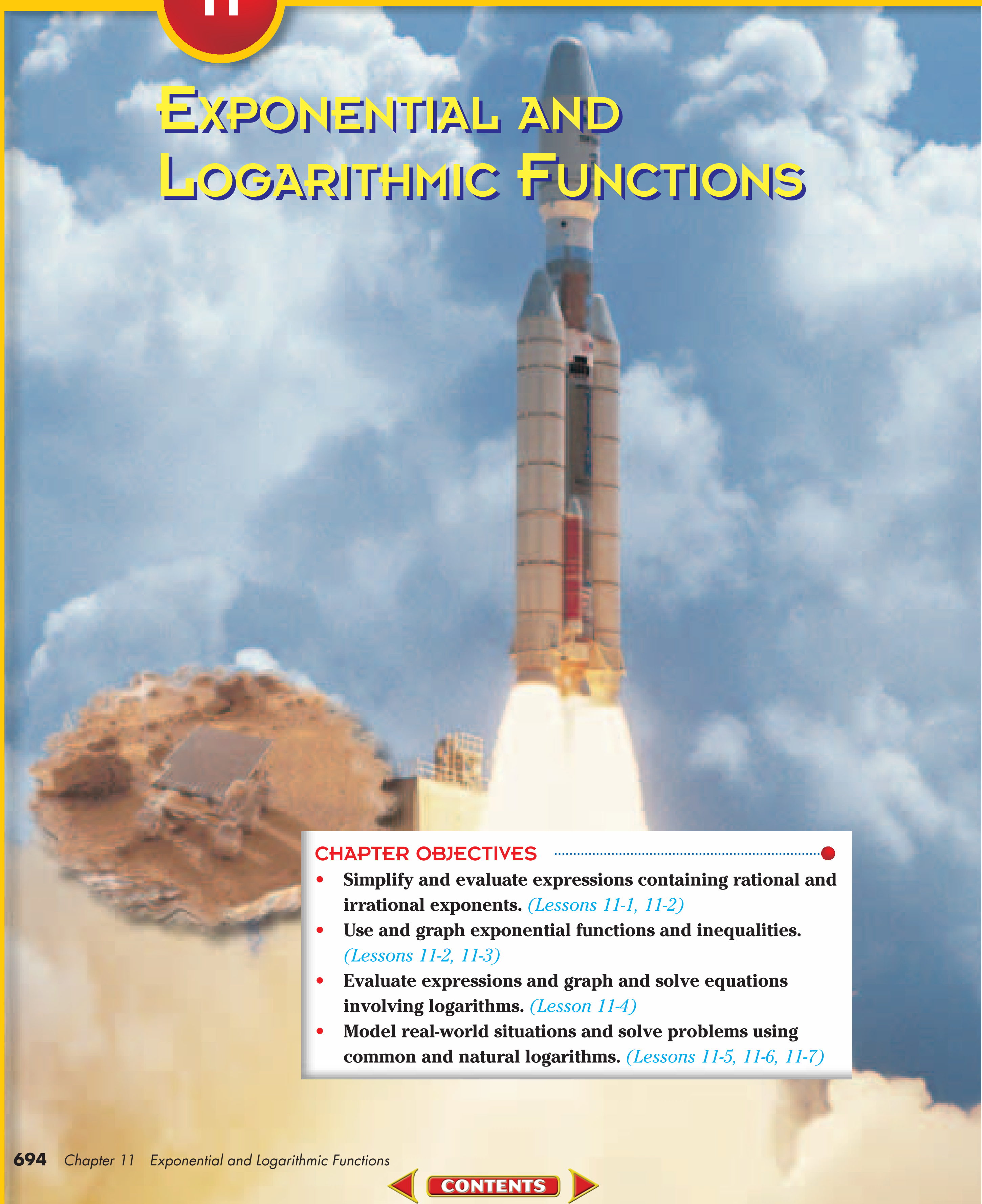
Exponential and Logarithmic Functions



CHAPTER OBJECTIVES

- **Simplify and evaluate expressions containing rational and irrational exponents.** *(Lessons 11-1, 11-2)*
- **Use and graph exponential functions and inequalities.** *(Lessons 11-2, 11-3)*
- **Evaluate expressions and graph and solve equations involving logarithms.** *(Lesson 11-4)*
- **Model real-world situations and solve problems using common and natural logarithms.** *(Lessons 11-5, 11-6, 11-7)*

11-1

Real Exponents

OBJECTIVES

- Use the properties of exponents.
- Evaluate and simplify expressions containing rational exponents.
- Solve equations containing rational exponents.

AEROSPACE On July 4, 1997, the Mars Pathfinder Lander touched down on Mars. It had traveled 4.013×10^8 kilometers from Earth. Two days later, the Pathfinder's Sojourner rover was released and transmitted data from Mars until September 27, 1997.

The distance Pathfinder traveled is written in **scientific notation.** A number is in scientific notation when it is in the form $a \times 10^n$ where $1 \leq a < 10$ and n is an integer. Working with numbers in scientific notation requires an understanding of the definitions and properties of integral exponents. For any real number b and a positive integer n, the following definitions hold.

Definition	Example
If $n = 1$, $b^n = b$.	$17^1 = 17$
If $n > 1$, $b^n = b \cdot b \cdot b \cdot \ldots \cdot b$. *n factors*	$15^4 = 15 \cdot 15 \cdot 15 \cdot 15$ or 50,625
If $b = 0$, $b^0 = 1$.	$400{,}785^0 = 1$
If $b \neq 0$, $b^{-n} = \frac{1}{b^n}$.	$7^{-3} = \frac{1}{7^3}$ or $\frac{1}{343}$

Example 1 **AEROSPACE** **At their closest points, Mars and Earth are approximately 7.5×10^7 kilometers apart.**

a. Write this distance in standard form.

$7.5 \times 10^7 = 7.5\,(10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10)$ or 75,000,000

b. How many times farther is the distance Mars Pathfinder traveled than the minimum distance between Earth and Mars?

Let n represent the number of times farther the distance is.

$$(7.5 \times 10^7)n = 4.013 \times 10^8$$

$$n = \frac{4.013 \times 10^8}{7.5 \times 10^7} \text{ or } 5.4$$

GRAPHING CALCULATOR EXPLORATION

Recall that if the graphs of two equations coincide, the equations are equivalent.

TRY THESE **Graph each set of equations on the same screen. Use the graphs and tables to determine whether Y1 is equivalent to Y2 or Y3.**

1. Y1 $= x^2 \cdot x^3$, Y2 $= x^5$, Y3 $= x^6$
2. Y1 $= (x^2)^3$, Y2 $= x^5$, Y3 $= x^6$

WHAT DO YOU THINK?

3. Make a conjecture about the value of $a^m \cdot a^n$.
4. Make a conjecture about the value of $(a^m)^n$.
5. Use the graphing calculator to investigate the value of an expression like $\left(\frac{a}{b}\right)^m$. What do you observe?

The Graphing Calculator Exploration leads us to the following properties of exponents. You can use the definitions of exponents to verify the properties.

Properties of Exponents

Suppose m and n are positive integers and a and b are real numbers. Then the following properties hold.

Property	Definition	Example
Product	$a^m a^n = a^{m+n}$	$16^3 \cdot 16^7 = 16^{3+7}$ or 16^{10}
Power of a Power	$(a^m)^n = a^{mn}$	$(9^3)^2 = 9^{3 \cdot 2}$ or 9^6
Power of a Quotient	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, where $b \neq 0$	$\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5}$ or $\frac{243}{1024}$
Power of a Product	$(ab)^m = a^m b^m$	$(5x)^3 = 5^3 \cdot x^3$ or $125x^3$
Quotient	$\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$	$\frac{15^6}{15^2} = 15^{6-2}$ or 15^4

Examples

2 Evaluate each expression.

a. $\frac{2^4 \cdot 2^8}{2^5}$

$$\frac{2^4 \cdot 2^8}{2^5} = 2^{(4+8)-5} \quad \textit{Product and Quotient Properties}$$
$$= 2^7$$
$$= 128$$

b. $\left(\frac{2}{5}\right)^{-1}$

$$\left(\frac{2}{5}\right)^{-1} = \frac{1}{\frac{2}{5}} \quad b^{-n} = \frac{1}{b^n}$$
$$= \frac{5}{2}$$

3 Simplify each expression.

a. $(s^2t^3)^5$

$$(s^2t^3)^5 = (s^2)^5(t^3)^5 \quad \textit{Power of a Product}$$
$$= s^{(2 \cdot 5)}t^{(3 \cdot 5)} \quad \textit{Power of a Power}$$
$$= s^{10}t^{15}$$

b. $\frac{x^3y}{(x^4)^3}$

$$\frac{x^3y}{(x^4)^3} = \frac{x^3y}{x^{12}} \quad \textit{Power of a Power}$$
$$= x^{(-3-12)}y \quad \textit{Quotient Property}$$
$$= x^{-9}y \text{ or } \frac{y}{x^9}$$

Expressions with rational exponents can be defined so that the properties of integral exponents are still valid. Consider the expressions $3^{\frac{1}{2}}$ and $1^{\frac{1}{3}}$. Extending the properties of integral exponents gives us the following equations.

$$3^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} = 3^{\frac{1}{2}+\frac{1}{2}}$$
$$= 3^1 \text{ or } 3$$

By definition, $\sqrt{3} \cdot \sqrt{3} = 3$. Therefore, $3^{\frac{1}{2}}$ and $\sqrt{3}$ are equivalent.

$$12^{\frac{1}{3}} \cdot 12^{\frac{1}{3}} \cdot 12^{\frac{1}{3}} = 12^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$
$$= 12^1 \text{ or } 12$$

We know that $\sqrt[3]{12} \cdot \sqrt[3]{12} \cdot \sqrt[3]{12} = 12$. Therefore, $12^{\frac{1}{3}}$ and $\sqrt[3]{12}$ are equivalent.

Exploring other expressions can reveal the following properties.

- If n is an odd number, then $b^{\frac{1}{n}}$ is the nth root of b.
- If n is an even number and $b \geq 0$, then $b^{\frac{1}{n}}$ is the non-negative nth root of b.
- If n is an even number and $b < 0$, then $b^{\frac{1}{n}}$ does not represent a real number, but a complex number.

In general, let $y = b^{\frac{1}{n}}$ for a real number b and a positive integer n. Then, $y^n = \left(b^{\frac{1}{n}}\right)^n = b^{\frac{n}{n}}$ or b. But $y^n = b$ if and only if $y = \sqrt[n]{b}$. Therefore, we can define $b^{\frac{1}{n}}$ as follows.

Definition of $b^{\frac{1}{n}}$

For any real number $b \geq 0$ and any integer $n > 1$,

$$b^{\frac{1}{n}} = \sqrt[n]{b}.$$

This also holds when $b < 0$ and n is odd.

In this chapter, b will be a real number greater than or equal to 0 so that we can avoid complex numbers that occur by taking an even root of a negative number.

The properties of integral exponents given on page 696 can be extended to rational exponents.

Examples

4 Evaluate each expression.

a. $125^{\frac{1}{3}}$

$125^{\frac{1}{3}} = (5^3)^{\frac{1}{3}}$ — *Rewrite 125 as 5^3.*

$= 5^{\frac{3}{3}}$ — *Power of a Power*

$= 5$

b. $\sqrt{14} \cdot \sqrt{7}$

$\sqrt{14} \cdot \sqrt{7} = 14^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$ — *$\sqrt[n]{b} = b^{\frac{1}{n}}$*

$= (2 \cdot 7)^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$ — *$14 = 2 \cdot 7$*

$= 2^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$ — *Power of a Product*

$= 2^{\frac{1}{2}} \cdot 7$ — *Product Property*

$= 7\sqrt{2}$

5 Simplify each expression.

a. $(81c^4)^{\frac{1}{4}}$

$(81c^4)^{\frac{1}{4}} = (3^4c^4)^{\frac{1}{4}}$ — *$81 = 3^4$*

$= (3^4)^{\frac{1}{4}} \cdot (c^4)^{\frac{1}{4}}$ — *Power of a Product*

$= 3^{\frac{4}{4}} \cdot c^{\frac{4}{4}}$ — *Power of a Power*

$= 3|c|$

b. $\sqrt[6]{9x^3}$

$\sqrt[6]{9x^3} = \sqrt[6]{9} \cdot \sqrt[6]{x^3}$ — *Power of a Product*

$= 9^{\frac{1}{6}} \cdot (x^3)^{\frac{1}{6}}$ — *$b^{\frac{1}{n}} = \sqrt[n]{b}$*

$= (3^2)^{\frac{1}{6}} \cdot (x^3)^{\frac{1}{6}}$ — *$9 = 3^2$*

$= 3^{\frac{1}{3}} \cdot x^{\frac{1}{2}}$ — *Power of a Power*

$= \sqrt[3]{3} \cdot \sqrt{x}$ — *$b^{\frac{1}{n}} = \sqrt[n]{b}$*

Rational exponents with numerators other than 1 can be evaluated by using the same properties. Study the two methods of evaluating $64^{\frac{5}{6}}$ below.

Method 1

$$46^{\frac{5}{6}} = \left(46^{\frac{1}{6}}\right)^5$$
$$= \left(\sqrt[6]{46}\right)^5$$

Method 2

$$46^{\frac{5}{6}} = (46^5)^{\frac{1}{6}}$$
$$= \sqrt[6]{46^5}$$

Therefore, $\left(\sqrt[6]{46}\right)^5$ and $\sqrt[6]{46^5}$ both equal $46^{\frac{5}{6}}$.

In general, we define $b^{\frac{m}{n}}$ as $\left(b^{\frac{1}{n}}\right)^m$ or $(b^m)^{\frac{1}{n}}$. Now apply the definition of $b^{\frac{1}{n}}$ to $\left(b^{\frac{1}{n}}\right)^m$ and $(b^m)^{\frac{1}{n}}$.

$$\left(b^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{b}\right)^m \qquad (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}$$

Rational Exponents

For any nonzero number b, and any integers m and n with $n > 1$, and m and n have no common factors

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

except where $\sqrt[n]{b}$ is not a real number.

Examples

6 **Evaluate each expression.**

a. $625^{\frac{3}{4}}$

$$625^{\frac{3}{4}} = (5^4)^{\frac{3}{4}}$$ *Write 625 as 5^4.*
$$= 5^3$$ *Power of a Product*
$$= 125$$

b. $\dfrac{16^{\frac{3}{4}}}{16^{\frac{1}{4}}}$

$$\frac{16^{\frac{3}{4}}}{16^{\frac{1}{4}}} = 16^{\frac{3}{4} - \frac{1}{4}}$$ *Quotient Property*
$$= 16^{\frac{1}{2}} \text{ or } 4$$

7 **a. Express $\sqrt[3]{64s^9t^{15}}$ using rational exponents.**

$$\sqrt[3]{64s^9t^{15}} = (64s^9t^{15})^{\frac{1}{3}}$$ *$b^{\frac{1}{n}} = \sqrt[n]{b}$*
$$= 64^{\frac{1}{3}}s^{\frac{9}{3}}t^{\frac{15}{3}}$$ *Power of a Product*
$$= 4s^3t^5$$

b. Express $12x^{\frac{2}{3}}y^{\frac{1}{2}}$ using a radical.

$$12x^{\frac{2}{3}}y^{\frac{1}{2}} = 12(x^4y^3)^{\frac{1}{6}}$$ *Power of a Product*
$$= 12\sqrt[6]{x^4y^3}$$

When you simplify a radical, use the product property to factor out the *n*th roots and use the smallest index possible for the radical. Remember that you should use caution when evaluating even roots to avoid negative values that would result in an imaginary number.

Example 8 **Simplify $\sqrt{r^7s^{25}t^3}$.**

For $r^7s^{25}t^3$ to be nonnegative, none or exactly two of the variables must be negative. Check the final answer to determine if an absolute value is needed.

$\sqrt{r^7s^{25}t^3} = (r^7s^{25}t^3)^{\frac{1}{2}}$	$b^{\frac{1}{n}} = \sqrt[n]{b}$
$= r^{\frac{7}{2}}s^{\frac{25}{2}}t^{\frac{3}{2}}$	*Power of a Product*
$= r^{\frac{6}{2}}r^{\frac{1}{2}}s^{\frac{24}{2}}s^{\frac{1}{2}}t^{\frac{2}{2}}t^{\frac{1}{2}}$	*Product Property*
$= \lvert r \rvert^3 s^{12} \lvert t \rvert \sqrt{rst}$	*Use $\lvert r \rvert$ and $\lvert t \rvert$ since rst must be nonnegative and there is no indication of which variables are negative.*

You can also use the properties of exponents to solve equations containing rational exponents.

Example 9 **Solve $734 = x^{\frac{3}{4}} + 5$.**

$734 = x^{\frac{3}{4}} + 5$	
$729 = x^{\frac{3}{4}}$	*Subtract 5 from each side.*
$729^{\frac{4}{3}} = \left(x^{\frac{3}{4}}\right)^{\frac{4}{3}}$	*Raise each side to the $\frac{4}{3}$ power.*
$6561 = x$	*Use a calculator.*

Graphing Calculator Tip

Use the [^] calculator key to enter rational exponents. For example, to evaluate $729^{\frac{4}{3}}$, press 729 [^] [(] 4 [÷] 3 [)] .

An expression can also have an irrational exponent, but what does it represent? Consider the expression $2^{\sqrt{3}}$. Since $1.7 < \sqrt{3} < 1.8$, it follows that $2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$. Closer and closer approximations for $\sqrt{3}$ allow us to find closer and closer approximations for $2^{\sqrt{3}}$. Therefore, we can define a value for a^x when x is an irrational number.

Irrational Exponents

If x is an irrational number and $b > 0$, then b^x is the real number between b^{x_1} and b^{x_2} for all possible choices of rational numbers x_1 and x_2, such that $x_1 < x < x_2$.

A calculator can be used to approximate the value of an expression with irrational exponents. For example, $2^{\sqrt{3}} \approx 3.321997085$. While operations with irrational exponents are rarely used, evaluating such expressions is useful in graphing exponential equations.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **State** whether -4^{-2} and $(-4)^{-2}$ represent the same quantity. Explain.
2. **Explain** why rational exponents are not defined when the denominator of the exponent in lowest terms is even and the base is negative.
3. **You Decide** Laura says that a number written in scientific notation with an exponent of -10 is between 0 and 1. Lina says that a number written in scientific notation with an exponent of -10 is a negative number with a very large absolute value. Who is correct and why?

Guided Practice

Evaluate each expression.

4. 5^{-4}
5. $\left(\frac{9}{16}\right)^{-2}$
6. $216^{\frac{1}{3}}$
7. $\sqrt{27} \cdot \sqrt{3}$
8. $32^{\frac{3}{5}}$

Simplify each expression.

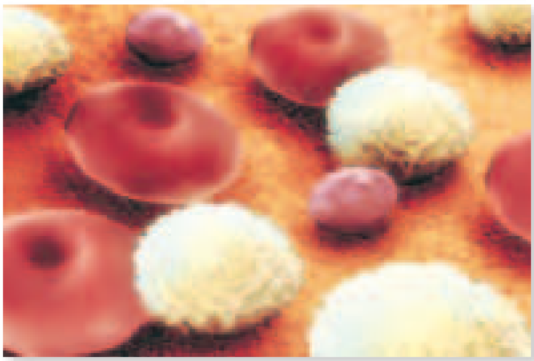
9. $(3a^{-2})^3 \cdot 3a^5$
10. $\sqrt{m^3n^2} \cdot \sqrt{m^4n^5}$
11. $\sqrt{\frac{8^n \cdot 2^7}{4^{-n}}}$
12. $(2x^4y^8)^{\frac{1}{2}}$

Express using rational exponents.

13. $\sqrt{169x^5}$
14. $\sqrt[4]{a^2b^3c^4d^5}$

Express using a radical.

15. $6^{\frac{1}{4}}b^{\frac{3}{4}}c^{\frac{1}{4}}$
16. $15x^{\frac{1}{3}}y^{\frac{1}{5}}$
17. Simplify $\sqrt[3]{p^4q^6r^5}$
18. Solve $y^{\frac{4}{5}} = 34$.



19. **Biology** Red blood cells are circular-shaped cells that carry oxygen through the bloodstream. The radius of a red blood cell is about 3.875×10^{-7} meters. Find the area of a red blood cell.

EXERCISES

Practice

Evaluate each expression.

20. $(-6)^{-4}$
21. -6^{-4}
22. $(5 \cdot 3)^2$
23. $\frac{2^4}{2^{-1}}$
24. $\left(\frac{7}{8}\right)^{-3}$
25. $(3^{-1} + 3^{-2})^{-1}$
26. $81^{\frac{1}{2}}$
27. $729^{\frac{1}{3}}$
28. $\frac{27}{27^{\frac{2}{3}}}$
29. $2^{\frac{1}{2}} \cdot 12^{\frac{1}{2}}$
30. $64^{\frac{1}{12}}$
31. $16^{-\frac{1}{4}}$
32. $\frac{(3^7)(9^4)}{\sqrt{27^6}}$
33. $\left(\sqrt[3]{216}\right)^2$
34. $81^{\frac{1}{2}} - 81^{-\frac{1}{2}}$
35. $\frac{1}{\sqrt[7]{(-128)^4}}$

Simplify each expression.

36. $(3n^2)^3$ **37.** $(y^2)^{-4} \cdot y^8$ **38.** $(4y^4)^{\frac{3}{2}}$ **39.** $(27p^3q^6r^{-1})^{\frac{1}{3}}$

40. $[(2x)^4]^{-2}$ **41.** $(36x^6)^{\frac{1}{2}}$ **42.** $\left(\frac{b^{2n}}{b^{-2n}}\right)^{\frac{1}{2}}$ **43.** $\frac{2n}{4n^{\frac{1}{2}}}$

44. $\left(3m^{\frac{1}{2}} \cdot 27n^{\frac{1}{4}}\right)^4$ **45.** $\left(\frac{f^{-16}}{256g^4h^{-4}}\right)^{-\frac{1}{4}}$ **46.** $\sqrt[6]{x^2\left(x^{\frac{3}{4}} + x^{-\frac{3}{4}}\right)}$ **47.** $\left(2x^{\frac{1}{4}}y^{\frac{1}{3}}\right)\left(3x^{\frac{1}{4}}y^{\frac{2}{3}}\right)$

48. Show that $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.

Express using rational exponents.

49. $\sqrt{m^6n}$ **50.** $\sqrt{xy^3}$ **51.** $\sqrt[3]{8x^3y^6}$

52. $17\sqrt[7]{x^{14}y^7z^{12}}$ **53.** $\sqrt[5]{a^{10}b^2} \cdot \sqrt[4]{c^2}$ **54.** $60\sqrt[8]{r^{80}s^{56}t^{27}}$

Express using a radical.

55. $16^{\frac{1}{5}}$ **56.** $(7a)^{\frac{5}{8}}b^{\frac{3}{8}}$ **57.** $p^{\frac{2}{3}}q^{\frac{1}{2}}r^{\frac{1}{3}}$

58. $\frac{2^{\frac{2}{3}}}{2^{\frac{1}{3}}}$ **59.** $13a^{\frac{1}{7}}b^{\frac{1}{3}}$ **60.** $(n^3m^9)^{\frac{1}{2}}$

61. What is the value of x in the equation $x = \sqrt[3]{(-245)^{-\frac{1}{5}}}$ to the nearest hundredth?

Simplify each expression.

62. $\sqrt{d^3e^2f^2}$ **63.** $\sqrt[3]{a^5b^7c}$ **64.** $\sqrt{20x^3y^6}$

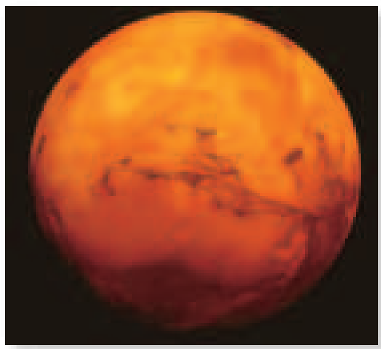
Solve each equation.

65. $14.2 = x^{-\frac{3}{2}}$ **66.** $724 = 15a^{\frac{5}{2}} + 12$ **67.** $\frac{1}{8}\sqrt{x^5} = 3.5$

Applications and Problem Solving

Real World Application

68. Aerospace Mars has an aproximate diameter of 6.794×10^3 kilometers. Assume that Mars is a perfect sphere. Use the formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$ to determine the volume of Mars.



Mars

69. Critical Thinking Consider the equation $y = 3^x$. Find the values of y for $x = -8, -6, -5, \frac{10}{33}, \frac{1}{2}, \frac{2}{3}, \frac{10}{9}, \frac{5}{3}$, and $\frac{7}{2}$.

a. What is the value of y if $x < 0$?

b. What is the value of y if $0 < x < 1$?

c. What is the value of y if $x > 1$?

d. Write a conjecture about the relationship between the value of the base and the value of the power if the exponent is greater than or less than 1. Justify your answer.

70. Chemistry The nucleus of an atom is the center portion of the atom that contains most of its mass. A formula for the radius r of a nucleus of an atom is $r = (1.2 \times 10^{-15})A^{\frac{1}{3}}$ meters, where A is the mass number of the nucleus. Suppose the radius of a nucleus is approximately 2.75×10^{-15} meters. Is the atom boron with a mass number of 11, carbon with a mass number of 12, or nitrogen with a mass number of 14?

71. Critical Thinking Find the solutions for $32^{(x^2 + 4x)} = 16^{(x^2 + 4x + 3)}$.

72. Meteorology Have you ever heard the meteorologist on the news tell the day's windchill factor? A model that approximates the windchill temperature for an air temperature 5°F and a wind speed of s miles per hour is $C = 69.2(0.94^s) - 50$.

a. Copy and complete the table.

b. How does the effect of a 5-mile per hour increase in the wind speed when the wind is light compare to the effect of a 5-mile per hour increase in the wind speed when the wind is heavy?

Wind Speed	Windchill
5	
10	
15	
20	
25	
30	

73. Communication Geosynchronous satellites stay over a single point on Earth. To do this, they rotate with the same period of time as Earth. The distance r of an object from the center of Earth, that has a period of t seconds, is given by $r^3 = \frac{GM_e t^2}{4\pi^2}$, where G is 6.67×10^{-11} N m^2/kg^2, M_e is the mass of Earth, and t is time in seconds. The mass of Earth is 5.98×10^{24} kilograms.

a. How many meters is a geosynchronous communications satellite from the center of Earth?

b. If the radius of Earth is approximately 6380 kilometers, how many kilometers is the satellite above the surface of Earth?

74. Critical Thinking Use the definition of an exponent to verify each property.

a. $a^m a^n = a^{m+n}$ b. $(a^m)^n = a^{mn}$ c. $(ab)^m = a^m b^m$

d. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, where $b \neq 0$ e. $\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$

Mixed Review

75. Graph the system of inequalities $x^2 + y^2 \leq 9$ and $x^2 + y^2 \geq 4$. *(Lesson 10-8)*

76. Find the coordinates of the focus and the equation of the directrix of the parabola with equation $y^2 = 12x$. Then graph the equation. *(Lesson 10-5)*

77. Find $(2\sqrt{3} + 2\boldsymbol{i})^{\frac{1}{5}}$. Express the answer in the form $a + b\boldsymbol{i}$ with a and b to the nearest hundredth. *(Lesson 9-8)*

78. Graph $r^2 = 9 \cos 2\theta$. Identify the classical curve it represents. *(Lesson 9-2)*

79. Baseball Lashon hits a baseball 3 feet above the ground with a force that produces an initial velocity of 105 feet per second at an angle of 42° above the horizontal. What is the elapsed time between the moment the ball is hit and the time it hits the ground? *(Lesson 8-7)*

80. Find an ordered triple that represents $\overrightarrow{TC}$ for $T(3, -4, 6)$ and $C(2, 6, -5)$. Then find the magnitude of $\overrightarrow{TC}$. *(Lesson 8-3)*

81. Find the numerical value of one trigonometric function of S if $\tan S \cos S = \frac{1}{2}$. *(Lesson 7-2)*

82. Find the values of θ for which $\cot \theta = 0$ is true. *(Lesson 6-7)*

83. Agriculture A center-pivot irrigation system with a 75-meter radial arm completes one revolution every 6 hours. Find the linear velocity of a nozzle at the end of the arm. *(Lesson 6-2)*

Extra Practice See p. A46.

84. Find the values of x in the interval $0° \leq x \leq 360°$ for which $x = \arccos 0$. *(Lesson 5-5)*

85. State the number of complex roots of the equation $x^3 - 25x = 0$. Then find the roots and graph. *(Lesson 4-1)*

86. Use a graphing calculator to graph $f(x) = 2x^2 - 3x + 3$ and to determine and classify its extrema. *(Lesson 3-6)*

87. **SAT/ACT Practice** If all painters work at the same rate and 8 painters can paint a building in 48 hours, how many hours will it take 16 painters to paint the building?

A 96 **B** 72 **C** 54 **D** 36 **E** 24

CAREER CHOICES

Food Technologist

Everyone needs to eat properly in order to stay healthy. Food technologists help ensure that foods taste their best, provide good nutrition, and are safe. They study the composition of food and help develop methods for processing, preserving, and packaging foods. Food technologists usually specialize in some aspect of food technology such as improving taste or increasing nutritional value.

Food safety is one very important area in this field. Foods need to be handled, processed, and packaged under strict conditions to insure that they are safe for human consumption. Food technologists generally work either in private industry or for the government.

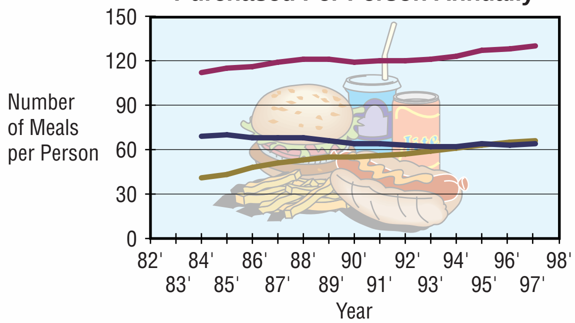
Career Overview

Degree Preferred:
bachelor's degree in Food Science, Food Management, or Human Nutrition

Related Courses:
mathematics, biology, chemistry, physics

Outlook:
better than average through the year 2006

Take-Out and On-Premise Meals Purchased Per Person Annually



interNET CONNECTION For more information on careers in food technology, visit: **www.amc.glencoe.com**

11-2

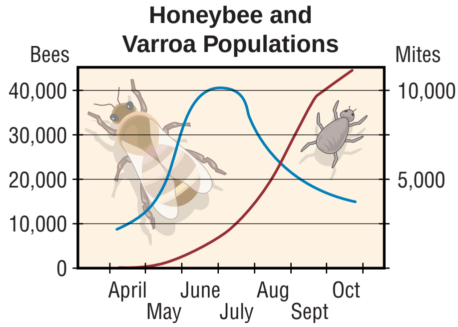
Exponential Functions

OBJECTIVES

- Graph exponential functions and inequalities.
- Solve problems involving exponential growth and decay.

Real World Application

ENTOMOLOGY In recent years, beekeepers have experienced a serious decline in the honeybee population in the United States. One of the causes for the decline is the arrival of varroa mites. Experts estimate that as much as 90% of the wild bee colonies have been wiped out.

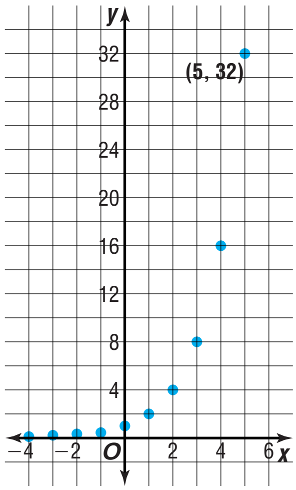


The graph shows typical honeybee and varroa populations over several months. A graph of the varroa population growth from April to September resembles an *exponential curve*. *A problem related to this will be solved in Example 2.*

You have evaluated functions in which the variable is the base and the exponent is any real number. For example, $y = x^5$ has x as the base and 5 as the power. Such a function is known as a **power function.** Functions of the form $y = b^x$, in which the base b is a positive real number and the exponent is a variable are known as **exponential functions.** In the previous lesson, the expression b^x was defined for integral and rational values of x. In order for us to graph $y = b^x$ with a continuous curve, we must define b^x for irrational values of x.

Consider the graph of $y = 2^x$, where x is an integer. This is a function since there is a unique y-value for each x-value.

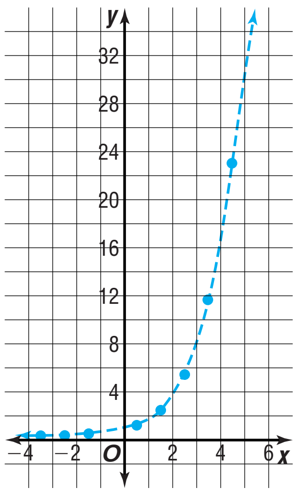
x	−4	−3	−2	−1	0	1	2	3	4	5
$\mathbf{2^x}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32



Notice that the vertical scale is condensed.

The graph suggests that the function is increasing. That is, for any values x_1 and x_2, if $x_1 < x_2$, then $2^{x_1} < 2^{x_2}$.

Suppose the domain of $y = 2^x$ is expanded to include all rational numbers. The additional points graphed seem to "fill in" the graph of $y = 2^x$. That is, if k is between x_1 and x_2, then 2^k is between 2^{x_1} and 2^{x_2}. The graph of $y = 2^x$, when x is a rational number, is indicated by the broken line on the graph at the right.



x	−3.5	−2.5	−1.5	−0.5	0.5	1.5	2.5	3.5	4.5
$\mathbf{2^x}$	0.09	0.18	0.35	0.71	1.41	2.83	5.66	11.31	22.63

Values in the table are approximate.

In Lesson 11-1, you learned that exponents can also be irrational. By including irrational values of x, we can explore the graph of $y = b^x$ for the domain of all real numbers.

GRAPHING CALCULATOR EXPLORATION

TRY THESE **Graph $y = b^x$ for $b = 0.5$, 0.75, 2, and 5 on the same screen.**

1. What is the range of each exponential function?
2. What point is on the graph of each function?
3. What is the end behavior of each graph?
4. Do the graphs have any asymptotes?

WHAT DO YOU THINK?

5. Is the range of every exponential function the same? Explain.
6. Why is the point at (0, 1) on the graph of every exponential function?
7. For what values of a is the graph of $y = a^x$ increasing and for what values is the graph decreasing? Explain.
8. Explain the existence or absence of the asymptotes in the graph of an exponential function.

The equations graphed in the Graphing Calculator Exploration demonstrate many properties of exponential graphs.

Characteristics of graphs of $y = b^x$

	$b > 1$	$0 < b < 1$
Domain	all real numbers	all real numbers
Range	all real numbers > 0	all real numbers > 0
y-intercept	(0, 1)	(0, 1)
behavior	continuous, one-to-one, and increasing	continuous, one-to-one, and decreasing
Horizontal asymptote	negative x-axis	positive x-axis
Vertical asymptote	none	none

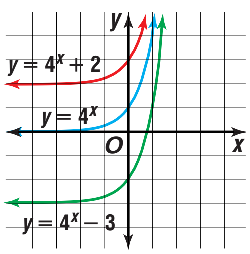
When $b = 1$ the graph of $y = b^x$ is the horizontal line $y = 1$.

As with other types of graphs, $y = b^x$ represents a parent graph. The same techniques used to transform the graphs of other functions you have studied can be applied to graphs of exponential functions.

Examples

To graph an exponential function using paper and pencil, you can use a calculator to find each range value for each domain value.

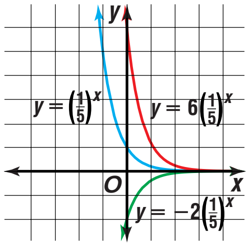
1 **a. Graph the exponential functions $y = 4^x$, $y = 4^x + 2$, and $y = 4^x - 3$ on the same set of axes. Compare and contrast the graphs.**



All of the graphs are continuous, increasing, and one-to-one. They have the same domain, and no vertical asymptote.

The y-intercept and the horizontal asymptotes for each graph are different from the parent graph $y = 4^x$. While $y = 4^x$ and $y = 4^x + 2$ have no x-intercept, $y = 4^x - 3$ has an x-intercept.

b. Graph the exponential functions $y = \left(\frac{1}{5}\right)^x$, $y = 6\left(\frac{1}{5}\right)^x$, and $y = -2\left(\frac{1}{5}\right)^x$ on the same set of axes. Compare and contrast the graphs.



All of the graphs are decreasing, continuous, and one-to-one. They have the same domain and horizontal asymptote. They have no vertical asymptote or x-intercept.

The y-intercepts for each graph are different from the parent graph $y = \left(\frac{1}{5}\right)^x$.

2 PHYSICS **According to Newton's Law of Cooling, the difference between the temperature of an object and its surroundings decreases in time exponentially. Suppose a certain cup of coffee is 95°C and it is in a room that is 20°C. The cooling for this particular cup can be modeled by the equation $y = 75(0.875)^t$ where y is the temperature difference and t is time in minutes.**

Real World Application

a. Find the temperature of the coffee after 15 minutes.

b. Graph the cooling function.

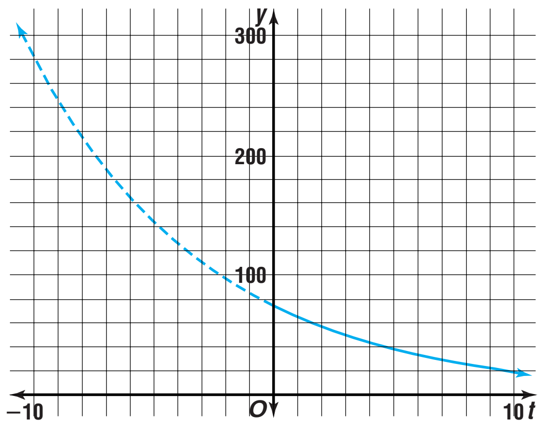
a. $y = 75(0.875)^t$

$y = 75(0.875)^{15}$ $\quad t = 15$

$y \approx 10.12003603$

The difference is about 10.1° C. So the coffee is 95° C − 10.1° C or 84.9° C.

b.



In a situation like cooling when a quantity loses value exponentially over time, the value exhibits what is called **exponential decay.** Many real-world situations involve quantities that increase exponentially over time. For example, the balance in a savings or money market account and the population of people or animals in a region often demonstrate what is called **exponential growth.** As you saw in Example 1, you can use the general form of an exponential function to describe exponential growth or decay. When you know the rate at which the growth or decay is occurring, the following equation may be used.

Exponential Growth or Decay

The equation $N = N_0(1 + r)^t$, where N is the final amount, N_0 is the initial amount, r is the rate of growth or decay per time period, and t is the number of time periods, is used for modeling exponential growth or decay.

Example 3 **ENTOMOLOGY** **Refer to the application at the beginning of the lesson. Suppose that a researcher estimates that the initial population of varroa in a colony is 500. They are increasing at a rate of 14% per week. What is the expected population in 22 weeks?**

$N = N_0(1 + r)^t$

$N = 500(1 + 0.14)^{22}$ *$N_0 = 500$, $r = 0.14$, $t = 22$*

$N \approx 8930.519719$ *Use a calculator.*

There will be about 8931 varroa in the colony in 22 weeks.

The general equation for exponential growth is modified for finding the balance in an account that earns compound interest.

Compound Interest

The compound interest equation is $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where P is the principal or initial investment, A is the final amount of the investment, r is the annual interest rate, n is the number of times interest is paid, or compounded each year, and t is the number of years.

Example 4 **FINANCE** **Determine the amount of money in a money market account providing an annual rate of 5% compounded daily if Marcus invested \$2000 and left it in the account for 7 years.**

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 2000\left(1 + \frac{0.05}{365}\right)^{365 \cdot 7}$ *$P = 2000$, $r = 0.05$, $n = 365$, $t = 7$*

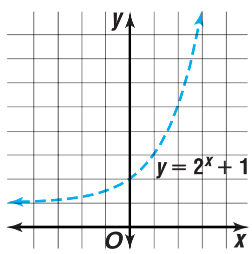
$A \approx 2838.067067$ *Use a calculator.*

After 7 years, the \$2000 investment will have a value of \$2838.06.
The value 2838.067067 is rounded to 2838.06 as banks generally round down when they are paying interest.

Graphing exponential inequalities is similar to graphing other inequalities.

Example 5 **Graph $y < 2^x + 1$**

First, graph $y = 2^x + 1$. Since the points on this curve are not in the solution of the inequality, the graph of $y = 2^x + 1$ is shown as a dashed curve.

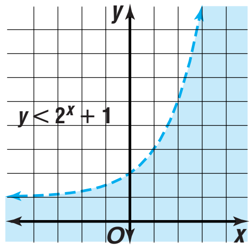


(continued on the next page)

Then, use (0, 0) as a test point to determine which area to shade.

$y < 2^x + 1$ ➡ $0 < 2^0 + 1$
$0 < 1 + 1$
$0 < 2$

Since (0, 0) satisfies the inequality, the region that contains (0, 0) should be shaded.



Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Tell** whether $y = x^4$ is an exponential or a power function. Justify your answer.
2. **Compare and contrast** the graphs of $y = b^x$ when $b > 1$ and when $0 < b < 1$.
3. **Write** about how you can tell whether an exponential function represents exponential growth or exponential decay.
4. **Describe** the differences between the graphs of $y = 4^x$ and $y = 4^x - 3$.

Guided Practice

Graph each exponential function or inequality.

5. $y = 3^x$
6. $y = 3^{-x}$
7. $y > 2^x - 4$
8. **Business** Business owners keep track of the value of their assets for tax purposes. Suppose the value of a computer depreciates at a rate of 25% a year. Determine the value of a laptop computer two years after it has been purchased for $3750.
9. **Demographics** In the 1990 U.S. Census, the population of Los Angeles County was 8,863,052. By 1997, the population had increased to 9,145,219.
   a. Find the yearly growth rate by dividing the average change in population by the 1990 population.
   b. Assuming the growth rate continues at a similar rate, predict the number of people who will be living in Los Angeles County in 2010.

Exercises

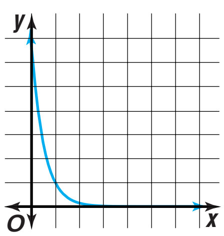
Practice

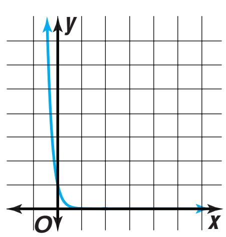
Graph each exponential function or inequality.

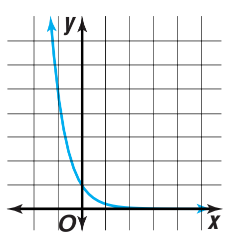
10. $y = 2^x$
11. $y = -2^x$
12. $y = 2^{-x}$
13. $y = 2^{x+3}$
14. $y = -2^{x+3}$
15. $y > -4^x + 2$
16. $y = \left(\frac{1}{5}\right)^x$
17. $y \leq \left(\frac{1}{2}\right)^x$
18. $y < 2^{x-4}$

Match each equation to its graph.

19. $y = 0.01^x$
20. $y = 5^{-x}$
21. $y = 7^{1-x}$

A


B


C


www.amc.glencoe.com/self_check_quiz

Graphing Calculator

22. Graph the functions $y = 5^x$, $y = -5^x$, $y = 5^{-x}$, $y = 5^x + 2$, $y = 5^x - 2$, and $y = 10^x$ on the same screen.
   a. Compare $y = -5^x$, $y = 5^{-x}$ to the parent graph $y = 5^x$. Describe the transformations of the functions.
   b. What transformation occurs with the graphs of $y = 5^x + 2$ and $y = 5^x - 2$?
   c. The function $y = 10^x$ can be expressed as $y = (2 \cdot 5)^x$. Compare this function to $y = 5^x$. Is the graph of $y = 5^{2x}$ the same as $y = 10^x$? Explain.

23. Without graphing, describe how each pair of graphs is related. Then use a graphing calculator to check your descriptions.
   a. $y = 6^x$ and $y = 6^x + 4$
   b. $y = -3^x$ and $y = 3^x$
   c. $y = 7^x$ and $y = 7^{-x}$
   d. $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$

Applications and Problem Solving

24. **Employment** Average national teachers' salaries can be modeled using the equation $y = 9.25(1.06)^n$, where y is the salary in thousands of dollars and n is the number of years since 1970.
   a. Graph the function.
   b. Using this model, what can a teacher expect to have as a salary in the year 2020?

25. **Aviation** When kerosene is purified to make jet fuel, pollutants are removed by passing the kerosene through a special clay filter. Suppose a filter is fitted in a pipe so that 15% of the impurities are removed for every foot that the kerosene travels.
   a. Write an exponential function to model the percent of impurity left after the kerosene travels x feet.
   b. Graph the function.
   c. About what percent of the impurity remains after the kerosene travels 12 feet?
   d. Will the impurities ever be completely removed? Explain.

26. **Demographics** Find the projected population of each location in 2015.
   a. In Honolulu, Hawaii, the population was 836,231 in 1990. The average yearly rate of growth is 0.7%.
   b. The population in Kings County, New York has demonstrated an average decrease of 0.45% over several years. The population in 1997 was 2,240,384.
   c. Janesville, Wisconsin had a population of 139,420 in 1980 and 139,510 in 1990.
   d. The population in Cedar Rapids, Iowa was 169,775 in the 1980 U.S. Census and 168,767 in the 1990 U.S. Census.

27. **Biology** Scientists who study Atlantic salmon have found that the oxygen consumption of a yearling salmon O is given by the function $O = 100\left(3^{\frac{3s}{5}}\right)$, where s is the speed that the fish is traveling in feet per second.

   a. What is the oxygen consumption of a fish that is traveling at 5 feet per second?
   b. If a fish has traveled 4.2 miles in an hour, what is its oxygen consumption?

interNET CONNECTION

Graphing Calculator Programs
For a graphing calculator program that calculates loan payments, visit **www.amc.glencoe.com**

28. Finance Bankers call a series of payments made at equal intervals an *annuity*. The present value of an annuity P_n is the sum of the present values of all of the periodic payments P. In other words, a lump-sum investment of P_n dollars now will provide payments of P dollars for n periods. The formula for the present value is $P_n = P\left[\frac{1-(1+i)^{-n}}{i}\right]$, where i is the interest rate for the period.

a. Use the present value formula to find the monthly payment you would pay on a home mortgage if the present value is \$121,000, the annual interest rate is 7.5%, and payments will be made for 30 years. (*Hint*: First find the interest rate for the period.)

b. How much is the monthly mortgage payment if the borrowers choose a loan with a 20-year term and an interest rate of 7.25%?

c. How much will be paid in interest over the life of each mortgage?

d. Explain why a borrower might choose each mortgage.

29. Finance The future value of an annuity F_n is the sum of all of the periodic payments P and all of the accumulated interest. The formula for the future value is $F_n = P\left[\frac{(1+i)^n - 1}{i}\right]$, where i is the interest rate for the period.

a. When Connie Hockman began her first job at the age of 22, she started saving for her retirement. Each year she places \$4000 in an account that will earn an average 4.75% annual interest until she retires at 65. How much will be in the account when she retires?

b. If Ms. Hockman had invested in an account that earns an average of 5.25% annual interest, how much more would her account be worth?

30. Critical Thinking Explain when the exponential function $y = a^x$ is undefined for $a < 0$.

31. Investments The number of times that interest is compounded has a dramatic effect on the total interest earned on an investment.

a. How much interest would you earn in one year on an \$1000 investment earning 5% interest if the interest is compounded once, twice, four times, twelve times, or 365 times in the year?

b. If you are making an investment that you will leave in an account for one year, which account should you choose to get the highest return?

Account	Rate	Compounded
Statement Savings	5.1%	Yearly
Money Market Savings	5.05%	Monthly
Super Saver	5%	Daily

c. Suppose you are a bank manager determining rates on savings accounts. If the account with interest compounded annually offers 5% interest, what interest rate should be offered on an account with interest compounded daily in order for the interest earned on equal investments to be the same?

Mixed Review

32. Simplify $4x^2(4x)^{-2}$. *(Lesson 11-1)*

33. Write the rectangular equation $y = 15$ in polar form. *(Lesson 9-4)*

34. Find the inner product of vectors $\langle -3, 9\rangle$ and $\langle 2, 1\rangle$. Are the vectors perpendicular? Explain. *(Lesson 8-4)*

35. Find $\frac{1}{3}\left(\cos\frac{7\pi}{8} + \boldsymbol{i}\sin\frac{7\pi}{8}\right) \cdot 3\sqrt{3}\left[\cos\left(-\frac{\pi}{4}\right) + \boldsymbol{i}\sin\left(-\frac{\pi}{4}\right)\right]$. Then express the product in rectangular form with a and b to the nearest hundredth. *(Lesson 9-7)*

36. Sports Suppose a baseball player popped a baseball straight up at an initial velocity v_0 of 72 feet per second. Its distance s above the ground after t seconds is described by $s = v_0t - 16t^2 + 4$. Find the maximum height of the ball. *(Lesson 10-5)*

37. Art Leigh Ann is drawing a sketch of a village. She wants the town square to be placed midway between the library and the fire station. Suppose the ordered pair for the library is $(-7, 6)$ and the ordered pair for the fire station is $(12, 8)$. *(Lesson 10-1)*

a. Draw a graph that represents this situation.

b. Determine the coordinates indicating where the town square should be placed.

38. Verify that $\sin^4 A + \cos^2 A = \cos^4 A + \sin^2 A$ is an identity. *(Lesson 7-2)*

39. Mechanics A circular saw 18.4 centimeters in diameter rotates at 2400 revolutions per second. What is the linear velocity at which a saw tooth strikes the cutting surface in centimeters per second? *(Lesson 6-2)*

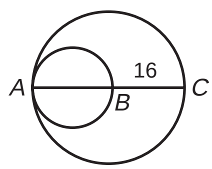
40. Travel Martina went to Acapulco, Mexico, on a vacation with her parents. One of the sights they visited was a cliff-diving exhibition into the waters of the Gulf of Mexico. Martina stood at a lookout site on top of a 200-foot cliff. A team of medical experts was in a boat below in case of an accident. The angle of depression to the boat from the top of the cliff was 21°. How far is the boat from the base of the cliff? *(Lesson 5-4)*

41. Salary Diane has had a part time job as a Home Chef demonstrator for 9 years. Her yearly income is listed in the table. Write a model that relates the income as a function of the number of years since 1990. *(Lesson 4-8)*

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998
Income ($)	4012	6250	7391	8102	8993	9714	10,536	11,362	12,429

42. Use the parent graph $f(x) = \frac{1}{x}$ to graph $f(x) = \frac{1}{x+3}$. Describe the transformation(s) that have taken place. Identify the new locations of the asymptotes. *(Lesson 3-7)*

43. SAT/ACT Practice In the figure, $\overline{AB}$ is the diameter of the smaller circle, and $\overline{AC}$ is the diameter of the larger circle. If the distance from B to C is 16 inches, then the circumference of the larger circle is approximately how many inches greater than the circumference of the smaller circle?



A 8 **B** 16 **C** 25 **D** 35 **E** 50

Extra Practice See p. A46.

11-3

The Number e

OBJECTIVES

- Use the exponential function $y = e^x$.

Real World Application

MEDICINE Swiss entomologist Dr. Paul Mueller was awarded the Nobel Prize in medicine in 1948 for his work with the pesticide DDT. Dr. Mueller discovered that DDT is effective against insects that destroy agricultural crops, mosquitoes that transmit malaria and yellow fever, as well as lice that carry typhus.

It was later discovered that DDT presented a risk to humans. Effective January 1, 1973, the United States Environmental Protection Agency banned all uses of DDT. More than 1.0×10^{10} kilograms of DDT had been used in the U.S. before the ban. How much will remain in the environment in 2005? *This problem will be solved in Example 1.*

DDT degrades into harmless materials over time. To find the amount of a substance that decays exponentially remaining after a certain amount of time, you can use the following formula for exponential growth or decay, which involves the number ***e***.

Exponential Growth or Decay (in terms of *e*)

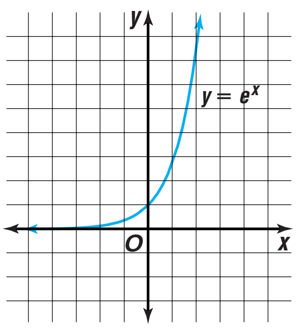
$N = N_0e^{kt}$, where N is the final amount, N_0 is the initial amount, k is a constant and t is time.

The number e in the formula is not a variable. It is a special irrational number. This number is the sum of the infinite series shown below.

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots + \frac{1}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} + \cdots$$

The following computation for e is correct to three decimal places.

$$\begin{aligned} e &= 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \\ &\quad \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} \\ &= 1 + 1 + 0.5 + 0.16667 + 0.04167 + 0.00833 + \\ &\quad 0.00139 + 0.000198 \\ &= 2.718 \end{aligned}$$



The function $y = e^x$ is one of the most important exponential functions. The graph of $y = e^x$ is shown at the right.

Example 1

Real World Application

MEDICINE **Refer to the application at the beginning of the lesson. Assume that there were 1.0×10^9 kilograms of DDT in the environment in 1973 and that for DDT, $k = -0.0211$.**

a. Write a function to model the amount of DDT remaining in the environment.

b. Find the amount of DDT that will be in the environment in 2005.

c. Graph the function and use the graph to verify your answer in part b.

Graphing Calculator Tip

To graph an equation or evaluate an expression involving *e* raised to a power, use the second function of ln on a calculator.

Example 1 is an example of chemical decay.

a. $y = ne^{kt}$

$y = (1 \times 10^9)e^{-0.0211t}$

$y = (10^9)e^{-0.0211t}$

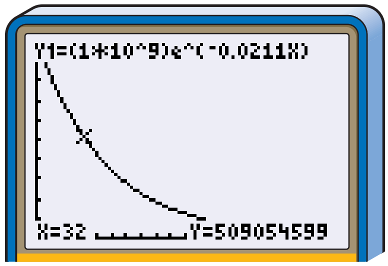
b. In 2005, it will have been 2005 − 1973 or 32 years since DDT was banned. Thus, $t = 32$.

$y = (10^9)e^{-0.0211t}$

$y = (10^9)e^{-0.0211(32)}$ *$t = 32$*

$y \approx (10^9)0.5090545995$

c. Use a graphing calculator to graph the function.



[−0, 200] scl:10, [0, (1 × 10^9] scl:1 × 10^8

If there were 1×10^9 kilograms of DDT in the environment in 1973, there will be about $0.51(1 \times 10^9)$ or 5.1×10^8 kilograms remaining in 2005.

Some banks offer accounts that compound the interest continuously. The formula for finding continuously compounded interest is different from the one used for interest that is compounded a specific number of times each year.

Continuously Compounded Interest

The equation $A = Pe^{rt}$, where P is the initial amount, A is the final amount, r is the annual interest rate, and t is time in years, is used for calculating interest that is compounded continuously.

Example 2

Real World Application

FINANCE **Compare the balance after 25 years of a \$10,000 investment earning 6.75% interest compounded continuously to the same investment compounded semiannually.**

In both cases, $P = 10{,}000$, $r = 0.0675$, $t = 25$. When the interest is compounded semiannually, $n = 2$. Use a calculator to evaluate each expression.

Continuously	Semiannually
$A = Pe^{rt}$	$A = P\left(1 + \frac{r}{n}\right)^{nt}$
$A = 10{,}000(e)^{(0.0675 \cdot 25)}$	$A = 10{,}000\left(1 + \frac{0.0675}{2}\right)^{2 \cdot 25}$
$A = 54{,}059.49$	$A = 52{,}574.62$

The same principal invested over the same amount of time yields \$54,059.49 if compounded continuously and \$52,574.62 when compounded twice a year. You would earn \$54059.49 − \$52574.62 = \$1484.87 more by choosing the account that compounds continuously.

CHECK FOR UNDERSTANDING

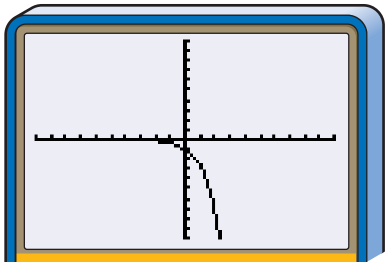
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Tell** which equation is represented by the graph.

   a. $y = e^x$ b. $y = e^{-x}$

   c. $y = -e^x$ d. $y = -e^{-x}$



[−10, 10] scl:1 by [−10, 10] scl:1

2. **Describe** the value of k when the equation $N = N_0e^{kt}$ represents exponential growth and when it represents exponential decay.
3. **Describe** a situation that could be modeled by the equation $A = 3000e^{0.055t}$.
4. **State** the domain and range of the function $f(x) = e^x$.
5. *Math Journal* **Write** a sentence or two to explain the difference between interest compounded continuously and interest compounded monthly.

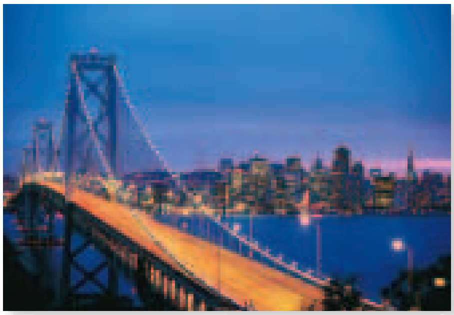
Guided Practice

6. **Demographics** Bakersfield, California was founded in 1859 when Colonel Thomas Baker planted ten acres of alfalfa for travelers going from Visalia to Los Angeles to feed their animals. The city's population can be modeled by the equation $y = 33{,}430e^{0.0397t}$, where t is the number of years since 1950.
   a. Has Bakersfield experienced growth or decline in population?
   b. What was Bakersfield's population in 1950?
   c. Find the projected population of Bakersfield in 2010.
7. **Financial Planning** The Kwans are saving for their daughter's college education. If they deposit $12,000 in an account bearing 6.4% interest compounded continuously, how much will be in the account when Ann goes to college in 12 years?

EXERCISES

Applications and Problem Solving

8. **Psychology** Without further study, as time passes you forget things you have learned. The Ebbinghaus model of human memory gives the percent p of acquired knowledge that a person retains after t weeks. The formula is $p = (100 - a)e^{-bt} + a$, where a and b vary from one person to another. If $a = 18$ and $b = 0.6$ for a certain student, how much information will the student retain two weeks after learning a new topic?
9. **Physics** Newton's Law of Cooling expresses the relationship between the temperature of a cooling object y and the time t elapsed since cooling began. This relationship is given by $y = ae^{-kt} + c$, where c is the temperature of the medium surrounding the cooling object, a is the difference between the initial temperature of the object and the surrounding temperature, and k is a constant related to the cooling object.
   a. The initial temperature of a liquid is 160°F. When it is removed from the heat, the temperature in the room is 76°F. For this object, $k = 0.23$. Find the temperature of the liquid after 15 minutes.
   b. Alex likes his coffee at a temperature of 135°. If he pours a cup of 170°F coffee in a 72°F room and waits 5 minutes before drinking, will his coffee be too hot or too cold? Explain. For Alex's cup, $k = 0.34$.



10. **Civil Engineering** Suspension bridges can span distances far longer than any other kind of bridge. The roadway of a suspension bridge is suspended from huge cables. When a flexible cable is suspended between two points, it forms a *catenary curve*.

a. Use a graphing calculator to graph the catenary $y = \frac{e^x + e^{-x}}{2}$.

b. What kind of symmetry is displayed by the graph?

11. **Banking** If your bank account earns interest that is compounded more than one time per year, the effective annual yield E is the interest rate that would give the same amount of interest earnings if the interest were compounded once per year. To find the effective annual yield, divide the interest earned by the principal.

a. Copy and complete the table to find the effective annual yield for each account if the principal is \$1000, the annual interest rate is 8%, and the term is one year.

Interest Compounded	Interest	Effective Annual Yield
Annually		
Semi-annually		
Quarterly		
Monthly		
Daily		
Continuously		

b. Which type of compounding provides the greatest effective annual yield?

c. If P represents the principal and A is the total value of the investment, the value of an investment is $A = P(1 + E)$. Find a formula for the effective annual yield for an account with interest compounded n times per year.

d. Write a formula for the effective annual yield of an account with interest compounded continuously.

12. **Sociology** Sociologists have found that information spreads among a population at an exponential rate. Suppose that the function $y = 525(1 - e^{-0.038t})$ models the number of people in a town of 525 people who have heard news within t hours of its distribution.

a. How many people will have heard about the opening of a new grocery store within 24 hours of the announcement?

b. Graph the function on a graphing calculator. When will 90% of the people have heard about the grocery store opening?

13. **Customer Service** The service-time distribution describes the probability P that the service time of the customer will be no more than t hours. If m is the mean number of customers serviced in an hour, then $P = 1 - e^{-mt}$.

a. Suppose a computer technical support representative can answer calls from 6 customers in an hour. What is the probability that a customer will be on hold less than 30 minutes?

b. A credit card customer service department averages 34 calls per hour. Use a graphing calculator to determine the amount of time after which it is 50% likely that a customer has been served?

14. **Critical Thinking** In 1997, inventor and amateur mathematician Harlan Brothers discovered some simple algebraic expressions that approximate e.

a. Use Brothers' expression $\left(\frac{2x+1}{2x-1}\right)^x$ for $x = 10$, $x = 100$, and $x = 1000$.

b. Compare each approximation to the value of e stored in a calculator. If each digit of the calculator value is correct, how accurate is each approximation?

c. Is the approximation always greater than e, always less than e, or sometimes greater than and sometimes less than e?

15. **Marketing** The probability P that a person has responded to an advertisement can be modeled by the exponential equation $P = 1 - e^{-0.047t}$ where t is the number of days since the advertisement began to appear in the media.

a. What are the probabilities that a person has responded after 5 days, 20 days, and 90 days?

b. Use a graphing calculator to graph the function. Use the graph to find when the probability that an individual has responded to the advertisement is 75%.

c. If you were planning a marketing campaign, how would you use this model to plan the introduction of new advertisements?

16. **Critical Thinking** Consider $f(x) = \frac{e^x}{e^x + c}$ if c is a constant greater than 0.

a. What is the domain of the function?

b. What is the range of the function?

c. How does the value of c affect the graph?

Mixed Review

17. **Finance** A *sinking fund* is a fund into which regular payments are made in order to pay off a debt when it is due. The Gallagher Construction Company foresees the need to buy a new cement truck in four years. At that time, the truck will probably cost $120,000. The firm sets up a sinking fund in order to accumulate the money. They will pay semiannual payments into a fund with an APR of 7%. Use the formula for the future value of an annuity $F_n = P\left[\frac{(1+i)^n - 1}{i}\right]$, where F_n is the future value of the annuity, P is the payment amount, i is the interest rate for the period, and n is the number of payments, to find the payment amount. *(Lesson 11-2)*

18. Express $x^{\frac{8}{5}} y^{\frac{3}{5}} z^{\frac{1}{5}}$ using radicals. *(Lesson 11-1)*

19. **Communications** A satellite dish tracks a satellite directly overhead. Suppose the equation $y = 6x^2$ models the shape of the dish when it is oriented in this position. Later in the day, the dish is observed to have rotated approximately 45°. Find an equation that models the new orientation of the dish. *(Lesson 10-7)*

20. Express $-5 - \boldsymbol{i}$ in polar form. *(Lesson 9-6)*

21. **Physics** Tony is pushing a cart weighing 150 pounds up a ramp 10 feet long at an incline of 28°. Find the work done to push the cart the length of the ramp. Assume that friction is not a factor. *(Lesson 8-5)*

22. **Safety** A model relating the average number of crimes reported at a shopping mall as a function of the number of years since 1991 is $y = -1.5x^2 + 13.3x + 19.4$. According to this model, what is the number of crimes for the year 2001? *(Lesson 4-8)*

Extra Practice See p. A46.

23. Solve $\sqrt{2x + 3} = 4$. *(Lesson 4-7)*

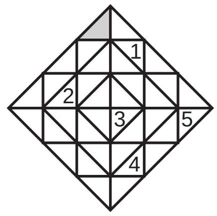
24. Solve $|3x + 2| \leq 6$. *(Lesson 3-3)*

25. Quadrilateral $JKLM$ has vertices at $J(-3, -2)$, $K(-2, 6)$, $L(2, 5)$ and $M(3, -1)$. Find the coordinates of the dilated quadrilateral $J'K'L'M'$ for a scale factor of 3. Describe the dilation. *(Lesson 2-4)*

26. Find the values of x and y for which $\begin{bmatrix} 4x + y \\ x \end{bmatrix} = \begin{bmatrix} 6 \\ 2y - 12 \end{bmatrix}$ is true. *(Lesson 2-3)*

27. State the domain and range of the relation $\{(2, 7), (-4, 5), (5, 7)\}$. Is this relation a function? *(Lesson 1-1)*

28. **SAT/ACT Practice** An artist wants to shade exactly 18 of the 36 smallest triangles in the pattern, including the one shown. If no two shaded triangles can have a side in common, which of the triangles indicated must *not* be shaded?



A 1 **B** 2 **C** 3 **D** 4 **E** 5

MID-CHAPTER QUIZ

Evaluate each expression. (Lesson 11-1)

1. $64^{\frac{1}{2}}$

2. $\left(\sqrt[3]{343}\right)^{-2}$

3. Simplify $\left(\frac{8x^3y^{-6}}{27w^6z^{-9}}\right)^{\frac{1}{3}}$

4. Express $\sqrt{a^6b^3}$ using rational exponents. (Lesson 11-1)

5. Express $(125a^2b^3)^{\frac{1}{3}}$ using radicals. (Lesson 11-1)

6. **Architecture** A soap bubble will enclose the maximum space with a minimum amount of surface material. Architects have used this principle to create buildings that enclose a great amount of space with a small amount of building material. If a soap bubble has a surface area of A, then its volume V is given by the equation $V = 0.094\sqrt{A^3}$. Find the surface area of a bubble with a volume of 1.75×10^2 cubic millimeters. (Lesson 11-1)

7. **Demographics** In 1990, the population of Houston, Texas was 1,637,859. In 1998, the population was 1,786,691. Predict the population of Houston in 2014. (Lesson 11-2)

8. **Finance** Determine the amount of money in a money market account at an annual rate of 5.2% compounded quarterly if Mary invested \$3500 and left it in the account for $3\frac{1}{2}$ years. (Lesson 11-2)

9. **Forestry** The yield in millions of cubic feet y of trees per acre is given by $y = 6.7e^{\frac{-48.1}{t}}$ for a forest that is t years old.
   a. Find the yield after 15 years.
   b. Find the yield after 50 years.
   (Lesson 11-3)

10. **Biology** It has been observed that the rate of growth of a population of organisms will increase until the population is half its maximum and then the rate will decrease. If M is the maximum population and b and c are constants determined by the type of organism, the population n after t years is given by $n = \frac{M}{1 + be^{-ct}}$. A certain organism yields the values of $M = 200$, $b = 20$, and $c = 0.35$. What is the population after 2, 15, and 60 years? (Lesson 11-3)

11-4

Logarithmic Functions

OBJECTIVES

- Evaluate expressions involving logarithms.
- Solve equations and inequalities involving logarithms.
- Graph logarithmic functions and inequalities.

Real World Application

CHEMISTRY Radioactive materials decay, or change to a non-radioactive material, in a predictable way. Archeologists use the decaying property of Carbon-14 in dating artifacts like dinosaur bones. Geologists use Thorium-230 in determining the age of rock formations, and medical researchers conduct tests using Arsenic-74.

A radioactive material's half-life is the time it takes for half of a given amount of the material to decay and can range from less than a second to billions of years. The half-life for various elements is shown in the table at the right.

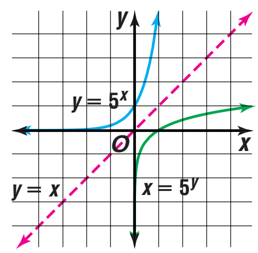
Element	Half-Life
Arsenic-74	17.5 days
Carbon-14	5730 years
Polonium-194	0.5 second
Thorium-230	80,000 years
Thorium-232	14 billion years
Thorium-234	25 days

How long would it take for 256,000 grams of Thorium-234 to decay to 1000 grams? Radioactive decay can be modeled by the equation $N = N_0\left(\frac{1}{2}\right)^t$ where N is the final amount of a substance, N_0 is the initial amount, and t represents the number of half-lifes. If you want to find the number of 25-day half-lifes that will pass, you would have to use an inverse function. *This problem will be solved in Example 4.*

Since the graphs of exponential functions pass the horizontal line test, their inverses are also functions. As with inverses of other functions, you can find the inverse of an exponential function by interchanging the x- and y- values in the ordered pairs of the function.

$f(x)$: $y = 5^x$	
x	y
−3	0.008
−2	0.004
−1	0.2
0	1
1	0
2	25
3	125

$f^{-1}(x)$: $x = 5^y$	
x	y
0.008	−3
0.004	−2
0.2	−1
1	0
0	1
25	2
125	3



The function $f^{-1}(x)$ can be defined as $x = 5^y$. The ordered pairs can be used to sketch the graphs $y = 5^x$ and $x = 5^y$ on the same axes. *Note that they are reflections of each other over the line $y = x$.*

This example can be applied to a general statement that the inverse of $y = a^x$ is $x = a^y$. In the function $x = a^y$, y is called the **logarithm** of x. It is usually written as $y = \log_a x$ and is read "y equals the log, base a, of x." The function $y = \log_a x$ is called a **logarithmic function.**

Logarithmic Function	The logarithmic function $y = \log_a x$, where $a > 0$ and $a \neq 1$, is the inverse of the exponential function $y = a^x$. So, $y = \log_a x$ if and only if $x = a^y$.

Each logarithmic equation corresponds to an equivalent exponential equation. That is, the logarithmic equation $y = \log_a x$ is equivalent to $a^y = x$.

exponent

$y = \log_2 32 \longleftrightarrow 2^y = 32.$

base

Since $2^5 = 32$, $y = 5$. Thus, $\log_2 32 = 5$.

Example 1 **Write each equation in exponential form.**

a. $\log_{125} 25 = \frac{2}{3}$

The base is 125, and the exponent is $\frac{2}{3}$.

$25 = 125^{\frac{2}{3}}$

b. $\log_8 2 = \frac{1}{3}$

The base is 8, and the exponent is $\frac{1}{3}$.

$2 = 8^{\frac{1}{3}}$

You can also write an exponential function as a logarithmic function.

Example 2 **Write each equation in logarithmic form.**

a. $4^3 = 64$

The base is 4, and the exponent or logarithm is 3.

$\log_4 64 = 3$

b. $3^{-3} = \frac{1}{27}$

The base is 3, and the exponent or logarithm is -3.

$\log_3 \frac{1}{27} = -3$

Using the fact that if $a^u = a^v$ then $u = v$, you can evaluate a logarithmic expression to determine its logarithm.

Example 3 **Evaluate the expression $\log_7 \frac{1}{49}$.**

Let $x = \log_7 \frac{1}{49}$.

$x = \log_7 \frac{1}{49}$

$7^x = \frac{1}{49}$ *Definition of logarithm.*

$7^x = (49)^{-1}$ $a^{-m} = \frac{1}{a^m}$

$7^x = (7^2)^{-1}$ $7^2 = 49$

$7^x = 7^{-2}$ $(a^m)^n = a^{mn}$

$x = -2$ *If $a^u = a^v$ then $u = v$.*

Example 4 Real World Application

CHEMISTRY **Refer to the application at the beginning of the lesson. How long would it take for 256,000 grams of Thorium-234, with a half-life of 25 days, to decay to 1000 grams?**

$N = N_0\left(\frac{1}{2}\right)^t$	*$N = N_0(1 + r)^t$ for $r = \frac{1}{2}$*
$1000 = 256{,}000\left(\frac{1}{2}\right)^t$	*$N = 1000$, $N_0 = 256{,}000$*
$\frac{1}{256} = \left(\frac{1}{2}\right)^t$	*Divide each side by 256,000.*
$\log_{\frac{1}{2}} \frac{1}{256} = t$	*Write the equation in logarithmic form.*
$\log_{\frac{1}{2}} \left(\frac{1}{2}\right)^8 = t$	*$256 = 2^8$*
$\log_{\frac{1}{2}} \left(\frac{1}{2}\right)^8 = t$	*$\left(\frac{1}{b^n}\right) = \left(\frac{1}{b}\right)^n$*
$\left(\frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^t$	*Definition of logarithm*
$8 = t$	It will take 8 half-lifes or 200 days.

Since the logarithmic function and the exponential function are inverses of each other, both of their compositions yield the identity function. Let $f(x) = \log_a x$ and $g(x) = a^x$. For $f(x)$ and $g(x)$ to be inverses, it must be true that $f(g(x)) = x$ and $g(f(x)) = x$.

Look Back
Refer to Lesson 1-2 to review composition of functions.

$f(g(x)) \stackrel{?}{=} x$	$g(f(x)) \stackrel{?}{=} x$
$f(a^x) \stackrel{?}{=} x$	$g(\log_a x) \stackrel{?}{=} x$
$\log_a a^x \stackrel{?}{=} x$	$a^{\log_a x} \stackrel{?}{=} x$
$x = x$	$x = x$

The properties of logarithms can be derived from the properties of exponents.

Properties of Logarithms

Suppose m and n are positive numbers, b is a positive number other than 1, and p is any real number. Then the following properties hold.

Property	Definition	Example
Product	$\log_b mn = \log_b m + \log_b n$	$\log_3 9x = \log_3 9 + \log_3 x$
Quotient	$\log_b \frac{m}{n} = \log_b m - \log_b n$	$\log_{\frac{1}{4}} \frac{4}{5} = \log_{\frac{1}{4}} 4 - \log_{\frac{1}{4}} 5$
Power	$\log_b m^p = p \cdot \log_b m$	$\log_2 8^x = x \cdot \log_2 8$
Equality	If $\log_b m = \log_b n$, then $m = n$.	$\log_8 (3x - 4) = \log_8 (5x + 2)$ so, $3x - 4 = 5x + 2$

Each of these properties can be verified using the properties of exponents. For example, suppose we want to prove the Product Property. Let $x = \log_b m$ and $y = \log_b n$. Then by definition $b^x = m$ and $b^y = n$.

You will be asked to prove other properties in Exercises 2 and 61.

$\log_b mn = \log_b (b^x \cdot b^y)$	*$b^x = m$, $b^y = n$*
$= \log_b (b^{x+y})$	*Product Property of Exponents*
$= x + y$	*Definition of logarithm*
$= \log_b m + \log_b n$	*Substitution*

Equations can be written involving logarithms. Use the properties of logarithms and the definition of logarithms to solve these equations.

Example 5 **Solve each equation.**

a. $\log_p 64^{\frac{1}{3}} = \frac{1}{2}$

$\log_p 64^{\frac{1}{3}} = \frac{1}{2}$

$p^{\frac{1}{2}} = 64^{\frac{1}{3}}$ *Definition of logarithm.*

$\sqrt{p} = \sqrt[3]{64}$ $b^{\frac{m}{n}} = \sqrt[n]{b^m}$

$\sqrt{p} = 4$

$(\sqrt{p})^2 = (4)^2$ *Square each side.*

$p = 16$

b. $\log_4 (2x + 11) = \log_4 (5x - 4)$

$\log_4 (2x + 11) = \log_4 (5x - 4)$

$2x + 11 = 5x - 4$ *Equality Property*

$-3x = -15$

$x = 5$

c. $\log_{11} x + \log_{11} (x + 1) = \log_{11} 6$

$\log_{11} x + \log_{11} (x + 1) = \log_{11} 6$

$\log_{11} [x(x + 1)] = \log_{11} 6$ *Product Property*

$(x^2 + x) = 6$ *Equality Property*

$x^2 + x - 6 = 0$

$(x - 2)(x + 3) = 0$ *Factor.*

$x - 2 = 0$ or $x + 3 = 0$

$x = 2$ $\quad x = -3$

By substituting $x = 2$ and $x = -3$ into the equation, we find that $x = -3$ is undefined for the equation $\log_{11} x + \log_{11} (x + 1) = \log_{11} 6$. When $x = -3$ we get an extraneous solution. So, $x = 2$ is the correct solution.

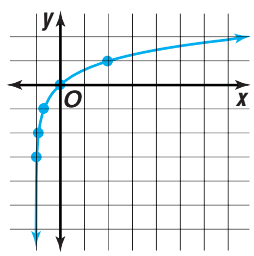
You can graph a logarithmic function by rewriting the logarithmic function as an exponential function and constructing a table of values.

Example 6 **Graph $y = \log_3 (x + 1)$.**

The graph of $y = \log_3 (x + 1)$ is a horizontal translation of the graph of $y = \log_3 x$.

The equation $y = \log_3 (x + 1)$ can be written as $3^y = x + 1$. Choose values for y and then find the corresponding values of x.

y	x + 1	x	(x, y)
−3	0.037	−0.963	(−0.963, −3)
−2	0.11	−0.89	(−0.89, −2)
−1	0.33	−0.67	(−0.67, −1)
0	1	0	(0, 0)
1	3	2	(2, 1)
2	9	8	(8, 2)
3	27	26	(26, 3)



You can graph logarithmic inequalities using the same techniques as shown in Example 6. Choose a test point to determine which region to shade.

Example 7 **Graph $y \leq \log_5 x - 2$.**

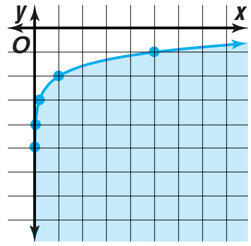
The boundary for the inequality $y \leq \log_5 x - 2$ can be written as $y = \log_5 x - 2$. Rewrite this equation in exponential form.

$$y = \log_5 x - 2$$
$$y + 2 = \log_5 x$$
$$5^{y+2} = x$$

The graph of $y = \log_5 x - 2$ is a vertical translation of the graph of $y = \log_5 x$.

Use a table of values to graph the boundary.

y	$y + 2$	x	(x, y)
−5	−3	0.008	(0.008, −5)
−4	−2	0.25	(0.25, −4)
−3	−1	0.2	(0.2, −3)
−2	0	1	(1, −2)
−1	1	5	(5, −1)
0	2	25	(25, 0)
1	3	125	(125, 1)



Test a point, for example (0, 0), to determine which region to shade.

$5^{y+2} \leq 0 \rightarrow 5^{0+2} \leq 0$ *False*

Shade the region that does not contain the point at (0, 0).

Remember that you must exclude values for which the log is undefined. For example, for $y \geq \log_5 x - 2$, the negative values for x must be omitted from the domain and no shading would occur in that area.

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** the graphs of $y = 3^x$ and $y = \log_3 x$.
2. **Show** that the Power Property of logarithms is valid.
3. **State** the difference between the graph of $y = \log_5 x$ and $y = \log_{\frac{1}{5}} x$.
4. **You Decide** Courtney and Sean are discussing the expansion of $\log_b mn$. Courtney states that $\log_b mn$ can be written as $\log_b m \times \log_b n$. Sean states that $\log_b mn$ can be written as $\log_b m + \log_b n$. Who is correct? Explain your answer.
5. **Explain** the relationship between the decay formula $N = N_0(1 + r)^t$ and the formula for determining half-life, $N = N_0\left(\frac{1}{2}\right)^t$.

Guided Practice

Write each equation in exponential form.

6. $\log_9 27 = \frac{3}{2}$
7. $\log_{\frac{1}{25}} 5 = -\frac{1}{2}$

Write each equation in logarithmic form.

8. $7^{-6} = y$
9. $8^{-\frac{2}{3}} = \frac{1}{4}$

Evaluate each expression.

10. $\log_2 \frac{1}{16}$

11. $\log_{10} 0.01$

12. $\log_7 \frac{1}{343}$

Solve each equation.

13. $\log_2 x = 5$

14. $\log_7 n = \frac{2}{3} \log_7 8$

15. $\log_6 (4x + 4) = \log_6 64$

16. $2 \log_6 4 - \frac{1}{4} \log_6 16 = \log_6 x$

Graph each equation or inequality.

17. $y = \log_{\frac{1}{2}} x$

18. $y \geq \log_6 x$

19. Biology The generation time for bacteria is the time that it takes for the population to double. The generation time G can be found using experimental data and the formula $G = \frac{t}{3.3 \log_b f}$, where t is the time period, b is the number of bacteria at the beginning of the experiment, and f is the number of bacteria at the end of the experiment. The generation time for mycobacterium tuberculosis is 16 hours. How long will it take four of these bacteria to multiply into 1024 bacteria?

EXERCISES

Write each equation in exponential form.

20. $\log_{27} 3 = \frac{1}{3}$

21. $\log_{16} 4 = \frac{1}{2}$

22. $\log_7 \frac{1}{2401} = -4$

23. $\log_4 32 = \frac{5}{2}$

24. $\log_e 65.98 = x$

25. $\log_{\sqrt{6}} 36 = 4$

Write each equation in logarithmic form.

26. $81^{\frac{1}{2}} = 9$

27. $36^{\frac{3}{2}} = 216$

28. $\left(\frac{1}{8}\right)^{-3} = 512$

29. $6^{-2} = \frac{1}{36}$

30. $16^0 = 1$

31. $x^{1.238} = 14.36$

Evaluate each expression.

32. $\log_8 64$

33. $\log_{125} 5$

34. $\log_2 32$

35. $\log_4 128$

36. $\log_9 9^6$

37. $\log_{49} 343$

38. $\log_8 16$

39. $\log_{\sqrt{8}} 4096$

40. $10^{4 \log_{10} 2}$

Solve each equation.

41. $\log_x 49 = 2$

42. $\log_3 3x = \log_3 36$

43. $\log_6 x + \log_6 9 = \log_6 54$

44. $\log_8 48 - \log_8 w = \log_8 6$

45. $\log_6 216 = x$

46. $\log_5 0.04 = x$

47. $\log_{10} \sqrt[3]{10} = x$

48. $\log_{12} x = \frac{1}{2} \log_{12} 9 + \frac{1}{3} \log_{12} 27$

49. $\log_5(x + 4) + \log_5 8 = \log_5 64$

50. $\log_4(x - 3) + \log_4(x + 3) = 2$

51. $\frac{1}{2}(\log_7 x + \log_7 8) = \log_7 16$

52. $2 \log_5(x - 2) = \log_5 36$

Graph each equation or inequality.

53. $y = \log_4 x$ **54.** $y = 3 \log_2 x$ **55.** $y = \log_5(x - 1)$

56. $y \leq \log_2 x$ **57.** $y \geq 2 \log_2 x$ **58.** $y > \log_{10}(x + 1)$

Applications and Problem Solving

Real World Application

59. Public Health Inspectors for the Fulton County Health Department routinely check food samples for the presence of the E. coli bacteria. When E. coli cells are placed in a medium that provides nutrients needed for growth, the bacteria population can increase exponentially, reproducing itself every 15 minutes. If an inspector has a sample containing 1000 bacteria cells, how long will it take the population to reach 64,000 cells?

60. Critical Thinking Using the definition of a logarithm where $y = \log_a x$, explain why the base a cannot equal 1.

61. Proof Prove the quotient property of logarithms, $\log_b \frac{m}{n} = \log_b m - \log_b n$, using the definition of a logarithm.

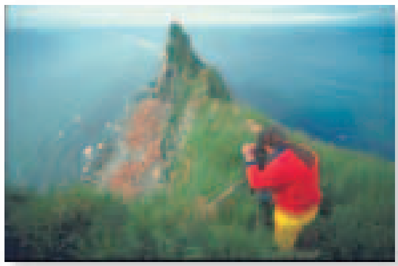
62. Finance Latasha plans to invest $2500 in an account that compounds quarterly, hoping to at least double her money in 10 years.

a. Write an inequality that can be used to find the interest rate at which Latasha should invest her money.

b. What is the lowest interest rate that will allow her to meet her goal?

c. Suppose she only wants to invest her money for 7 years. What interest rate would allow her to double her money?

63. Photography The aperture setting of a camera, or *f*-stop, controls the amount of light exposure on film. Each step up in *f*-stop setting allows twice as much light exposure as the previous setting. The formula $n = \log_2 \frac{1}{p}$ where p is the fraction of sunlight, represents the change in the *f*-stop setting n to use in less light.



a. A nature photographer sets her camera's *f*-stop at *f*/6.7 while taking outdoor pictures in direct sunlight. If the amount of sunlight on a cloudy day is $\frac{1}{4}$ as bright as direct sunlight, how many *f*-stop settings should she move to accommodate less light?

b. If she moves down 3 *f*-stop settings from her original setting, is she allowing more or less light into the camera? What fraction of daylight is she accommodating?

64. Critical Thinking Show that $\log_a x = \frac{\log_b x}{\log_b a}$.

65. Meteorology Atmospheric pressure decreases as altitude above sea level increases. Atmospheric pressure P, measured in pounds per square inch, at altitude h miles can be represented by the logarithmic function $\log_{2.72} \frac{P}{14.7} = -0.02h$.

a. Graph the function for atmospheric pressure.

b. The elevation of Denver, Colorado, is about 1 mile. What is the atmospheric pressure in Denver?

c. The lowest elevation on Earth is in the Mariana Trench in the Atlantic Ocean about 6.8 miles below sea level. If there were no water at this point, what would be the atmospheric pressure at this elevation?

66. **Radiation Safety** Radon is a naturally ocurring radioactive gas which can collect in poorly ventilated structures. Radon gas is formed by the decomposition of radium-226 which has a half-life of 1622 years. The half-life of radon is 3.82 days. Suppose a house basement contained 38 grams of radon gas when a family moved in. If the source of radium producing the radon gas is removed so that the radon gas eventually decays, how long will it take until there is only 6.8 grams of radon gas present?

Mixed Review

67. Find the value of $e^{4.243}$ to the nearest ten thousandth. *(Lesson 11-3)*

68. **Finance** What is the monthly principal and interest payment on a home mortgage of \$90,000 for 30 years at 11.5%? *(Lesson 11-2)*

69. Identify the conic section represented by $9x^2 - 18x + 4y^2 - 16y - 11 = 0$. Then write the equation in standard form and graph the equation. *(Lesson 10-6)*

70. **Engineering** A wheel in a motor is turning counterclockwise at 2 radians per second. There is a small hole in the wheel 3 centimeters from its center. Suppose a model of the wheel is drawn on a rectangular coordinate system with the wheel centered at the origin. If the hole has initial coordinates (3, 0), what are its coordinates after t seconds? *(Lesson 10-6)*

71. Find the lengths of the sides of a triangle whose vertices are $A(-1, 3)$, $B(-1, -3)$, and $C(3, 0)$. *(Lesson 10-1)*

72. Find the product $5\left(\cos \frac{3\pi}{4} + \boldsymbol{i} \sin \frac{3\pi}{4}\right) \cdot 2\left(\cos \frac{2\pi}{3} + \boldsymbol{i} \sin \frac{2\pi}{3}\right)$. Then express it in rectangular form. *(Lesson 9-7)*

73. **Electricity** A circuit has a current of $(3 - 4\boldsymbol{j})$ amps and an impedance of $(12 + 7\boldsymbol{j})$ ohms. Find the voltage of this circuit. *(Lesson 9-5)*

74. State whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are *opposite, parallel,* or *neither of these* for $A(-2, 5)$, $B(3, -1)$, $C(2, 6)$, and $D(7, 0)$. *(Lesson 8-1)*

75. Find $\cos (A + B)$ if $\cos A = \frac{5}{13}$ and $\cos B = \frac{35}{37}$ and A and B are first quadrant angles. *(Lesson 7-3)*

76. **Weather** The maximum normal daily temperatures in each season for New Orleans, Louisiana, are given below.

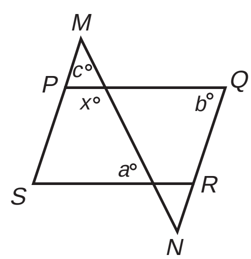
Winter	Spring	Summer	Fall
64°	78°	90°	79°

Source: Rand McNally & Company

Write a sinusoidal function that models the temperatures, using $t = 1$ to represent winter. *(Lesson 6-6)*

77. Solve $\triangle ABC$ if $C = 105°18'$, $a = 6.11$, and $b = 5.84$. *(Lesson 5-8)*

78. **SAT/ACT Practice** If $PQRS$ is a parallelogram and MN is a segment, then x must equal



A $180 - b$

B $180 - c$

C $a + b$

D $a + c$

E $b + c$

Extra Practice See p. A46.

11-5

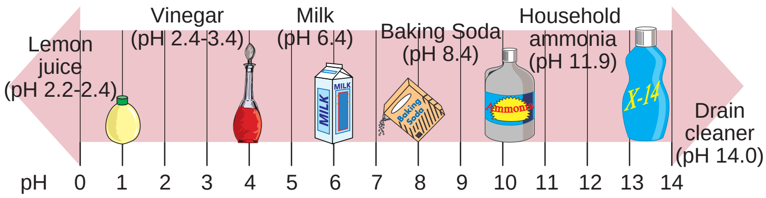
Common Logarithms

OBJECTIVES

- Find common logarithms and antilogarithms of numbers.
- Solve equations and inequalities using common logarithms.
- Solve real-word applications with common logarithmic functions.

Real World Application

CHEMISTRY Drinking water is routinely tested for its pH level. The pH of a solution is a measure of acidity and is related to the concentration of hydrogen ions measured in moles per liter.



Testing the water's pH level may indicate the presence of a contaminant that is harmful to life forms. *This information will be used in Examples 2 and 6.*

The pH of a solution can be modeled by the equation pH $= \log \frac{1}{H^+}$, where H^+ is the number of moles of hydrogen ions per liter. Notice that no base is indicated for the logarithm in the equation for pH. When no base is indicated, the base is assumed to be 10. So, $\log \frac{1}{H^+}$ means $\log_{10} \frac{1}{H^+}$. Logarithms with base 10 are called **common logarithms.** You can easily find the common logarithms of integral powers of ten.

$$\begin{array}{llll} \log 1000 = 3 & \text{since} & 1000 = 10^3 \\ \log 100 = 2 & \text{since} & 100 = 10^2 \\ \log 10 = 1 & \text{since} & 10 = 10^1 \\ \log 1 = 0 & \text{since} & 1 = 10^0 \\ \log 0.1 = -1 & \text{since} & 0.1 = 10^{-1} \\ \log 0.01 = -2 & \text{since} & 0.01 = 10^{-2} \\ \log 0.001 = -3 & \text{since} & 0.001 = 10^{-3} \end{array}$$

The common logarithms of numbers that differ by integral powers of ten are closely related. Remember that a logarithm is an exponent. For example, in the equation $y = \log x$, y is the power to which 10 is raised to obtain the value of x.

$\log x = y$	**means**	$10^y = x$
$\log 1 = 0$	since	$10^0 = 1$
$\log 10 = 1$	since	$10^1 = 10$
$\log 10^m = m$	since	$10^m = 10^m$

Example 1 **Given that log 7 = 0.8451, evaluate each logarithm.**

a. log 7,000,000

$$\begin{aligned} \log 7{,}000{,}000 &= \log(1{,}000{,}000 \times 7) \\ &= \log 10^6 + \log 7 \\ &= 6 + 0.8451 \\ &= 6.8451 \end{aligned}$$

b. log 0.0007

$$\begin{aligned} \log 0.0007 &= \log(0.0001 \times 7) \\ &= \log 10^{-4} + \log 7 \\ &= -4 + 0.8451 \\ &= -3.1549 \end{aligned}$$

A common logarithm is made up of two parts, the **characteristic** and the **mantissa.** The mantissa is the logarithm of a number between 1 and 10. Thus, the mantissa is greater than 0 and less than 1.

In Example 1, the mantissa is log 7 or 0.8451. The characteristic is the exponent of ten that is used to write the number in scientific notation. So, in Example 1, the characteristic of 1,000,000 is 6, and the characteristic of 0.0001 is −4. Traditionally, a logarithm is expressed as the indicated sum of the mantissa and the characteristic.

You can use a calculator to solve certain equations containing common logarithms.

Example 2 Real World Application

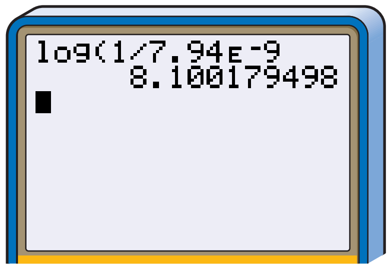
CHEMISTRY **Refer to the application at the beginning of the lesson. If the water being tested contains 7.94×10^{-9} moles of H^+ per liter, what is the pH level of the water?**

$\text{pH} = \log \frac{1}{H^+}$

$\text{pH} = \log \frac{1}{7.94 \times 10^{-9}}$ *$H^+ = 7.94 \times 10^{-9}$*

Evaluate with a calculator.

[LOG] 1 [÷] 7.94 [2nd] **[EE]** [(–)] 9 [ENTER]



$\text{pH} \approx 8.1$

The pH level of the water is about 8.1

Graphing Calculator Tip

The key marked [LOG] on your calculator will display the common logarithm of a number.

The properties of logarithmic functions you learned in Lesson 11-4 apply to common logarithms. You can use these properties to evaluate logarithmic expressions.

Example 3 **Evaluate each expression.**

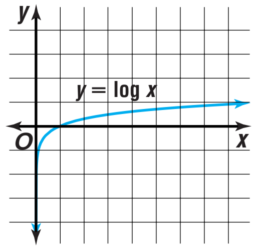
a. $\log 5(2)^3$

$$\begin{aligned} \log 5(2)^3 &= \log 5 + 3 \log 2 && \textit{Product Property, Power Property} \\ &\approx 0.6990 + 3(0.3010) && \textit{Use a calculator.} \\ &\approx 0.6990 + 0.9031 \\ &\approx 1.6021 \end{aligned}$$

b. $\log \frac{19^2}{6}$

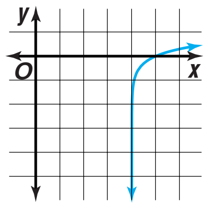
$$\begin{aligned} \log \frac{19^2}{6} &= 2 \log 19 - \log 6 && \textit{Quotient Property, Power Property} \\ &\approx 2(1.2788) - 0.7782 && \textit{Use a calculator.} \\ &\approx 2.5576 - 0.7782 \\ &\approx 1.7794 \end{aligned}$$

The graph of $y = \log x$ is shown at the right. As with other logarithmic functions, it is continuous and increasing. It has a domain of positive real numbers and a range of all real numbers. It has an x-intercept at (1, 0), and there is a vertical asymptote at $x = 0$, or the y-axis, and no horizontal asymptote. The parent graph of $y = \log x$ can be transformed.

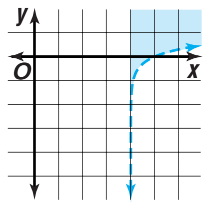


Example 4 **Graph $y > \log(x - 4)$.**

The boundary of the inequality is the graph of $y = \log(x - 4)$. The graph is translated 4 units to the right of $y = \log x$ with a domain of all real numbers greater than 4, an x-intercept at (5, 0) and a vertical asymptote at $x = 4$.



Choose a test point, (6, 1) for example, to determine which region to shade.



$y > \log(x - 4)$

$1 \stackrel{?}{>} \log(6 - 1)$ *Substitution*

$1 \stackrel{?}{>} \log 5$

$1 \stackrel{?}{>} 0.6990$ *Use a calculator.*

The inequality is true, so shade the region containing the point at (6, 1).

In Lesson 11-4, you evaluated logarithms in various bases. In order to evaluate these with a calculator, you must convert these logarithms in other bases to common logarithms using the following formula.

Change of Base Formula

If a, b, and n are positive numbers and neither a nor b is 1, then the following equation is true.

$$\log_a n = \frac{\log_b n}{\log_b a}$$

Example 5 **Find the value of $\log_9 1043$ using the change of base formula.**

$$\log_a n = \frac{\log_b n}{\log_b a}$$

$$\log_9 1043 = \frac{\log_{10} 1043}{\log_{10} 9}$$

$$\approx \frac{3.0183}{0.9542} \quad \textit{Use a calculator.}$$

$$\approx 3.1632$$

The value of $\log_9 1043$ is about 3.1632.

Sometimes the logarithm of x is known to have a value of a, but x is not known. Then x is called the **antilogarithm** of a, written antilog a. If $\log x = a$, then $x = \text{antilog } a$. Remember that the inverse of a logarithmic function is an exponential function.

Example 6 Real World Application

CHEMISTRY **Refer to the application at the beginning of the lesson. Technicians at a water treatment plant determine that the water supply has a pH of 6.7. What is the concentration of hydrogen ions in the tested water?**

$$\text{pH} = \log \frac{1}{H^+}$$

$$6.7 = \log \frac{1}{H^+} \qquad \textit{pH = 6.7}$$

$$\text{antilog } 6.7 = \frac{1}{H^+} \qquad \textit{Take the antilogarithm of each side.}$$

$$H^+ = \frac{1}{\text{antilog } 6.7}$$

$$H^+ \approx 1.9953 \times 10^{-7} \qquad \textit{Use a calculator.}$$

The concentration of hydrogen ions is about 1.9953×10^{-7} moles per liter.

Logarithms can be used to solve exponential equations.

Example 7 Real World Application

Solve $6^{3x} = 81$.

$$6^{3x} = 81$$

$$\log 6^{3x} = \log 81 \qquad \textit{Take the logarithm of each side.}$$

$$3x \log 6 = \log 81 \qquad \log_b m^p = p \cdot \log_b m$$

$$3x = \frac{\log 81}{\log 6} \qquad \textit{Divide each side by log 6.}$$

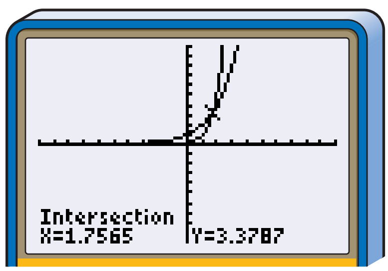
$$x \approx 0.8175 \qquad \textit{Use a calculator.}$$

The solution is approximately 0.8175.

Graphing is an alternate way of finding approximate solutions of exponential or logarithmic equations. To do this, graph each side of the equation as a function and find the coordinates of the intersection of the graphs.

Example 8 **Solve $5^{x-1} = 2^x$ by graphing.**

Graph $y = 5^{x-1}$ and $y = 2^x$ on the same set of axes. The graphs appear to intersect at about (1.76, 3.38).



[−10, 10] scl:1 by [−10, 10] scl:1

Therefore, $x \approx 1.76$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Demonstrate** the fact that $\log 1 = 0$ and $\log 10 = 1$.
2. **Write** an explanation of how you would determine the characteristic of $\log 38{,}272.65$.
3. **Compare** the value of antilog 2.835 and $10^{2.835}$.
4. **Show** that $\log 15 = \log 5 + \log 3$.

Guided Practice

Given that $\log 8 = 0.9031$ and $\log 3 = 0.4771$, evaluate each logarithm.

5. $\log 80{,}000$
6. $\log 0.003$
7. $\log 0.0081$

Evaluate each expression.

8. $\log 424$
9. antilog 4.8740

10. Graph $y \leq \log(x + 3)$.

Find the value of each logarithm using the change of base formula.

11. $\log_{12} 18$
12. $\log_8 15$

Solve each equation or inequality.

13. $2.2^{x-5} = 9.32$
14. $6^{x-2} = 4^x$
15. $4.3^x < 76.2$
16. $3^{x-3} \geq 2\sqrt[4]{4^{x-1}}$

17. Solve $12^{x-4} = 3^{x-2}$ by graphing.

Graphing Calculator

18. **Seismology** The intensity of an earthquake is described by a number on the Richter scale. The Richter scale number R of an earthquake is given by the formula $R = \log\left(\frac{a}{T}\right) + B$, where a is the amplitude of the vertical ground motion in microns, T is the period of the seismic wave in seconds, and B is a factor that accounts for the weakening of seismic waves.
    a. Find the intensity of an earthquake to the nearest tenth if a recording station measured the amplitude as 200 microns and the period as 1.6 seconds, and $B = 4.2$.
    b. How much more intense is an earthquake of magnitude 5 on the Richter scale than an earthquake that measures 4 on the Richter scale? Explain how you determined this result.

EXERCISES

Practice

Given that $\log 4 = 0.6021$, $\log 9 = 0.9542$, and $\log 12 = 1.0792$, evaluate each logarithm.

19. $\log 400{,}000$
20. $\log 0.00009$
21. $\log 1.2$
22. $\log 0.06$
23. $\log 36$
24. $\log 108{,}000$
25. $\log 0.0048$
26. $\log 4.096$
27. $\log 1800$

Evaluate each expression.

28. $\log 98.2$
29. $\log 894.3$
30. antilog -0.0600
31. antilog 0.3012
32. $\log 1891.91$
33. antilog 0.33736

Find the value of each logarithm using the change of base formula.

34. $\log_2 8$ **35.** $\log_5 625$ **36.** $\log_6 24$

37. $\log_7 4$ **38.** $\log_{0.5} 0.0675$ **39.** $\log_{\frac{1}{2}} 15$

Solve each equation or inequality.

40. $2^x = 95$ **41.** $5^x = 4^{x+3}$ **42.** $\frac{1}{3}\log x = \log 8$

43. $0.16^{4+3x} = 0.3^{8-x}$ **44.** $4\log(x+3) = 9$ **45.** $0.25 = \log 16^x$

46. $3^{x-1} \leq 2^{x-7}$ **47.** $\log_x 6 > 1$ **48.** $4^{2x-5} \leq 3^{x-3}$

49. $0.5^{2x-4} \leq 0.1^{5-x}$ **50.** $\log_2 x = -3$ **51.** $x < \log_3 52.7$

Graphing Calculator

Graph each equation or inequality.

52. $y = 2\log(x+3)$ **53.** $y = \frac{1}{2}\log(x^3 - 1)$ **54.** $y \leq 5^{x+3}$

Solve each equation by graphing.

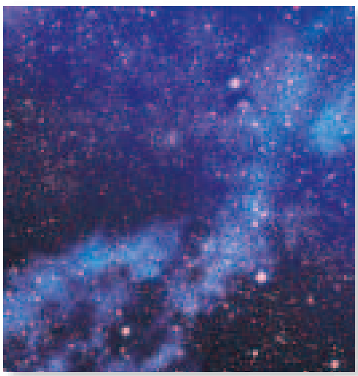
55. $8^{2x+1} = 30.4$ **56.** $9^{2x} - 2 = 8^x$ **57.** $4^{2x-1} = 2^{3x}$

Applications and Problem Solving

58. Aviation The altitude of an aircraft is in part affected by the outside air pressure and can be determined by the equation $h = -\frac{100}{9}\log\frac{P}{B}$, where h is the altitude in miles, P is the air pressure outside the aircraft, and B is the air pressure at sea level. Normally $B = 14.7$ pounds per square inch (psi).

a. Suppose the air pressure outside an airplane is 10.3. What is the altitude of the plane?

b. If a jet's altitude is 4.3 miles above sea level, what is the air pressure outside the jet?



59. Astronomy The parallax of a stellar object is the difference in direction of the object as seen from two distantly separated points. Astronomers use the parallax to determine an object's distance from Earth. The apparent magnitude of an object is its brightness as observed from Earth. Astronomers use parsecs, which stands for *par*allax of a *sec*ond, to measure distances in interstellar space. One parsec is about 3.26 light years or 19.2 trillion miles. The absolute magnitude is the magnitude an object would have if it were 10 parsecs from Earth. The greater the magnitude of an object the fainter it appears. For objects more than 30 parsecs, or 576 trillion miles from Earth, the formula relating the parallax p, the absolute magnitude M, and the apparent magnitude m, is $M = m + 5 + 5\log p$.

a. The object M35 in the constellation Gemini has an apparent magnitude of 5.3 and a parallax of about 0.018. Find the absolute magnitude of this object.

b. Stars with apparent magnitudes greater than 5 can only be seen with a telescope. If a star has an apparent magnitude of 8.6 and an absolute magnitude of 5.3, find its parallax to four decimal places.

60. Sales After t years, the annual sales in hundreds of thousands of units of a product q is given by $q = \left(\frac{1}{2}\right)^{0.8^t}$.

a. Find the annual sales of a product after 9 years.

b. After how many years will the annual sales be about 95,350 units?

61. Critical Thinking If $x = \log 372$, the value of x is between two consecutive integers. Name these two integers and explain how you determined the values.

62. Sound The loudness of a sound is often given on the decibel (dB) scale. The loudness of a sound L in dB is defined in terms of its intensity I by the equation $L = 10 \log \frac{I}{I_0}$ where I_0 is the minimum intensity audible to the average person (the "threshold of hearing"), 1.0×10^{-12} W/m^2.

a. A typical rock concert can have an intensity of 1 W/m^2. What is the loudness in decibels of such a concert?

b. A whisper has a loudness of 20 dB. What is the intensity of the whisper?

63. Archaeology Stonehenge is an ancient megalithic site in southern England. Charcoal samples taken from a series of pits at Stonehenge have about 630 micrograms of Carbon-14 in a 1-milligram sample. Assuming the half-life of Carbon-14 is 5730 years, what is the age of the charcoal pits at Stonehenge?

64. Critical Thinking If $\log_a y = \log_a p - \log_a q + \log_a r$, express y in terms of p, q, and r.

Mixed Review

65. Solve $\log_x 243 = 5$. *(Lesson 11-4)*

66. Use a graphing calculator to graph $y = 4e^x$. Then describe the interval in which it is increasing or decreasing. *(Lesson 11-3)*

67. Express $(a^4b^2)^{\frac{1}{3}}c^{\frac{2}{3}}$ using radicals. *(Lesson 11-1)*

68. Write the standard form of the equation of the circle that passes through points at (5, 0), (1, −2), and (4, −3). *(Lesson 10-2)*

69. Find the midpoint of the segment that has endpoints at $(-2\sqrt{5}, -4)$ and $(6\sqrt{5}, -18)$. *(Lesson 10-1)*

70. Write the polar equation $r = 6$ in rectangular form. *(Lesson 9-3)*

71. Graph $3r = 12$ on a polar plane. *(Lesson 9-1)*

72. Find an ordered pair that represents $\overrightarrow{AB}$ for $A(5, -6)$, and $B(6, -5)$. Then find the magnitude of $\overrightarrow{AB}$. *(Lesson 8-2)*

73. Geometry Find the area of a regular pentagon that is inscribed in a circle with a diameter of 7.3 centimeters. *(Lesson 5-4)*

74. Determine the binomial factors of $x^3 - 2x^2 - 11x + 12$. *(Lesson 4-3)*

75. Is the function $y = 5x^3 - 2x + 5$ odd, even, or neither? Explain. *(Lesson 3-1)*

76. SAT/ACT Practice According to the graph, the total annual nut consumption in California projected for 2000 is approximately how many pounds?

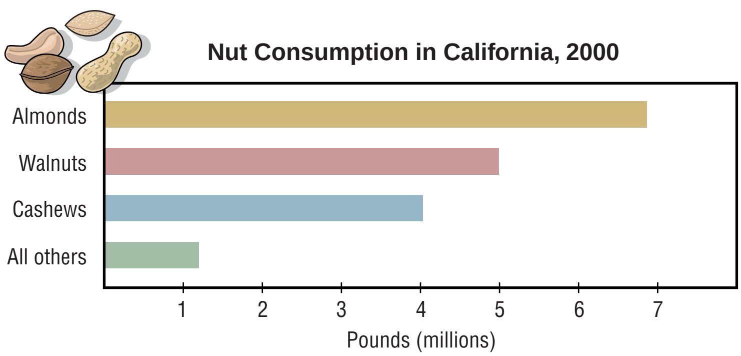
A 11,000,000

B 12,000,000

C 13,000,000

D 17,000,000

E 19,000,000



Extra Practice See p. A47.

11-6

Natural Logarithms

OBJECTIVES

- Find natural logarithms of numbers.
- Solve equations and inequalities using natural logarithms.
- Solve real-world applications with natural logarithmic functions.

Real World Application

SOCIOLOGY Rumors often spread quickly through a group of people as one person hears something then tells other people about it, who in turn tell more people. Sociologists studying a group of people determine that the fraction of people in the group p who have heard a rumor after t days can be represented by the equation $p = \frac{p_0 e^{kt}}{1 - p_0(1 - e^{kt})}$, where p_0 is the fraction of people who have heard the rumor at time $t = 0$, and k is a constant. *This information will be used in Example 1.*

Often when you solve problems involving the number e, it is best to use logarithms to the base e. These are called **natural logarithms** and are usually written **ln x**. In other words, $\log_e x = \ln x$. Since e is a positive number between 2 and 3, all of the properties of logarithms also hold for natural logarithms. Note that if $\ln e = x$ and $e^x = e$, then $x = 1$. Thus, $\ln e = 1$.

Example 1 Real World Application

SOCIOLOGY **Refer to the application above. In a particular study, sociologists determine that at time $t = 0$, 5% of the people in a group have heard a certain rumor. After 2 days, 25% of the people in the group had heard the rumor. How long will it take for 80% to hear the rumor?**

First, find the value of the constant k in the formula $p = \frac{p_0 e^{kt}}{1 - p_0(1 - e^{kt})}$. To do this, substitute the known values.

$$p = \frac{p_0 e^{kt}}{1 - p_0(1 - e^{kt})}$$

$$0.25 = \frac{0.05e^{2k}}{1 - 0.05(1 - e^{2k})} \qquad \textit{p = 25\% or 0.25, } p_0 \textit{ = 5\% or 0.05, t = 2}$$

$$0.2375 + 0.0125e^{2k} = 0.05e^{2k} \qquad \textit{Multiply each side by } 1 - 0.05(1 - e^{2k}).$$

$$0.2375 = 0.0375e^{2k} \qquad \textit{Subtract 0.0125 } e^{2k} \textit{ from each side.}$$

$$6.3333 = e^{2k} \qquad \textit{Divide each side by 0.0375}$$

$$\ln 6.3333 = \ln e^{2k} \qquad \textit{Take the natural logarithm of each side.}$$

$$\ln 6.3333 = 2k \ln e \qquad \ln a^n = n \ln a$$

$$1.8458 = 2k \qquad \textit{Use a calculator.}$$

$$0.9229 = k$$

Then, find the time when the fraction of the people in the group who have heard the rumor is 80%.

$$0.8 = \frac{0.05e^{0.9229t}}{1 - 0.05(1 - e^{0.9229t})} \qquad \textit{p = 80\% or 0.8. } p_0 \textit{ = 5\% or 0.05, k = 0.9229}$$

$$0.76 + 0.04e^{0.9229t} = 0.05e^{0.9229t} \qquad \textit{Multiply each side by } 1 - 0.05(1 - e^{0.9229t})$$

$$0.76 = 0.01e^{0.9229t} \qquad \textit{Subtract } 0.04e^{0.9229t} \textit{ from each side.}$$

$$76 = e^{0.9229t} \qquad \textit{Divide each side by 0.01}$$

(continued on the next page)

CONTENTS

$$\ln 76 = \ln e^{0.9229t}$$ *Take the natural logarithm of each side*

$$\ln 76 = 0.9229t \ln e$$ *$\ln a^n = n \ln a$*

$$4.3307 = 0.9229t$$ *Use a calculator.*

$$4.6925 = t$$

In about 4.7 days, 80% of the people in the group will hear the rumor.

Logarithms with a base other than e can be converted to natural logarithms using the change of base formula.

Example 2 **Convert $\log_6 254$ to a natural logarithm and evaluate.**

Use the change of base formula to convert the logarithm to a base e logarithm.

$$\log_6 254 = \frac{\log_e 254}{\log_e 6}$$ *$\log_a n = \frac{\log_b n}{\log_b a}$, $a = 6$, $b = e$, $n = 254$*

$$= \frac{\ln 254}{\ln 6}$$ *$\log_e x = \ln x$*

$$\approx 3.0904$$ *Use a calculator.*

So, $\log_6 254$ is about 3.0904.

Sometimes you know the natural logarithm of a number x and must find x. The antilogarithm of a natural logarithm is written **antiln x**. If $\ln x = a$, then $x = \text{antiln } a$. Use the e^x function, which is the [2nd] function of [LN] to find the antilogarithm of a natural logarithm. *(antiln x is often written as e^x)*

Examples 3 **Solve $6.5 = -16.25 \ln x$.**

$$6.5 = -16.25 \ln x$$

$$-0.4 = \ln x$$ *Divide each side by −16.25.*

$$\text{antiln } (-0.4) = x$$ *Take the antilogarithm of each side.*

$$0.67 \approx x$$ *Use a calculator.*

4 **Solve each equation or inequality by using natural logarithms.**

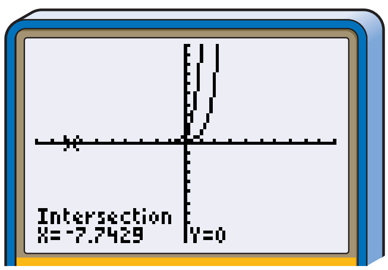
a. $3^{2x} = 7^{x-1}$

Method 1: Algebra

$$3^{2x} = 7^{x-1}$$
$$\ln 3^{2x} = \ln 7^{x-1}$$
$$2x \ln 3 = (x-1)\ln 7$$
$$2x(1.0986) = (x-1)(1.9459)$$
$$2.1972x = 1.9459x - 1.9459$$
$$0.2513x = -1.9459$$
$$x = -7.7433$$

Method 2: Graphing

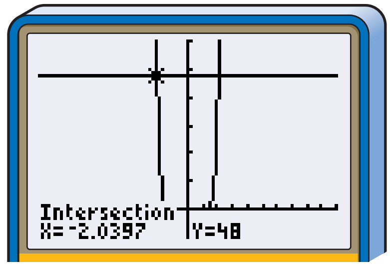
$y_1 = 3^{2x}$, $y_2 = 7^{x-1}$



[−10, 10] scl:1 by [−10, 10] scl:1

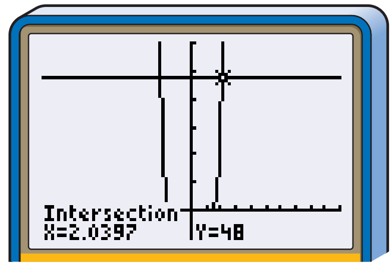
b. $6^{x^2 - 2} < 48$

$$\begin{aligned} 6^{x^2 - 2} &< 48 \\ \ln 6^{x^2 - 2} &< \ln 48 \\ (x^2 - 2)\ln 6 &< \ln 48 \\ x^2 - 2 &< \frac{\ln 48}{\ln 6} \\ x^2 - 2 &< 2.1606 \\ x^2 &< 4.1606 \end{aligned}$$



[−10, 10] scl:1 by [−10, 60] scl:10

Notice that there are two intersection points, about −2.04 and 2.04. Since $6^{x^2 - 2}$ is less than 48, the solution is all points between $x = -2.04$ and $x = 2.04$. From this, we can write the inequalities $x > -2.04$ and $x < 2.04$, or $-2.04 < x < 2.04$.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why $\ln e = 1$.
2. **Show** that $\log 17 \neq \ln 17$.
3. **Verify** that $\ln 64 = \ln 16 + \ln 4 = 2 \ln 8$.
4. *Math Journal* **Describe** the similarities between the equation $A = Pe^{rt}$ and the equation $N = N_0 e^{kt}$. How are the two equations alike?

Guided Practice

Evaluate each expression.

5. $\ln 0.0089$
6. $\ln \frac{1}{0.32}$
7. $\ln 0.21$
8. antiln (-0.7831)

Convert each logarithm to a natural logarithm and evaluate.

9. $\log_5 132$
10. $\log_3 64$

Use natural logarithms to solve each equation or inequality.

11. $18 = e^{3x}$
12. $10 = 5e^{5k}$
13. $25e^x < 100$
14. $4.5 \geq e^{0.031t}$

Graphing Calculator

Solve each equation or inequality by graphing. Round solutions to the nearest hundredth.

15. $6^{x+2} = 14^{x-3}$
16. $5^{x+3} > 10^{x-6}$

17. **Meteorology** The atmospheric pressure varies with the altitude above the surface of Earth. Meteorologists have determined that for altitudes up to 10 kilometers, the pressure p in millimeters of mercury or torrs, is given by $p = 760e^{-0.125a}$, where a is the altitude in kilometers.
    a. What is the atmospheric pressure at 3.3 kilometers?
    b. At what altitude will the atmospheric pressure be 450 torrs?

EXERCISES

Practice

Evaluate each expression.

18. $\ln 243$ **19.** $\ln 0.763$ **20.** $\ln 980$ **21.** $\ln 125^{\frac{1}{5}}$

22. $\ln 1$ **23.** $\ln 9.32$ **24.** antiln 2.3456 **25.** antiln 0.1934

26. antiln (-3.7612) **27.** antiln (-0.0034) **28.** antiln 4.987 **29.** antiln (-1.42)

Convert each logarithm to a natural logarithm and evaluate.

30. $\log_{12} 56$ **31.** $\log_5 36$ **32.** $\log_4 83$

33. $\log_8 0.512$ **34.** $\log_6 323$ **35.** $\log_5 \sqrt{288}$

Use natural logarithms to solve each equation or inequality.

36. $6^x = 72$ **37.** $2^x = 27$ **38.** $9^{x-4} = 7.13$ **39.** $3^x = 3\sqrt{2}$

40. $25e^x = 1000$ **41.** $60.3 < e^{0.1t}$ **42.** $6.2e^{0.64t} = 3e^{t+1}$ **43.** $22 = 44(1 - e^{2x})$

44. $25 < e^{0.075y}$ **45.** $5^x \leq 7\sqrt{6}$ **46.** $12^{x-4} > 4^x$ **47.** $x^{\frac{2}{3}} \geq 27.6$

Graphing Calculator

Solve each equation or inequality by graphing. Round solutions to the nearest hundredth.

48. $15e^{-x} = 645$ **49.** $2^x = \sqrt{3^{x-2}}$ **50.** $4e^{0.045t} < 1600$

51. $x = \log_4 19.5$ **52.** $10^{x-3} \geq 52$ **53.** $\log_3 \sqrt[4]{5} \leq x$

Applications and Problem Solving

54. Electricity If a charged capacitor is placed in a circuit where the only other component is a resistor, then the charge will flow out of the capacitor. The charge Q on the capacitor after t seconds is given by $Q = Q_0 e^{-\frac{t}{RC}}$, where R is the resistance, C is the capacitance, and Q_0 is the initial charge on the capacitor. In a circuit with $R = 20{,}000$ ohms and $C = 4 \times 10^{-11}$ farads, how long will it take the capacitor to lose 40% of its initial charge?

55. Medicine Nuclear medicine technologists use the iodine isotope I-131, with a half-life of 8 days, to check thyroid function of patients. After ingesting a tablet containing the iodine, the isotopes collect in a patient's thyroid, and a special camera is used to view its function. Suppose a patient ingests a tablet containing 9 microcuries of I-131. To the nearest hour, how long will it be until there are only 2.8 microcuries in the patient's thyroid? *A microcurie is a measure of radiation.*

56. Temperature According to Newton's Law of Cooling, the temperature T of an object at time t satisfies the equation $\ln|T - T_0| = -kt + C$ where T_0 is the temperature of the surrounding medium and k and C are constants. Suppose a cup of coffee with temperature of 180° F is placed in a 72° F room at time $t = 0$.

a. Find the value of C to the nearest ten thousandth.

b. After 2 minutes, the temperature of the coffee has dropped to 150° F. Find the value of k to the nearest ten thousandth.

c. How much longer will it take for the temperature of the coffee to drop to 100° F?

57. Critical Thinking Solve the equation $e^{-2x} - 4e^{-x} + 3 = 0$ without graphing.

58. Finance Jenny Oiler deposited some money in an investment account that earns 6.3% interest compounded continuously.

a. How long will it take to double the money in the account?

b. The Rule of 72 states that if you divide 72 by the interest rate of an account that compounds interest continuously, the result is the approximate number of years that it will take for the money in the account to double. Do you think the Rule of 72 is accurate? Explain your reasoning.

59. Linguistics If two languages have evolved separately from a common ancestral language, the number of years since the split $n(r)$, is modeled by the formula $n(r) = -5000 \ln r$, where r is the percent of the words from the ancestral language that are common to both languages now. If two languages split off from a common ancestral language about 1800 years ago, what portion of the words from the ancestral language would you expect to find in both languages today?

60. Chemistry Radium 226, which had been used for cancer treatment, decays with a half-life of 1622 years.

a. Find the constant k using 1 gram as the original amount of Radium 226.

b. Suppose there is a 2.3-gram sample of Radium 226. How long will it take to have 1.7 grams remaining?

61. Critical Thinking Using the table below, determine if y is a logarithmic function of x or if x is a logarithmic function of y. Explain your answer.

x	1	3	9	27
y	0	1	2	3

Mixed Review

62. Use a calculator to find the common logarithm of 19.25 to the nearest ten thousandth. *(Lesson 11-5)*

63. Write $\log_{16} 8 = \frac{3}{4}$ in exponential form. *(Lesson 11-4)*

64. Solve the system $x^2 = y + 4$, $x^2 + 4y^2 = 8$, algebraically. Round to the nearest tenth. Check the solutions by graphing each system. *(Lesson 10-8)*

65. Chemistry Boyle's Law states that the pressure P exerted by a gas varies inversely as the volume V of the gas at constant temperature. If a specific gas is collected in a 146 cubic centimeter container and the pressure is 52.4 pascals (1 pascal $= 1\ \text{N/m}^2$), use the formula $PV = c$ to find c. *(Lesson 10-4)*

66. Find the rectangular coordinates of the point with polar coordinates $(0.25, \pi)$. *(Lesson 9-3)*

67. Find an ordered pair to represent $\vec{\mathbf{a}}$ in the equation $\vec{\mathbf{a}} = \vec{\mathbf{b}} + 3\vec{\mathbf{c}}$ if $\vec{\mathbf{b}} = \langle 1, -2 \rangle$ and $\vec{\mathbf{c}} = \langle 4, 3 \rangle$. *(Lesson 8-2)*

68. Change $2x - 5y + 3 = 0$ to normal form. Then find p, the measure of the normal, and ϕ, the angle it makes with the positive x-axis. *(Lesson 7-6)*

69. Write an equation of a cosine function with amplitude 70 and period 90°. *(Lesson 6-4)*

70. SAT Practice **Grid-In** Doreen sets out on an 800-mile car trip. Her car can travel for 10 hours at 55 miles per hour on a full tank of gas. If she starts with a full tank of gas and stops only once to refuel, what is the least number of miles Doreen would have remaining after stopping to refuel?

Extra Practice See p. A47.

GRAPHING CALCULATOR EXPLORATION

11-6B Natural Logarithms and Area

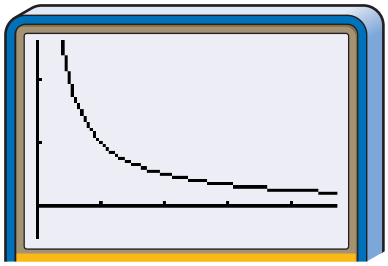
An Extension of Lesson 11-6

OBJECTIVES

- Investigate the relationship between area of regions below the graph of $y = \frac{1}{x}$ and natural logarithms.

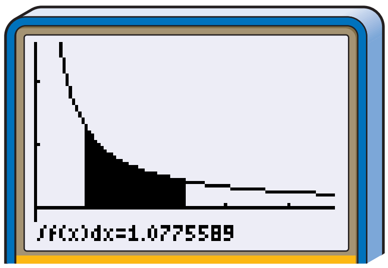
There is an interesting and important relationship between the function $y = \ln x$ and the areas of regions below the graph of $y = \frac{1}{x}$. (You will learn more about areas under graphs or curves in Chapter 15.) You can explore this relationship with a graphing calculator by using the **∫ f(x) dx** feature under the **CALC** menu. Using this feature, you can have the calculator display the area of the region between the graph of a function, the x-axis, and a pair of vertical lines at $x = a$ and $x = b$, where a and b are any real numbers, called the lower and upper boundaries.

Graph the equation $y = \frac{1}{x}$ and select the **∫ f(x) dx** feature under the **CALC** menu. Move the cursor to the left until it is at the point where $x = 0.8$. Press ENTER to set 0.8 as the lower boundary for x. Then move the cursor to the right and set 2.35 as the upper boundary for x.



[0, 4.7] scl:1, [−0.5, 2.6] scl:1

The calculator automatically shades the region between the graph, the x-axis, and the lines $x = 0.8$ and $x = 2.35$. At the bottom of the screen, the calculator displays the area of the shaded region, which is approximately 1.0775589 square units.



TRY THESE

1. Use the procedure described above to find the area of the region between the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 2$ and $x = 4$. Use $x = 2$ as the lower boundary and $x = 4$ as the upper boundary. What area does the calculator display for the shaded region?

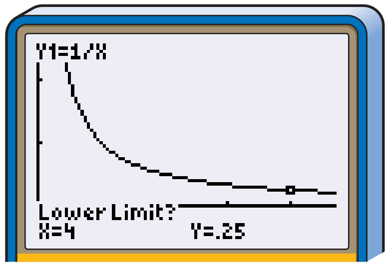
2. Press 2nd **[QUIT]** to go to the home screen. Then press 2nd **[ANS]** ENTER. What number does the calculator display? How is this number related to the area you found in Exercise 1?

3. Press GRAPH to redisplay the graph. Clear the shading from the graph using the **ClrDraw** command on the **DRAW** menu to display a fresh graph.

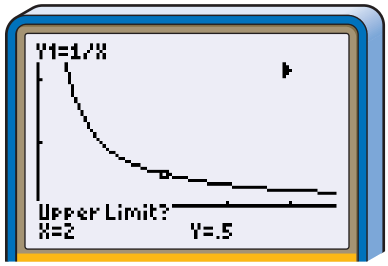
Repeat Exercise 1 setting $x = 4$ as the *lower* limit and $x = 2$ as the *upper* limit.

a. Describe how this result is related to the answer for Exercise 1.

b. What kind of result do you think the calculator will display when the value of the lower limit is greater than the value of the upper limit?



[0, 4.7] scl:1, [−0.5, 2.6] scl:1



4. Graph $y = \frac{1}{x}$.

a. Set $x = 1$ as the lower boundary and $x = 2$ as the upper boundary. Record the displayed area.

b. Set $x = 1$ as the lower boundary and $x = 3$ as the upper boundary. Record the displayed area.

c. Set $x = 1$ as the lower boundary and $x = 4$ as the upper boundary. Record the displayed area.

d. Use your calculator to find the values of ln 2, ln 3, and ln 4.

e. Compare each area with each natural logarithm.

5. Find the areas of the regions under the graph of $y = \frac{1}{x}$ that result when you use $x = 1$ as the lower limit and $x = 0.6$, 0.5, and 0.4 as upper limits. How do the values compare with ln 0.6, ln 0.5, and ln 0.4?

WHAT DO YOU THINK?

6. Consider the region between the graph of $y = \frac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = k$, where $k > 0$. Make a conjecture about how the area of this region is related to ln k.

7. Suppose you enter the areas you found in Exercises 4 and 5 as the elements of list **L1** and the corresponding upper-boundary values for x in list **L2**. If you use the **ExpReg** on the **STAT CALC** menu to display an exponential regression equation of the form $y = ab^x$, what values would you expect the calculator to display for the constants a and b? Enter the numbers for lists **L1** and **L2** and display the regression equation to see if your prediction is correct.

8. One way to approach the concept of natural logarithms is to define e as the limiting value of $\left(1 + \frac{1}{n}\right)^n$ as n approaches ∞ and then define natural logarithms as logarithms to the base e. Make a conjecture of another way to define natural logarithms and e based on the ideas you have explored in the preceding exercises.

11-7

Modeling Real-World Data with Exponential and Logarithmic Functions

OBJECTIVES

- Find the doubling time of an exponential quantity.
- Find exponential and logarithmic functions to model real-world data.
- Linearize data.

Real World Application

INVESTMENT Latrell, a freshman at Finneytown High School, is given a gift of $4000 by his great aunt. He would like to invest the money so that he will have enough to purchase a car that costs twice that amount, or $8000, when he graduates in four years. If he invests the $4000 in an account that pays 9.5% compounded continuously, will he have enough money to buy the car? *This problem will be solved in Example 1.*

There are situations, such as investments and population, where it is of interest to know how long it takes a quantity modeled by an exponential function to reach twice its initial amount. This amount of time is called the **doubling time** of the quantity. We can derive a formula for doubling time from the exponential equation $N = N_0e^{kt}$. If t is the doubling time, then $N = 2N_0$.

$N = N_0e^{kt}$

$2N_0 = N_0e^{kt}$ *Replace N with $2N_0$.*

$2 = e^{kt}$ *Divide each side by N_0.*

$\ln 2 = \ln e^{kt}$ *Take the natural logarithm of each side.*

$\ln 2 = kt$ *$\ln e^x = x$*

$\frac{\ln 2}{k} = t$ *Divide each side by k to solve for t.*

Doubling Time

The amount of time t required for a quantity modeled by the exponential equation $N = N_0e^{kt}$ to double is given by the following equation.

$$t = \frac{\ln 2}{k}$$

Example 1 Real World Application

INVESTMENT **Refer to the application at the beginning of the lesson.**

a. Will Latrell have enough money after 4 years to buy the car?

Find the doubling time for Latrell's investment. For continuously compounded interest, the constant k is the interest rate, written as a decimal.

$t = \frac{\ln 2}{k}$

$= \frac{\ln 2}{0.095}$ *The decimal for 9.5% is 0.095.*

≈ 7.30 years *Use a calculator.*

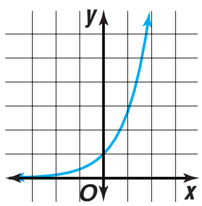
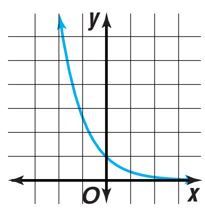
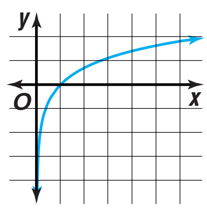
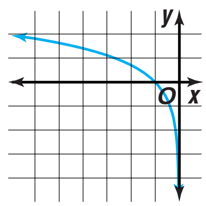
Four years is not enough time for Latrell's money to double. He will not have enough money to buy the car.

b. What interest rate is required for an investment with continuously compounded interest to double in 4 years?

$$t = \frac{\ln 2}{k}$$
$$4 = \frac{\ln 2}{k}$$
$$k = \frac{\ln 2}{4} \quad \textit{Solve for k.}$$
$$k \approx 0.1733$$

An interest rate of 17.33% is required for an investment with continuously compounded interest to double in 4 years.

In Lesson 1-6, you learned how to fit a linear function to a set of data. Sometimes the scatter plot of a set of data may suggest a nonlinear model. The process of fitting an equation to nonlinear data is called **nonlinear regression.** In this lesson, we will use a graphing calculator to perform two types of nonlinear regression, exponential regression and logarithmic regression. When deciding which type of model is more appropriate, you will need to keep in mind the basic shapes of exponential and logarithmic functions.

Exponential Functions: $y = ab^x$		**Logarithmic Functions: $y = a + b \ln x$**	
Growth	**Decay**	**Growth**	**Decay**
			

Examples

Real World Application

interNET CONNECTION

Data Update For the latest data on population and other applications studied in this lesson, visit: **www.amc.glencoe.com**

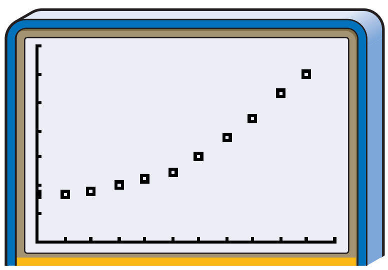
2 **POPULATION** **The table below gives the population of the world in billions for selected years during the 1900s.**

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Years since 1900	0	10	20	30	40	50	60	70	80	90	100
Population	1.65	1.75	1.86	2.07	2.30	2.52	3.02	3.70	4.44	5.27	6.06

Source: United Nations

a. Find a function that models the population data shown.

Enter the data on the **STAT EDIT** screen. In order to work with smaller numbers, use the number of years since 1900 as the independent variable. Then draw a scatter plot. The scatter plot suggests an exponential model.



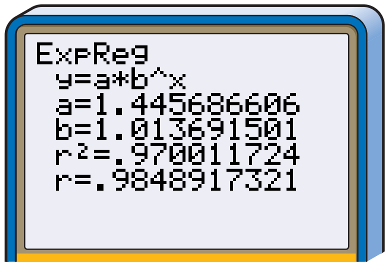
[0, 100] scl:1 10 by [0, 6] scl:1

(continued on the next page)

Graphing Calculator Tip

Select **DiagnosticOn** from the **CATALOG** to display r and r^2.

To have the calculator find a regression equation of the form $y = ab^x$, use **ExpReg** from the **STAT CALC** screen.



The equation of an exponential function that models the population data is $y = 1.4457(1.0137)^x$.

The equation can be written in terms of base e as follows.

$y = 1.4457(1.0137)^x$

$y = 1.4457(e^{\ln 1.0137})^x$ $e^{\ln a} = a$

$y = 1.4457e^{(\ln 1.0137)x}$ $(a^m)^n = a^{mn}$

$y = 1.4457e^{0.0136x}$ $\ln 1.0137 \approx 0.0136$

b. Use the equation to predict the population of the world in 2050.

To predict the population in 2050, observe that 2050 is 150 years after 1900.

$y = 1.4457(1.0137)^x$

$= 1.4457(1.0137)^{150}$ *Replace x with 150.*

≈ 11.13

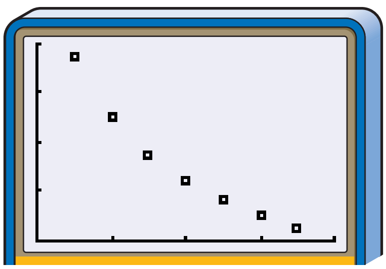
The model predicts that the population of the world in 2050 will be 11.13 billion.

3 **SKATING** **An ice skater begins to coast with an initial velocity of 4 meters per second. The table below gives the times required for the skater to slow down to various velocities. Find an equation that models the data.**

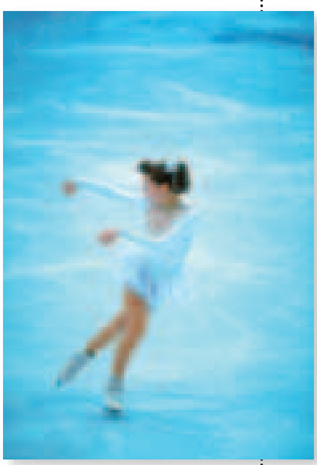
Real World Application

velocity (m/s)	3.5	3	2.5	2	1.5	1	0.5
time (s)	2.40	5.18	8.46	12.48	17.66	24.95	37.43

Make a scatter plot of the data. Let the velocity be the independent variable and let time be the dependent variable.

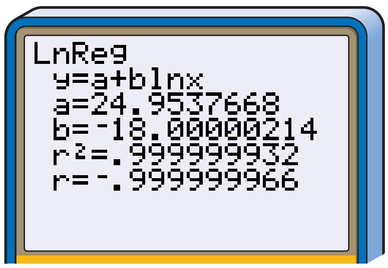


[0, 4] scl:1 by [0, 40] scl:10



Since the scatter plot does not appear to flatten out as much as a decaying exponential function would, we will look for a logarithmic model of the form $y = a + b \ln x$ where $b < 0$ for the data.

To have the calculator find a regression equation of the form $y = a + b \ln x$, use **LnReg** from the **STAT CALC** screen.



The equation of a logarithmic function that models the data is $y = 24.95 - 18 \ln x$.

Another approach to modeling curved data is the concept of **linearizing data.** Suppose a set of data can be modeled by the exponential function $y = ae^{bx}$. We can rewrite this equation using the properties of logarithms.

$y = ae^{bx}$	
$\ln y = \ln ae^{bx}$	*Take the natural logarithm of each side.*
$\ln y = \ln a + \ln e^{bx}$	*Product property of logarithms*
$\ln y = \ln a + bx$	*$\ln e^x = x$*

The last equation shows that $\ln y$ is a linear function of x. In other words, if the ordered pairs $(x, \ln y)$ are graphed, the result will be a straight line. This means that exponential data can be analyzed by linearizing the data and then applying linear regression.

Example 4 Real World Application

ECONOMICS **The Consumer Price Index (CPI) measures inflation. It is based on the average prices of goods and services in the United States, with the average for the years 1982–1984 set at an index of 100. The table below gives some CPI values from 1950 to 1996.**

Year	1950	1960	1970	1980	1990	1996
CPI	24.1	29.6	38.8	82.4	130.7	156.9

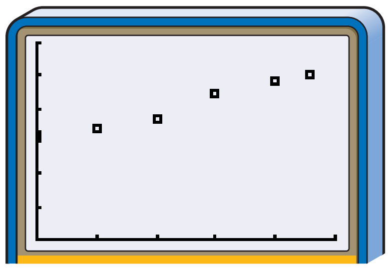
Source: Bureau of Labor Statistics

a. Linearize the data. That is, make a table with x- and $\ln y$-values, where x is the number of years since 1950 and y is the CPI. Then make a scatter plot of the linearized data.

Subtract 1950 from each year and find the natural logarithm of each CPI-value.

x	0	10	20	30	40	46
$\ln y$	3.18	3.39	3.66	4.41	4.87	5.06

The scatter plot suggests that there may be a linear relationship between x and $\ln y$.

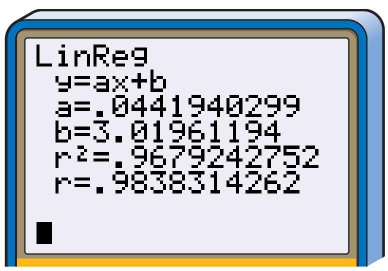


[0, 50] scl:10 by [0, 6] scl:1

b. Find a regression equation for the linearized data.

Use **LinReg(ax+b)** on the **STAT CALC** screen to find the linear regression equation.

We can use the equation $\ln y = 0.0442x + 3.0196$ to model $\ln y$.



c. Use the linear regression equation to find an exponential model for the original data.

To find a model solve the regression equation in part **b** for y.

$\ln y = 0.0442x + 3.0196$

$e^{\ln y} = e^{0.0442x + 3.0196}$ *Raise e to each side.*

$y = e^{0.0442x + 3.0196}$ $e^{\ln y} = y$

$y = e^{0.0442x} \cdot e^{3.0196}$ *Product Property of Exponents*

$y = 20.48e^{0.0442x}$ $e^{3.0196} \approx 20.48$

The Consumer Price Index between 1950 and 1996 can be modeled by the exponential function $y = 20.48e^{0.0442x}$.

d. Use the exponential model to predict the CPI in 2020.

The year 2020 is 70 years after 1950, so replace x with 70 in the exponential function.

$y = 20.48e^{0.0442x}$

$= 20.48e^{0.0442(70)}$ or about 451.9

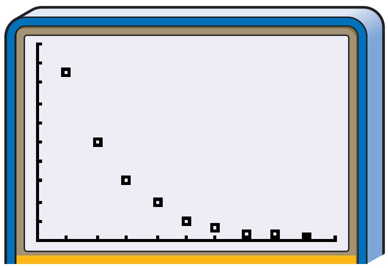
The model predicts that in 2020 the CPI will be 451.9.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how you would determine the time needed to quadruple an investment if the interest is compounded continuously.
2. **State** whether the data in the scatter plot should be modeled with an exponential function or a logarithmic function. Explain.
3. **Write** the function $y = 2 \cdot 4^x$ as an exponential function with base e. Then write $\ln y$ as a linear function of x.



Guided Practice

Find the amount of time required for an amount to double at the given rate if the interest is compounded continuously.

4. 1.75%

5. 8%

6. **Radioactivity** Exponential regression can be used in the experimental determination of the half-life of a radioactive element. A scientist starts with a 10-gram sample of uranium-239 and records the measurements shown below.

Time (min)	0	5	10	15	20
U-239 present (g)	10	8.6	7.5	6.3	5.5

a. Find a regression equation for the amount y of uranium as a function of time x.

b. Write the regression equation in terms of base e.

c. Use the equation from part b to estimate the half-life of uranium-239.

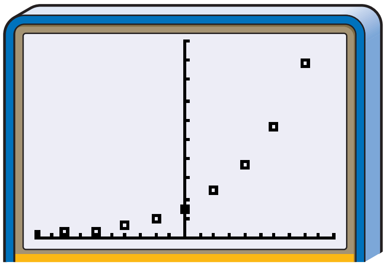
EXERCISES

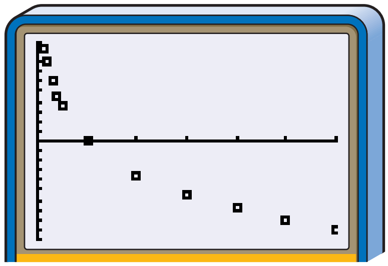
Practice

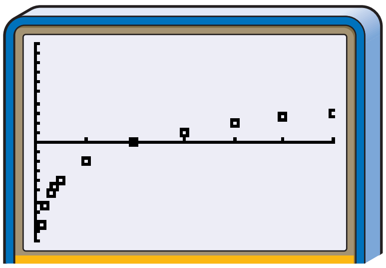
Find the amount of time required for an amount to double at the given rate if the interest is compounded continuously.

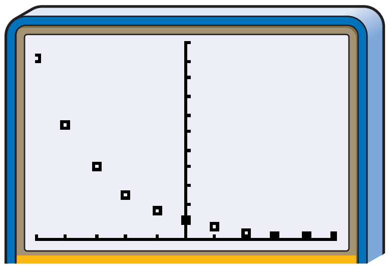
7. 2.25% **8.** 5% **9.** 7.125%

Determine whether the data in each scatter plot should be modeled with an exponential function or a logarithmic function. Explain your choices.

10.


11.


12.


13.


Applications and Problem Solving

Real World Application

14. Biology The data below give the number of bacteria, in millions, found in a certain culture.

Time (hours)	0	1	2	3	4
Bacteria	5	8	15	26	48

a. Find an exponential function that models the data.

b. Write the equation from part **a** in terms of base e.

c. Use the model to estimate the doubling time for the culture.

15. Radioactivity A scientist starts with a 1-gram sample of lead-211. The amount of the sample remaining after various times is shown in the table below.

Time (min)	10	20	30	40
Pb-211 present (g)	0.83	0.68	0.56	0.46

a. Find a regression equation for the amount y of lead as a function of time x.

b. Write the regression equation in terms of base e.

c. Use the equation from part **b** to estimate the half-life of lead-211.

16. **Banking** Elayna found an old savings passbook in her grandparents' attic. It contained the following account balances.

Date	Balance
Jan. 1, 1955	2137.52
Jan. 1, 1956	2251.61
Jan. 1, 1957	2371.79
Jan. 1, 1958	2498.39
Jan. 1, 1959	2631.74

a. Find a function that models the amount in the account. Use the number of years after Jan. 1, 1955 as the independent variable.

b. Write the equation from part **a** in terms of base e.

c. What was the interest rate on the account if the interest was compounded continuously and no deposits or withdrawals were made during the period in question?

17. **Sound** The human impression of the strength of a sound depends, among other things, on the frequency of the sound. The loudness level of a sound, in *phons,* is the decibel level of an equally loud 1000 Hz tone. The *sone* is another unit of comparative loudness. The table below lists some equivalent values of phons and sones. Find an equation that gives the phon level of a sound in terms of the number of sones.

sones	0.5	1	2	4
phons	30	40	50	60

18. **Education** The table below lists the number of bachelor's degrees granted in the United States for certain years.

Year	1960	1965	1970	1975	1980	1985	1990	1994
Degrees (1000s)	392	494	792	923	930	980	1052	1169

Source: U.S. National Center for Education Statistics

a. Find a function that models the data. Let x be the number of years after 1950 and let y be the number of degrees granted, in thousands.

b. Why can x not be the number of years since 1960?

19. **Critical Thinking** Consider the equation $y = cx^2$, where c is a constant and $x \geq 0$. What operation could be applied to each side of the equation to obtain a linear function of x?

20. **Banking** Tomasita deposited \$1000 in an account at the beginning of 2001. The account had a 3% interest rate, compounded quarterly. The table below shows her balance after each quarter of the year.

Date	Jan. 1	Mar. 31	June 30	Sept. 30	Dec. 31
Balance	\$1000	1007.50	1015.06	1022.67	1030.34

a. In an account where the interest is only compounded once per year, the balance after x years is given by $P(1 + r)^x$, where P is the amount invested and r is the interest rate as a decimal. Find the interest rate with an annual compounding that would result in the balance that Tomasita had at the end of one year. This interest rate is called the *effective rate*.

b. Find a function that models the amount in Tomasita's account. Let x be the number of years since the beginning of 2001.

c. Write the equation from part **b** in terms of base e.

d. What interest rate with continuous compounding would result in the balances shown for Tomasita's account?

21. **Population Density** The table below gives the population density (persons per square mile) of the United States at various times in its history.

Year	1800	1850	1900	1950	1990
Population Density	6.1	7.9	25.6	42.6	70.3

Source: U. S. Bureau of the Census

a. Let x be the number of years since 1800 and let y be the population density in persons per square mile. Copy and complete the table below to linearize the data.

x	0	50			
$\ln y$					

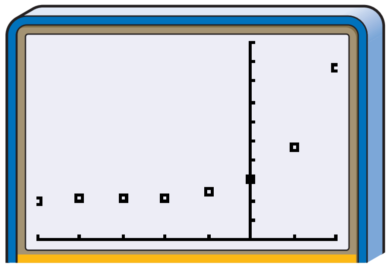
b. Find a regression equation for the linearized data.

c. Solve the regression equation for y.

d. Use the answer to part **c** to predict the population density of the United States in 2025.

22. **Critical Thinking** The graphing calculator screen shows a scatter plot of the data in the table below.

x	y	x	y
−5	2.01	−1	2.38
−4	2.02	0	3
−3	2.06	1	4.6
−2	2.15	2	8.76



a. What must you do to the data before you can use a graphing calculator to perform exponential regression? Explain.

b. Find a regression equation for the data.

23. **Critical Thinking** A *power function* is a function of the form $y = c \cdot x^a$, where c and a are constants.

a. What is the relationship between $\ln y$ and x for a power function?

b. The distance that a person can see from an airplane depends on the altitude of the airplane, as shown in the table below.

Altitude (m)	500	1000	5000	10,000	15,000
Viewing Distance (km)	89	126	283	400	490

Let x be the altitude of the airplane in meters and let y be the viewing distance in kilometers. Linearize the data.

c. Find a linear regression equation for the linearized data.

d. Solve the linear regression equation for y to obtain a power function that gives y in terms of x.

Mixed Review

24. **Biology** Under ideal conditions, the population of a certain bacterial colony will double in 85 minutes. How much time will it take for the population to increase 12 times? *(Lesson 11-6)*

25. Use a calculator to find the antilogarithm of -2.1654 to the nearest hundredth. *(Lesson 11-5)*

26. Solve $\log_5(7x) = \log_5(5x + 16)$. *(Lesson 11-4)*

27. **Entertainment** A theater has been staging children's plays during the summer. The average attendance at each performance is 400 people, and the cost of a ticket is \$3. Next summer, the theater manager would like to increase the cost of the tickets, while maximizing profits. The manager estimates that for every \$1 increase in ticket price, the attendance at each performance will decrease by 20. *(Lesson 10-5)*

a. What price should the manager propose to maximize profits?

b. What maximum profit might be expected?

28. Express $5\left(\cos \frac{\pi}{6} + \boldsymbol{i} \sin \frac{\pi}{6}\right)$ in rectangular form. *(Lesson 9-6)*

29. **Surveying** A building 60 feet tall is on top of a hill. A surveyor is at a point on the hill and observes that the angle of elevation to the top of the building measures $42°$ and the angle of elevation to the bottom measures $18°$. How far is the surveyor from the bottom of the building? *(Lesson 5-6)*

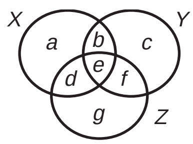
30. Find the discriminant of $5x^2 - 8x + 12 = 0$. Describe the nature of its roots. Solve the equation. *(Lesson 4-2)*

31. Describe the transformation(s) of the parent graph of $f(x) = x^2$ that are required to graph $f(x) = (x + 4)^2 - 8$. *(Lesson 3-2)*

32. Find the maximum and minimum values of the function $f(x) = 2x + 8y + 10$ for the polygonal convex set determined by the system of inequalities. *(Lesson 2-6)*

$x \geq 2$
$x \leq 4$
$y \geq 1$
$x - 2y \leq -4$

33. **SAT Practice** In the figure, circles X, Y, and Z overlap to form regions a through g. How many of these regions are contained in either of the circles X or Z?



A 2

B 5

C 6

D 7

E None of the above

Extra Practice See p. A47.

CONTENTS

VOCABULARY

Modeling

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term to best complete each sentence.

1. A logarithm with base 10 is called a(n) __?__.
2. A real-world situation that involves a quantity that increases exponentially over time exhibits __?__.
3. The inverse of an exponential function is a(n) __?__.
4. A number is in __?__ when it is in the form $a \times 10^n$ where $1 \leq a < 10$ and n is an integer.
5. A common logarithm is made up of two parts, the characteristic and the __?__.
6. A logarithm to the base e is called a __?__.
7. Exponential data can be analyzed by __?__ and then applying linear regression.
8. A function of the form $y = b^x$, where b is a positive real number and the exponent is a variable, is a(n) __?__.
9. The process of fitting an equation to nonlinear data is called __?__.
10. A(n) __?__ can be solved using logarithms.

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 11-1 Evaluate and simplify expressions containing rational exponents.

- Evaluate $16^{\frac{1}{4}}$.

$16^{\frac{1}{4}} = (2^4)^{\frac{1}{4}}$
$= 2^{\frac{4}{4}}$ *Power of a power*
$= 2$

Simplify $(6c^4d)^2(3c)^{-2}$.

$(6c^4d)^2(3c)^{-2} = \dfrac{36c^8d^2}{9c^2}$
$= 4c^{8-2}d^2$ *Quotient Property*
$= 4c^6d^2$

REVIEW EXERCISES

Evaluate each expression.

11. $\left(\frac{1}{4}\right)^{-2}$ **12.** $64^{\frac{1}{2}}$

13. $27^{\frac{4}{3}}$ **14.** $(\sqrt[4]{256})^3$

Simplify each expression.

15. $3x^2(3x)^{-2}$ **16.** $(6a^{\frac{1}{3}})^3$

17. $\left(\frac{1}{2}x^4\right)^3$ **18.** $(w^3)^4 \cdot (4w^2)^2$

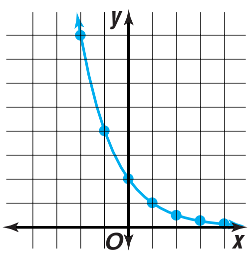
19. $\left((2a)^{\frac{1}{3}}(a^2b)^{\frac{1}{3}}\right)^3$ **20.** $(3x^{\frac{1}{2}}y^{\frac{1}{4}})(4x^2y^2)$

Lesson 11-2 Graph exponential functions and inequalities.

- Graph $y = \left(\frac{1}{2}\right)^{x-1}$.

Plot the points shown in the table below. Connect the points to graph the function.

x	$\left(\frac{1}{2}\right)^{x-1}$
−2	8
−1	4
0	2
1	1
2	0.5
3	0.25
4	0.125



Graph each exponential function or inequality.

21. $y = 3^{-x}$

22. $y = \left(\frac{1}{2}\right)^x$

23. $y = 2^{x-1}$

24. $y = 2^x + 2$

25. $y \geq -2^x + 1$

26. $y < 2^{x+2}$

Lesson 11-3 Use the exponential function $y = e^x$.

- Find the balance after 15 years of a \$5000 investment earning 5.8% interest compounded continuously.

$A = Pe^{rt}$
$A = 5000(e)^{(0.058 \cdot 15)}$
$A \approx \$11,934.55$

Find the balance for each account after 10 years if the interest is compounded continuously.

27. \$2500 invested at 6.5%

28. \$6000 invested at 7.25%

29. \$12,000 invested at 5.9%

OBJECTIVES AND EXAMPLES

Lesson 11-4 Evaluate expressions involving logarithms.

Evaluate the expression $\log_5 25$.

Let $x = \log_5 25$.

$x = \log_5 25$
$5^x = 25$ *Definition of logarithm*
$5^x = 5^2$
$x = 2$

REVIEW EXERCISES

Write each equation in exponential form.

30. $\log_8 4 = \frac{2}{3}$ **31.** $\log_3 \frac{1}{81} = -4$

Write each equation in logarithmic form.

32. $2^4 = 16$ **33.** $5^{-2} = \frac{1}{25}$

Evaluate each expression.

34. $\log_2 32$ **35.** $\log_{10} 0.001$
36. $\log_4 \frac{1}{16}$ **37.** $\log_2 0.5$
38. $\log_6 216$ **39.** $\log_9 \frac{1}{9}$
40. $\log_4 1024$ **41.** $\log_8 512$

Lesson 11-4 Solve equations involving logarithms.

Solve $\log_5 4 + \log_5 x = \log_5 36$.

$\log_5 4 + \log_5 x = \log_5 36$
$\log_5 4x = \log_5 36$
$4x = 36$
$x = 9$

Solve each equation.

42. $\log_x 81 = 4$
43. $\log_{\frac{1}{2}} x = -4$
44. $\log_3 3 + \log_3 x = \log_3 45$
45. $2 \log_6 4 - \frac{1}{3} \log_6 8 = \log_6 x$
46. $\log_2 x = \frac{1}{3} \log_2 27$

47. Graph $y = \log_{10} x$

Lesson 11-5 Find common logarithms and antilogarithms of numbers.

Given that $\log 8 = 0.9031$, evaluate $\log 80{,}000$.

$\log 80{,}000 = \log(10{,}000 \times 8)$
$= \log 10^4 + \log 8$
$= 4 + 0.9031$
$= 4.9031$

Solve $3^{x+2} = 4^{x-1}$.

$3^{x+2} = 4^{x-1}$
$\log 3^{x+2} = \log 4^{x-1}$
$(x+2) \log 3 = (x-1) \log 4$
$x \log 3 + 2 \log 3 = x \log 4 - \log 4$
$x \log 3 - x \log 4 = -\log 4 - 2 \log 3$
$x (\log 3 - \log 4) = -(\log 4 + 2 \log 3)$
$x = -\frac{(\log 4 + 2 \log 3)}{(\log 3 - \log 4)}$
$x \approx 12.4565$

Given that $\log 3 = 0.4771$ and $\log 14 = 1.1461$, evaluate each logarithm.

48. $\log 300{,}000$ **49.** $\log 0.0003$
50. $\log 140$ **51.** $\log 0.014$

Solve each equation or inequality.

52. $4^x = 6^{x+2}$ **53.** $12^{0.5x} = 8^{0.1x-4}$
54. $\left(\frac{1}{4}\right)^{3x} < 6^{x-2}$ **55.** $0.1^{2x+8} \geq 7^{x+4}$
56. $\log(2x+3) = -\log(3-x)$

Graph each equation or inequality.

57. $y = 3 \log(x-2)$ **58.** $y \geq 7^{x-2}$

59. Solve $9^x = 4^{x-2}$ by graphing.

OBJECTIVES AND EXAMPLES

Lesson 11-6 Find natural logarithms of numbers.

- Convert $\log_2 14$ to a natural logarithm and evaluate.

$$\log_2 14 = \frac{\ln 14}{\ln 2} \approx 3.8074$$

REVIEW EXERCISES

Convert each logarithm to a natural logarithm and evaluate.

60. $\log_4 15$

61. $\log_8 24$

62. $\log_9 100$

63. $\log_{15} 125$

Lesson 11-6 Solve equations using natural logarithms.

- Use natural logarithms to solve $2^{x-1} = 5^{3x}$.

$$\begin{aligned} 2^{x-1} &= 5^{3x} \\ \ln 2^{x-1} &= \ln 5^{3x} \\ (x-1)\ln 2 &= 3x \ln 5 \\ (x-1)(0.6931) &= 3x(1.6094) \\ 0.6931x - 0.6931 &= 4.8282x \\ -0.6931 &= 4.1351x \\ -0.1676 &= x \end{aligned}$$

Use natural logarithms to solve each equation or inequality.

64. $4^x = 100$

65. $6^{x-2} = 30$

66. $3^{x+1} = 4^{2x}$

67. $9^{4x} = 5^{x-4}$

68. $24 < e^{2x}$

69. $15e^x \geq 200$

Solve each equation or inequality by graphing. Round solutions to the nearest hundredth.

70. $2e^x = 56$

71. $9 > e^x$

Lesson 11-7 Find the doubling time of an exponential quantity.

- Find the amount of time required for an investment to double at a rate of 3.5% if the interest is compounded continuously.

Use $t = \frac{\ln 2}{k}$, where k is the interest rate written as a decimal.

$$\begin{aligned} t &= \frac{\ln 2}{k} \\ &= \frac{\ln 2}{0.035} \\ &\approx 19.80 \text{ years} \end{aligned}$$

It will take 19.8 years for the amount to double.

Find the amount of time required for an amount to double at the given rate if the interest is compounded continuously.

72. 2.8%

73. 5.125%

74. Finance What was the interest rate on an account that took 18 years to double if interest was compounded continuously and no deposits or withdrawals were made during the 18-year period?

APPLICATIONS AND PROBLEM SOLVING

75. Archaeology Carbon-14 tests are often performed to determine the age of an organism that died a long time ago. Carbon-14 has a half-life of 5730 years. If a turtle shell is found and tested to have 65% of its original carbon-14, how old is it? *(Lesson 11-4)*

76. Sound The decibel, dB, is the unit of measure for intensity of sound. The equation to determine the decibel rating of a sound is $\beta = 10 \log \frac{I}{I_0}$, where $I_0 = 10^{-12}$ W/m^2, the reference intensity of the faintest audible sound. *(Lesson 11-5)*

- **a.** Find the decibel rating, in dB, of a whisper with an intensity of 1.15×10^{-10} W/m^2.
- **b.** Find the decibel rating of a teacher's voice with an intensity of 9×10^{-9} W/m^2.
- **c.** Find the decibel rating of a rock concert with an intensity of 8.95×10^{-3} W/m^2.

77. Population A certain city has a population of $P = 142{,}000e^{0.014t}$ where t is the time in years and $t = 0$ is the year 1990. In what year will the city have a population of 200,000? *(Lesson 11-6)*

78. Data Entry The supervisor of the data entry department found that the average number of words per minute, N, input by trainees after t weeks was $N = 65 - 30e^{-0.20t}$. *(Lesson 11-6)*

- **a.** What was the average number of words per minute after 2 weeks?
- **b.** What was the average number of words per minute after 15 weeks?
- **c.** When was the average trainee able to input 50 words per minute?

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. The simplest form of an expression that contains at least two rational exponents is $\frac{n^2}{4m}$. If one of the exponents is negative and another is a fraction, find the original expression.

2. Melissa used the power and the product properties of logarithms to solve an equation involving logarithms. She got a solution of 1. What is the equation Melissa solved?

PORTFOLIO

Suggest real-world data that can be modeled with an exponential or a logarithmic function. Record the function that models the data. Explain why your function is a good model.

Unit 3 *inter*NET Project

SPACE — THE FINAL FRONTIER

Kepler is Still King!

- Search the Internet to find out about Kepler's Laws for planetary motion.
- Kepler's Third Law relates the distance of planets from the sun and the period of each planet. Use the Internet to find the distance each planet is from the sun and to find each planet's period.
- Use the information about distance and period of the planets to verify Kepler's Third Law. You can use a spreadsheet, if you prefer.
- Write a summary describing Kepler's Third Law and how you verified his computations.

Additional Assessment See p. A66 for Chapter 11 practice test.

Average Problems

Calculating the average of a set of n numbers is easy if you are given the numbers and know the definition of *average* (*arithmetic mean*).

$$\text{average} = \frac{\text{sum of } n \text{ numbers}}{n} \text{ or average} = \frac{\text{sum}}{n}$$

SAT and ACT problems that use averages are more challenging. They often involve missing numbers, evenly-spaced numbers, or weighted averages.

THE PRINCETON REVIEW

To find *weighted average*, multiply each number in the set by the number of times it appears. Add the products and divide the sum by the total number of numbers.

SAT EXAMPLE

1. In one month, a pet shop sold 5 red parrots that weighed 2 pounds each and 4 blue parrots that weighed 3 pounds each. What was the average (arithmetic mean) weight of a parrot in pounds that this pet shop sold this month?

A 2 **B** $2\frac{4}{9}$ **C** $2\frac{1}{2}$

D 5 **E** 9

HINT Eliminate choices that are impossible or unreasonable.

Solution Estimate the average weight of a parrot. The parrots sold weigh 2 and 3 pounds. So, the average weight cannot be 5 or 9 pounds. Eliminate choices D and E.

This is a weighted-average problem since there are different numbers of parrots with different weights. You cannot just average 2 pounds and 3 pounds. Notice that answer choice C is an obvious wrong answer as it is the average of 2 and 3.

Find the sum of the red-parrot weights, 5×2 or 10, and the sum of the blue-parrot weights, 4×3 or 12.

The total weight of the parrots is $10 + 12$ or 22. The total number of parrots is 9.

So the average weight is $22 \div 9$ or $2\frac{4}{9}$ pounds.

The correct answer is choice **B**.

ACT EXAMPLE

2. Louann computed the average of her test scores by adding the 5 scores together and dividing the sum by the number of tests. The average was 88. But she completely forgot a sixth test, which had a score of 82. What is the true average of Louann's six tests?

A 82 **B** 85 **C** 86

D 87 **E** 88

HINT Use the formula sum $= n \cdot$ average to find the sum when you know the average and the number of data.

Solution First estimate the answer. The missing score is 82, which is less than the average, 88, so the new average must be less than 88. Eliminate choice E. Also, the new average must be greater than 82, so eliminate choice A.

Next, find the sum of the 5 scores, using the formula for average. 5×88 or 440.

Then find the sum of all 6 scores by adding the sixth score to the sum of the first five scores. $440 + 82$ or 522.

Finally, divide the sum of the six scores by 6. $552 \div 6$ or 87.

The answer is choice **D**.

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. If the average (arithmetic mean) of four distinct positive integers is 11, what is the greatest possible value of any one of the integers?

   A 35 B 38 C 40 D 41 E 44

2. Which quadratic equation has roots of $\frac{1}{2}$ and $\frac{1}{3}$?

   A $5x^2 - 5x - 2 = 0$ B $5x^2 - 5x + 1 = 0$

   C $6x^2 + 5x - 1 = 0$ D $6x^2 - 6x + 1 = 0$

   E $6x^2 - 5x + 1 = 0$

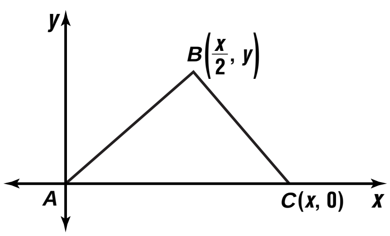
3. Brad tried to calculate the average of his 5 test scores. He mistakenly divided the correct total (T) of his scores by 6. The result was 14 less than what it should have been. Which of the following equations would determine the value of T?

   A $5T + 14 = 6T$ B $\frac{T}{6} = \frac{(T - 14)}{5}$

   C $\frac{T}{6} - 14 = \frac{T}{5}$ D $\frac{(T - 14)}{5} = \frac{T}{5}$

   E $\frac{T}{6} + 14 = \frac{T}{5}$

4. In the figure below, the ratio $\frac{\text{area of } \triangle ABC}{\tan A} =$



   A $\frac{1}{2y^2}$ B $\frac{1}{y^2}$ C $\frac{2}{x^2}$

   D $\frac{4}{x^2}$ E $\frac{x^2}{4}$

5. If the ratio of x to y is equal to the ratio of 10 to $2y$, then what is the value of x?

   A $\frac{1}{5}$ B 5 C 8 D 12 E 20

6. If the circumference of a circle is $\frac{2\pi}{3}$, then what is half of its area?

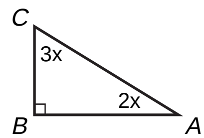
   A $\frac{\pi}{18}$ B $\frac{\pi}{9}$ C $\frac{4\pi}{9}$

   D $\frac{2\pi^2}{9}$ E $\frac{4\pi^2}{9}$

7. If the average (arithmetic mean) of eight numbers is 20 and the average of five of these numbers is 14, what is the average of the other three numbers?

   A 14 B 17 C 20 D 30 E 34

8. In the triangle below, what is the measure of $\angle A$?



   A 9° B 18° C 36° D 54° E 108°

9. A is the average (arithmetic mean) of three consecutive positive even integers. Which of the following could be the remainder when A is divided by 6?

   A 1

   B 3

   C 4

   D 5

   E It cannot be determined from the information given.

10. **Grid-In** If b is a prime integer such that $3b > 10 > \frac{5}{6}b$, what is one possible value of b?

interNET CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**