8. The ratio of $\frac{1}{7}$ to $\frac{1}{5}$ is equal to the ratio of 100 to

   A $\frac{20}{7}$ **B** 20 **C** 35 **D** 100 **E** 140

9. If there are 4 more nickels in a jar than there are dimes, which could be the ratio of dimes to nickels in the jar?

   A $\frac{8}{10}$
   B 1
   C $\frac{14}{10}$
   D 4
   E None of the above

10. **Grid-In** Twenty bottles contain a total of 8 liters of apple juice. If each bottle contains the same amount of apple juice, how much juice (in liters) is in each bottle?

interNET CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**