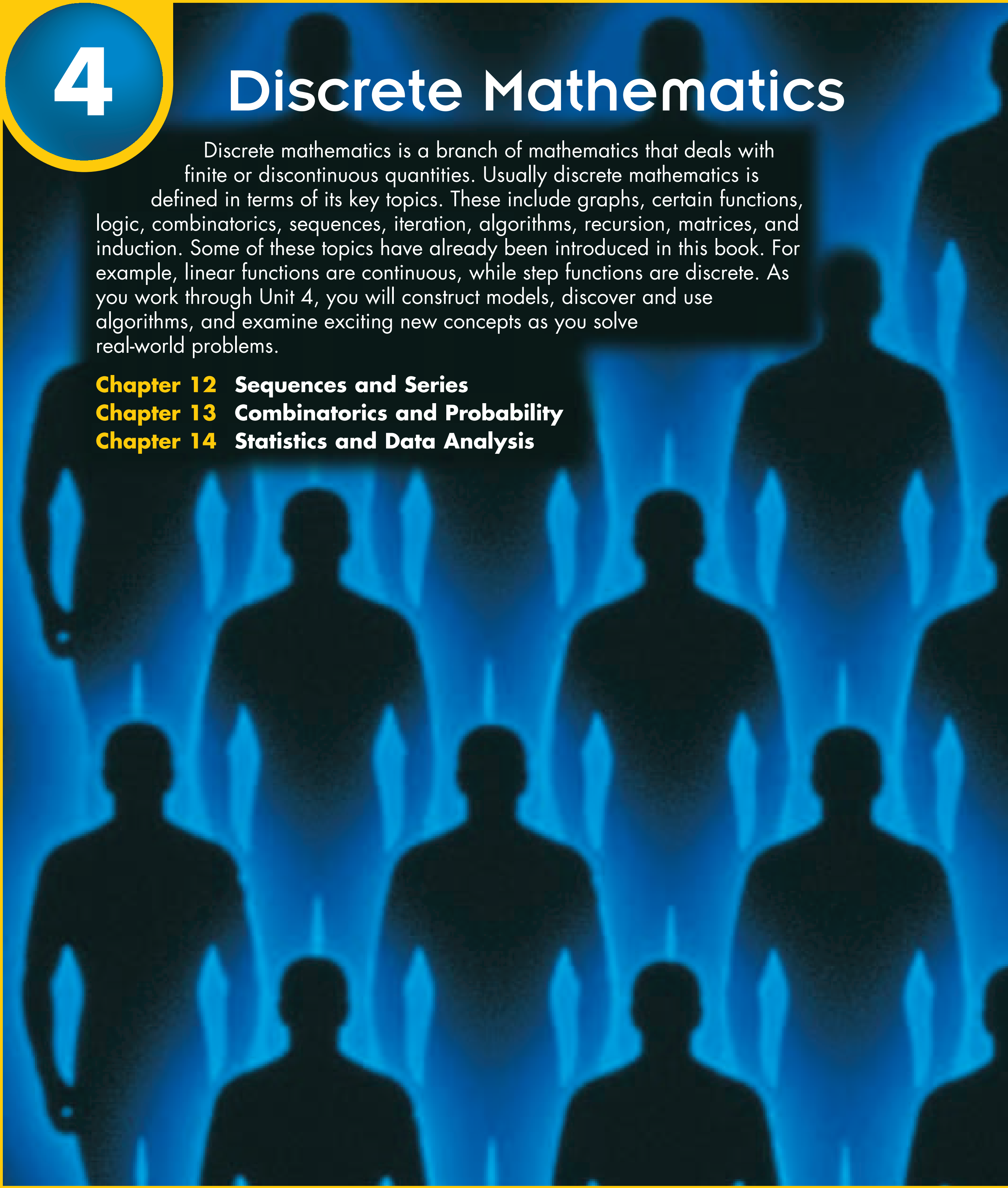
UNIT

4

Discrete Mathematics

Discrete mathematics is a branch of mathematics that deals with finite or discontinuous quantities. Usually discrete mathematics is defined in terms of its key topics. These include graphs, certain functions, logic, combinatorics, sequences, iteration, algorithms, recursion, matrices, and induction. Some of these topics have already been introduced in this book. For example, linear functions are continuous, while step functions are discrete. As you work through Unit 4, you will construct models, discover and use algorithms, and examine exciting new concepts as you solve real-world problems.



Unit 4 interNET Projects

THE UNITED STATES CENSUS BUREAU

Did you know that the very first United States Census was conducted in 1790? The United States Census Bureau completed this task in less than nine months. This is remarkable considering they had to do so without benefit of cars, telephones, or computers! As the U.S. population grew, it took more and more time to complete the census. The 1880 census took seven years to finish. Luckily, advancements in technology have helped the people employed by the Census Bureau analyze the data and prepare reports in a reasonable amount of time. At the end of each chapter in Unit 4, you will be given tasks to explore the data collected by the Census Bureau.

CHAPTER 12
(page 833)

That's a lot of people! In addition to determining how many people are in the United States every ten years, the Census Bureau also attempts to estimate the population at any particular time and in the future. On their web site, you can see estimates of the U. S. and world population for the current day. How does the Census Bureau estimate population?
Math Connection: Use the Internet to find population data. Model the population of the U.S. by using an arithmetic and then a geometric sequence. Predict the population for a future date using your models.

CHAPTER 13
(page 885)

Radically random! During a census year, the Census Bureau collects many type of data about people. This information includes age, ethnic background, and income, to name just a few. What types of data about the people of the United States can be found using the Internet?
Math Connection: Use data from the Internet to find the probability that a randomly-selected person in the U.S. belongs to a particular age group.

CHAPTER 14
(page 937)

More and more models! Did you know that in 1999 a person was born in the United States about every eight seconds? In that same year, about one person died every 15 seconds, and one person migrated every 20 seconds. Birth, deaths, and other factors directly affect the population. In Chapter 12, you modeled the U.S. population using sequences. What other types of population models could you use to predict population growth?
Math Connection: Use data from the Internet to write and graph several functions representing the U.S. population growth. Predict the population for a future data using your models.

interNET CONNECTION For more information on the Unit Project, visit:
www.amc.glencoe.com

CONTENTS

Chapter 12

SEQUENCES AND SERIES

CHAPTER OBJECTIVES

- **Identify and find *n*th terms of arithmetic, geometric, and infinite sequences.** *(Lessons 12-1, 12-2)*
- **Find sums of arithmetic, geometric, and infinite series.** *(Lessons 12-1, 12-2, 12-3)*
- **Determine whether a series is convergent or divergent.** *(Lesson 12-4)*
- **Use sigma notation.** *(Lesson 12-5)*
- **Use the Binomial Theorem to expand binomials.** *(Lesson 12-6)*
- **Evaluate expressions using exponential, trigonometric, and iterative series.** *(Lessons 12-7, 12-8)*
- **Use mathematical induction to prove the validity of mathematical statements.** *(Lesson 12-9)*

CONTENTS

12-1

Arithmetic Sequences and Series

OBJECTIVES
- Find the *n*th term and arithmetic means of an arithmetic sequence.
- Find the sum of *n* terms of an arithmetic series.

REAL ESTATE Ofelia Gonzales sells houses in a new development. She makes a commission of $3750 on the sale of her first house. To encourage aggressive selling, Ms. Gonzales' employer promises a $500 increase in commission for each additional house sold. Thus, on the sale of her next house, she will earn $4250 commission. How many houses will Ms. Gonzales have to sell for her total commission in one year to be at least $65,000? *This problem will be solved in Example 6.*

The set of numbers representing the amount of money earned for each house sold is an example of a **sequence.** A sequence is a function whose domain is the set of natural numbers. The **terms** of a sequence are the range elements of the function. The first term of a sequence is denoted a_1, the second term is a_2, and so on up to the nth term a_n.

Look Back
Refer to Lesson 4-1 for more about natural numbers.

Symbol	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
Term	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-1\frac{1}{2}$

The sequence given in the table above is an example of an **arithmetic sequence.** The difference between successive terms of an arithmetic sequence is a constant called the **common difference,** denoted *d*. In the example above, $d = \frac{1}{2}$.

Arithmetic Sequence

An arithmetic sequence is a sequence in which each term after the first, a_1, is equal to the sum of the preceding term and the common difference, *d*. The terms of the sequence can be represented as follows.

$$a_1,\ a_1 + d,\ a_1 + 2d,\ \ldots$$

To find the next term in an arithmetic sequence, first find the common difference by subtracting any term from its succeeding term. Then add the common difference to the last term to find the next term in the sequence.

Example 1 **Find the next four terms in the arithmetic sequence −5, −2, 1, …**

First, find the common difference.

$a_2 - a_1 = -2 - (-5)$ or 3
$a_3 - a_2 = 1 - (-2)$ or 3

Find the difference between pairs of consecutive terms to verify the common difference.

The common difference is 3.

Add 3 to the third term to get the fourth term, and so on.

$a_4 = 1 + 3$ or 4 $\quad a_5 = 4 + 3$ or 7 $\quad a_6 = 7 + 3$ or 10 $\quad a_7 = 10 + 3$ or 13

The next four terms are 4, 7, 10, and 13.

By definition, the nth term is also equal to $a_{n-1} + d$, where a_{n-1} is the $(n - 1)$th term. That is, $a_n = a_{n-1} + d$. This type of formula is called a **recursive formula.** This means that each succeeding term is formulated from one or more previous terms.

The nth term of an arithmetic sequence can also be found when only the first term and the common difference are known. Consider an arithmetic sequence in which $a = -3.7$ and $d = 2.9$. Notice the pattern in the way the terms are formed.

first term	a_1	a	-3.7
second term	a_2	$a + d$	$-3.7 + 1(2.9) = -0.8$
third term	a_3	$a + 2d$	$-3.7 + 2(2.9) = 2.1$
fourth term	a_4	$a + 3d$	$-3.7 + 3(2.9) = 5.0$
fifth term	a_5	$a + 4d$	$-3.7 + 4(2.9) = 7.9$
⋮	⋮	⋮	⋮
nth term	a_n	$a + (n - 1)d$	$-3.7 + (n - 1)2.9$

The *n*th Term of an Arithmetic Sequence

The nth term of an arithmetic sequence with first term a_1 and common difference d is given by $a_n = a_1 + (n - 1)d$.

Notice that the preceding formula has four variables: a_n, a_1, n, and d. If any three of these are known, the fourth can be found.

Examples

2 **Find the 47th term in the arithmetic sequence −4, −1, 2, 5, ….**

First, find the common difference.

$a_2 - a_1 = -1 - (-4)$ or 3 $\quad$ $a_3 - a_2 = 2 - (-1)$ or 3 $\quad$ $a_4 - a_3 = 5 - 2$ or 3

The common difference is 3.

Then use the formula for the nth term of an arithmetic sequence.

$$a_n = a_1 + (n - 1)d$$
$$a_{47} = -4 + (47 - 1)3 \quad n = 47,\ a_1 = -4, \text{ and } d = 3$$
$$a_{47} = 134$$

3 **Find the first term in the arithmetic sequence for which $a_{19} = 42$ and $d = -\frac{2}{3}$.**

$$a_n = a_1 + (n - 1)d$$
$$a_{19} = a_1 + (19 - 1)\left(-\frac{2}{3}\right) \quad n = 19 \text{ and } d = -\frac{2}{3}$$
$$42 = a_1 + (-12) \quad a_{19} = 42$$
$$a_1 = 54$$

Sometimes you may know two terms of an arithmetic sequence that are not in consecutive order. The terms between any two nonconsecutive terms of an arithmetic sequence are called **arithmetic means.** In the sequence below, 38 and 49 are the arithmetic means between 27 and 60.

5, 16, 27, 38, 49, 60

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Example 4 **Write an arithmetic sequence that has five arithmetic means between 4.9 and 2.5.**

The sequence will have the form 4.9, $\underline{\ ?\ }$, $\underline{\ ?\ }$, $\underline{\ ?\ }$, $\underline{\ ?\ }$, $\underline{\ ?\ }$, 2.5. Note that 2.5 is the 7th term of the sequence or a_7.

First, find the common difference, using $n = 7$, $a_7 = 2.5$, and $a_1 = 4.9$

$$a_n = a_1 + (n - 1)d$$
$$2.5 = 4.9 + (7 - 1)d$$
$$2.5 = 4.9 + 6d$$
$$d = -0.4$$

Then determine the arithmetic means.

$$a_2 = 4.9 + (-0.4) \text{ or } 4.5$$
$$a_3 = 4.5 + (-0.4) \text{ or } 4.1$$
$$a_4 = 4.1 + (-0.4) \text{ or } 3.7$$
$$a_5 = 3.7 + (-0.4) \text{ or } 3.3$$
$$a_6 = 3.3 + (-0.4) \text{ or } 2.9$$

The sequence is 4.9, 4.5, 4.1, 3.7, 3.3, 2.9, 2.5.

An indicated sum is 1 + 2 + 3 + 4. The sum 1 + 2 + 3 + 4 is 10.

An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence. The lists below show some examples of arithmetic sequences and their corresponding arithmetic series.

Arithmetic Sequence	Arithmetic Series
$-9, -3, 3, 9, 15$	$-9 + (-3) + 3 + 9 + 15$
$3, \frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}$	$3 + \frac{5}{2} + 2 + \frac{3}{2} + 1 + \frac{1}{2}$
$a_1, a_2, a_3, a_4, \ldots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

The symbol S_n, called the ***n*th partial sum**, is used to represent the sum of the first n terms of a series. To develop a formula for S_n for a finite arithmetic series, a series can be written in two ways and added term by term, as shown below. The second equation for S_n given below is obtained by reversing the order of the terms in the series.

$$\begin{array}{rl}
S_n = & a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n \\
+\ S_n = & a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1 \\
\hline
2S_n = & (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) \\
2S_n = & n(a_1 + a_n) \qquad \text{There are } n \text{ terms in the series, all of which are } (a_1 + a_n).
\end{array}$$

Therefore, $S_n = \frac{n}{2}(a_1 + a_n)$.

Sum of a Finite Arithmetic Series

The sum of the first n terms of an arithmetic series is given by $S_n = \frac{n}{2}(a_1 + a_n)$.

Example 5 **Find the sum of the first 60 terms in the arithmetic series 9 + 14 + 19 + ⋯ + 304.**

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$S_{60} = \frac{60}{2}(9 + 304) \qquad n = 60,\ a_1 = 9,\ a_{60} = 304$$
$$= 9390$$

When the value of the last term, a_n, is not known, you can still determine the sum of the series. Using the formula for the nth term of an arithmetic sequence, you can derive another formula for the sum of a finite arithmetic series.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ S_n &= \frac{n}{2}[a_1 + (a_1 + (n-1)d)] && a_n = a_1 + (n-1)d \\ S_n &= \frac{n}{2}[2a_1 + (n-1)d] \end{aligned}$$

Example 6 Real World Application

REAL ESTATE **Refer to the application at the beginning of the lesson. How many houses will Ms. Gonzales have to sell for her total commission in one year to be at least $65,000?**

Let S_n = the amount of her desired commission, $65,000.
Let a_1 = the first commission, $3750.
In this example, $d = 500$.

We want to find n, the number of houses that Ms. Gonzales has to sell to have a total commission greater than or equal to $65,000.

$$\begin{aligned} S_n &= \frac{n}{2}[2a_1 + (n-1)d] \\ 65{,}000 &= \frac{n}{2}[2(3750) + (n-1)(500)] && S_n = 65{,}000,\ a_1 = 3750 \\ 130{,}000 &= n(7500 + 500n - 500) && \text{Multiply each side by 2.} \\ 130{,}000 &= 7000n + 500n^2 && \text{Simplify.} \\ 0 &= 500n^2 + 7000n - 130{,}000 \\ 0 &= 5n^2 + 70n - 1300 && \text{Divide each side by 100.} \\ n &= \frac{-70 \pm \sqrt{70^2 - 4(5)(-1300)}}{2(5)} && \text{Use the Quadratic Formula.} \\ n &= \frac{-70 \pm \sqrt{30{,}900}}{10} \\ n &\approx 10.58 \text{ and } -24.58 && -24.58 \text{ is not a possible answer.} \end{aligned}$$

Ms. Gonzales must sell 11 or more houses for her total commission to be at least $65,000.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** the first five terms of the sequence defined by $a_n = 6 - 4n$. Is this an arithmetic sequence? Explain.
2. Consider the arithmetic sequence defined by $a_n = \frac{5 - 2n}{2}$.
   a. **Graph** the first five terms of the sequence. Let n be the x-coordinate and a_n be the y-coordinate, and connect the points.
   b. **Describe** the graph found in part **a**.
   c. **Find** the common difference of the sequence and determine its relationship to the graph found in part **a**.

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3. Refer to Example 6.

a. Explain why −24.58 is *not* a possible answer.

b. Determine how much money Ms. Gonzales will make if she sells 10 houses.

4. Describe the common difference for an arithmetic sequence in which the terms are decreasing.

5. You Decide Ms. Brooks defined two sequences, $a_n = (-1)^n$ and $b_n = (-2)^n$, for her class. She asked the class to determine if they were arithmetic sequences. Latonya said the second was an arithmetic sequence and that the first was not. Diana thought the reverse was true. Who is correct? Explain.

Guided Practice

Find the next four terms in each arithmetic sequence.

6. 6, 11, 16, …

7. −15, −7, 1, …

8. $a - 6, a - 2, a + 2, \ldots$

For Exercises 9-15, assume that each sequence or series is arithmetic.

9. Find the 17th term in the sequence for which $a_1 = 10$ and $d = -3$.

10. Find n for the sequence for which $a_n = 37$, $a_1 = -13$, and $d = 5$.

11. What is the first term in the sequence for which $d = -2$ and $a_7 = 3$?

12. Find d for the sequence for which $a_1 = 100$ and $a_{12} = 34$.

13. Write a sequence that has two arithmetic means between 9 and 24.

14. What is the sum of the first 35 terms in the series $7 + 9 + 11 + \cdots$?

15. Find n for a series for which $a_1 = 30$, $d = -4$, and $S_n = -210$.

16. Theater Design The right side of the orchestra section of the Nederlander Theater in New York City has 19 rows, and the last row has 27 seats. The numbers of seats in each row increase by 1 as you move toward the back of the section. How many seats are in this section of the theater?

EXERCISES

Practice

Find the next four terms in each arithmetic sequence.

17. 5, −1, −7, …

18. −18, −7, 4, …

19. 3, 4.5, 6, …

20. 5.6, 3.8, 2, …

21. $b, b + 4, b + 8, \ldots$

22. $-x, 0, x, \ldots$

23. $5n, -n, -7n, \ldots$

24. $5 + k, 5, 5 - k, \ldots$

25. $2a - 5, 2a + 2, 2a + 9, \ldots$

26. Determine the common difference and find the next three terms of the arithmetic sequence $3 + \sqrt{7}, 5, 7 - \sqrt{7}, \ldots$.

For Exercises 27-34, assume that each sequence or series is arithmetic.

27. Find the 25th term in the sequence for which $a_1 = 8$ and $d = 3$.

28. Find the 18th term in the sequence for which $a_1 = 1.4$ and $d = 0.5$.

29. Find n for the sequence for which $a_n = -41$, $a_1 = 19$, and $d = -5$.

30. Find n for the sequence for which $a_n = 138$, $a_1 = -2$, and $d = 7$.

31. What is the first term in the sequence for which $d = -3$, and $a_{15} = 38$?

32. What is the first term in the sequence for which $d = \frac{1}{3}$ and $a_7 = 10\frac{2}{3}$?

33. Find d for the sequence in which $a_1 = 6$ and $a_{14} = 58$.

34. Find d for the sequence in which $a_1 = 8$ and $a_{11} = 26$.

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For Exercises 35-49, assume that each sequence or series is arithmetic.

35. What is the eighth term in the sequence $-4 + \sqrt{5}, -1 + \sqrt{5}, 2 + \sqrt{5}, \ldots$?

36. What is the twelfth term in the sequence $5 - \boldsymbol{i}, 6, 7 + \boldsymbol{i}, \ldots$?

37. Find the 33rd term in the sequence 12.2, 10.5, 8.8, ….

38. Find the 79th term in the sequence $-7, -4, -1, \ldots$.

39. Write a sequence that has one arithmetic mean between 12 and 21.

40. Write a sequence that has two arithmetic means between -5 and 4.

41. Write a sequence that has two arithmetic means between $\sqrt{3}$ and 12.

42. Write a sequence that has three arithmetic means between 2 and 5.

43. Find the sum of the first 11 terms in the series $\frac{3}{2} + 1 + \frac{1}{2} + \ldots$.

44. Find the sum of the first 100 terms in the series $-5 - 4.8 - 4.6 - \ldots$.

45. Find the sum of the first 26 terms in the series $-19 - 13 - 7 - \ldots$.

46. Find n for a series for which $a_1 = -7$, $d = 1.5$, and $S_n = -14$.

47. Find n for a series for which $a_1 = -3$, $d = 2.5$, and $S_n = 31.5$.

48. Write an expression for the nth term of the sequence 5, 7, 9, ….

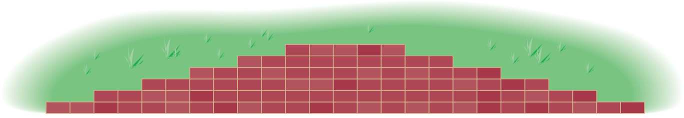
49. Write an expression for the nth term of the sequence $6, -2, -10, \ldots$.

Applications and Problem Solving

50. Keyboarding Antonio has found that he can input statistical data into his computer at the rate of 2 data items faster each half hour he works. One Monday, he starts work at 9:00 A.M., inputting at a rate of 3 data items per minute. At what rate will Antonio be inputting data into the computer by lunchtime (noon)?

51. Critical Thinking Show that if x, y, z, and w are the first four terms of an arithmetic sequence, then $x + w - y = z$.

52. Construction The Arroyos are planning to build a brick patio that approximates the shape of a trapezoid. The shorter base of the trapezoid needs to start with a row of 5 bricks, and each row must increase by 2 bricks on each side until there are 25 rows. How many bricks do the Arroyos need to buy?



53. Critical Thinking The measures of the angles of a convex polygon form an arithmetic sequence. The least measurement in the sequence is 85°. The greatest measurement is 215°. Find the number of sides in this polygon.

54. Geometry The sum of the interior angles of a triangle is 180°.

a. What are the sums of the interior angles of polygons with 4, 5, 6, and 7 sides?

b. Show that these sums (beginning with the triangle) form an arithmetic sequence.

c. Find the sum of the interior angles of a 35-sided polygon.

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55. **Critical Thinking** Consider the sequence of odd natural numbers.

a. What is S_5?

b. What is S_{10}?

c. Make a conjecture as to the pattern that emerges concerning the sum. Write an algebraic proof verifying your conjecture.

56. **Sports** At the 1998 Winter X-Games held in Crested Butte, Colorado, Jennie Waara, from Sweden, won the women's snowboarding slope-style competition. Suppose that in one of the qualifying races, Ms. Waara traveled 5 feet in the first second, and the distance she traveled increased by 7 feet each subsequent second. If Ms. Waara reached the finish line in 15 seconds, how far did she travel?

57. **Entertainment** A radio station advertises a contest with ten cash prizes totaling \$5510. There is to be a \$100 difference between each successive prize. Find the amounts of the least and greatest prizes the radio station will award.

58. **Critical Thinking** Some sequences involve a pattern but are not arithmetic. Find the sum of the first 72 terms in the sequence 6, 8, 2, …, where $a_n = a_{n-1} - a_{n-2}$.

Mixed Review

59. **Personal Finance** If Parker Hamilton invests \$100 at 7% compounded continuously, how much will he have at the end of 15 years? *(Lesson 11-3)*

60. Find the coordinates of the center, foci, and vertices of the ellipse whose equation is $4x^2 + 25y^2 + 250y + 525 = 0$. Then graph the equation. *(Lesson 10-3)*

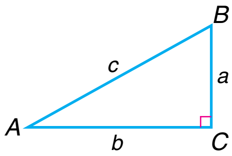
61. Find $6\left(\cos \frac{5\pi}{8} + \boldsymbol{i} \sin \frac{5\pi}{8}\right) \div 12\left(\cos \frac{\pi}{2} + \boldsymbol{i} \sin \frac{\pi}{2}\right)$. Then express the quotient in rectangular form. *(Lesson 9-7)*

62. Find the inner product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if $\vec{\mathbf{u}} = \langle 2, -1, 3\rangle$ and $\vec{\mathbf{v}} = \langle 5, 3, 0\rangle$. *(Lesson 8-4)*

63. Write the standard form of the equation of a line for which the length of the normal is 5 units and the normal makes an angle of 30° with the positive x-axis. *(Lesson 7-6)*

64. Graph $y = \sec 2\theta - 3$. *(Lesson 6-7)*

65. Solve triangle ABC if $B = 19°32'$ and $c = 4.5$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-5)*



66. Find the discriminant of $4p^2 - 3p + 2 = 0$. Describe the nature of its roots. *(Lesson 4-2)*

67. Determine the slant asymptote of $f(x) = \frac{x^2 - 4x + 2}{x - 3}$. *(Lesson 3-7)*

68. Triangle ABC is represented by the matrix $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 3 & -4 \end{bmatrix}$. Find the image of the triangle after a rotation of 270° counterclockwise about the origin. *(Lesson 2-4)*

69. **SAT/ACT Practice** If $a - 4b = 15$ and $4a - b = 15$, then $a - b = $?

A 3 **B** 4 **C** 6 **D** 15 **E** 30

Extra Practice See p. A48.

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12-2

Geometric Sequences and Series

OBJECTIVES

- Find the *n*th term and geometric means of a geometric sequence.
- Find the sum of *n* terms of a geometric series.

Real World Application

ACCOUNTING Bertha Blackwell is an accountant for a small company. On January 1, 1996, the company purchased $50,000 worth of office copiers. Since this equipment is a company asset, Ms. Blackwell needs to determine how much the copiers are presently worth. She estimates that copiers depreciate at a rate of 45% per year. What value should Ms. Blackwell assign the copiers on her 2001 year-end accounting report? *This problem will be solved in Example 3.*

The following sequence is an example of a **geometric sequence.**

10, 2, 0.4, 0.08, 0.016, … *Can you find the next term?*

The ratio of successive terms in a geometric sequence is a constant called the **common ratio,** denoted *r*.

Geometric Sequence

A geometric sequence is a sequence in which each term after the first, a_1, is the product of the preceding term and the common ratio, r. The terms of the sequence can be represented as follows, where a_1 is nonzero and r is not equal to 1 or 0.

$$a_1,\ a_1r,\ a_1r^2,\ \ldots.$$

You can find the next term in a geometric sequence as follows.

- First divide any term by the preceding term to find the common ratio.
- Then multiply the last term by the common ratio to find the next term in the sequence.

Example 1 **Determine the common ratio and find the next three terms in each sequence.**

a. $1, -\frac{1}{2}, \frac{1}{4}, \ldots$

First, find the common ratio.

$a_2 \div a_1 = -\frac{1}{2} \div 1$ or $-\frac{1}{2}$ $\qquad$ $a_3 \div a_2 = \frac{1}{4} \div \left(-\frac{1}{2}\right)$ or $-\frac{1}{2}$

The common ratio is $-\frac{1}{2}$.

Multiply the third term by $-\frac{1}{2}$ to get the fourth term, and so on.

$a_4 = \frac{1}{4} \cdot \left(-\frac{1}{2}\right)$ or $-\frac{1}{8}$ $\qquad$ $a_5 = -\frac{1}{8} \cdot \left(-\frac{1}{2}\right)$ or $\frac{1}{16}$ $\qquad$ $a_6 = \frac{1}{16} \cdot \left(-\frac{1}{2}\right)$ or $-\frac{1}{32}$

The next three terms are $-\frac{1}{8}$, $\frac{1}{16}$, and $-\frac{1}{32}$.

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b. $r - 1, -3r + 3, 9r - 9, \ldots$

First, find the common ratio.

$a_2 \div a_1 = \frac{-3r + 3}{r - 1}$

$a_2 \div a_1 = \frac{-3(r - 1)}{r - 1}$ *Factor.*

$a_2 \div a_1 = -3$ *Simplify.*

$a_3 \div a_2 = \frac{9r - 9}{-3r + 3}$

$a_3 \div a_2 = \frac{9(r - 1)}{-3(r - 1)}$ *Factor.*

$a_3 \div a_2 = -3$ *Simplify.*

The common ratio is -3.

Multiply the third term by -3 to get the fourth term, and so on.

$a_4 = -3(9r - 9)$ or $-27r + 27$

$a_5 = -3(-27r + 27)$ or $81r - 81$

$a_6 = -3(81r - 81)$ or $-243r + 243$

The next three terms are $-27r + 27$, $81r - 81$, and $-243r + 243$.

As with arithmetic sequences, geometric sequences can also be defined recursively. By definition, the nth term is also equal to $a_{n-1}r$, where a_{n-1} is the $(n - 1)$th term. That is, $a_n = a_{n-1}r$.

Since successive terms of a geometric sequence can be expressed as the product of the common ratio and the previous term, it follows that each term can be expressed as the product of a_1 and a power of r. The terms of a geometric sequence for which $a_1 = -5$ and $r = 7$ can be represented as follows.

first term	a_1	a_1	-5
second term	a_2	a_1r	$-5 \cdot 7^1 = -35$
third term	a_3	a_1r^2	$-5 \cdot 7^2 = -245$
fourth term	a_4	a_1r^3	$-5 \cdot 7^3 = -1715$
fifth term	a_5	a_1r^4	$-5 \cdot 7^4 = -12{,}005$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth term	a_n	ar^{n-1}	$-5 \cdot 7^{n-1}$

The nth Term of a Geometric Sequence

The nth term of a geometric sequence with first term a_1 and common ratio r is given by $a_n = a_1r^{n-1}$.

Example 2 **Find an approximation for the 23rd term in the sequence 256, −179.2, 125.44, …**

First, find the common ratio.

$a_2 \div a_1 = -179.2 \div 256$ or -0.7

$a_3 \div a_2 = 125.44 \div (-179.2)$ or -0.7

The common ratio is -0.7.

Then, use the formula for the nth term of a geometric sequence.

$a_n = a_1r^{n-1}$

$a_{23} = 256(-0.7)^{23-1}$ *$n = 23, a_1 = 256, r = -0.7$*

$a_{23} \approx 0.1000914188$ *Use a calculator.*

The 23rd term is about 0.1.

CONTENTS

Geometric sequences can represent growth or decay.

- For a common ratio greater than 1, a sequence may model growth. Applications include compound interest, appreciation of property, and population growth.
- For a positive common ratio less than 1, a sequence may model decay. Applications include some radioactive behavior and depreciation.

Example 3 Real World Application

ACCOUNTING **Refer to the application at the beginning of the lesson. Compute the value of the copiers at the end of the year 2001.**

Since the copiers were purchased at the beginning of the first year, the original purchase price of the copiers represents a_1. If the copiers depreciate at a rate of 45% per year, then they retain 100 − 45 or 55% of their value each year.

Use the formula for the nth term of a geometric sequence to find the value of the copiers six years later or a_7.

$a_n = a_1 r^{n-1}$

$a_7 = 50{,}000 \cdot (0.55)^{7-1}$ *$a_1 = 50{,}000$, $r = 0.55$, $n = 7$*

$a_7 \approx 1384.032031$ *Use a calculator.*

Ms. Blackwell should list the value of the copiers on her report as $1384.03.

The terms between any two nonconsecutive terms of a geometric sequence are called **geometric means.**

Example 4 **Write a sequence that has two geometric means between 48 and −750.**

This sequence will have the form 48, ___?___, ___?___, −750.

First, find the common ratio.

$a_n = a_1 r^{n-1}$

$a_4 = a_1 r^3$ *Since there will be four terms in the sequence, $n = 4$.*

$-750 = 48r^3$ *$a_4 = -750$ and $a_1 = 48$*

$\frac{-125}{8} = r^3$ *Divide each side by 48 and simplify.*

$\sqrt[3]{-\frac{125}{8}} = r$ *Take the cube root of each side.*

$-2.5 = r$

Then, determine the geometric sequence.

$a_2 = 48(-2.5)$ or -120 $a_3 = -120(-2.5)$ or 300

The sequence is 48, −120, 300, −750.

A **geometric series** is the indicated sum of the terms of a geometric sequence. The lists below show some examples of geometric sequences and their corresponding series.

Geometric Sequence	**Geometric Series**
$3, 9, 27, 81, 243$	$3 + 9 + 27 + 81 + 243$
$16, 4, 1, \frac{1}{4}, \frac{1}{16}$	$16 + 4 + 1 + \frac{1}{4} + \frac{1}{16}$
$a_1, a_2, a_3, a_4, \ldots, a_n$	$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$

To develop a formula for the sum of a finite geometric sequence, S_n, write an expression for S_n and for rS_n, as shown below. Then subtract rS_n from S_n and solve for S_n.

$$\begin{aligned}
S_n &= a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} \\
rS_n &= \phantom{a_1 + {}} a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n \\
\hline
S_n - rS_n &= a_1 - a_1r^n && \textit{Subtract.} \\
S_n(1 - r) &= a_1 - a_1r^n && \textit{Factor.} \\
S_n &= \frac{a_1 - a_1r^n}{1 - r} && \textit{Divide each side by } 1 - r,\ r \neq 1.
\end{aligned}$$

Sum of a Finite Geometric Series

The sum of the first n terms of a finite geometric series is given by $S_n = \frac{a_1 - a_1r^n}{1 - r}$.

Example 5 **Find the sum of the first ten terms of the geometric series $16 - 48 + 144 - 432 + \cdots$.**

The formula for the sum of a geometric series can also be written as $S_n = a_1 \frac{1 - r^n}{1 - r}$.

First, find the common ratio.

$a_2 \div a_1 = -48 \div 16$ or -3 $\qquad$ $a_4 \div a_3 = -432 \div 144$ or -3

The common ratio is -3.

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

$$S_{10} = \frac{16 - 16(-3)^{10}}{1 - (-3)} \qquad n = 10,\ a_1 = 16,\ r = -3.$$

$$S_{10} = -236{,}192 \qquad \textit{Use a calculator.}$$

The sum of the first ten terms of the series is $-236{,}192$.

Banks and other financial institutions use compound interest to determine earnings in accounts or how much to charge for loans. The formula for compound interest is $A = P\left(1 + \frac{r}{n}\right)^{tn}$, where

A = the account balance,
P = the initial deposit or amount of money borrowed,
r = the annual percentage rate (APR),
n = the number of compounding periods per year, and
t = the time in years.

Suppose at the beginning of each quarter you deposit \$25 in a savings account that pays an APR of 2% compounded quarterly. Most banks post the interest for each quarter on the last day of the quarter. The chart below lists the additions to the account balance as a result of each successive deposit through the rest of the year. Note that $1 + \frac{r}{n} = 1 + \frac{0.02}{4}$ or 1.005.

Date of Deposit	$A = P\left(1 + \frac{r}{n}\right)^{tn}$	**1st Year Additions (to the nearest penny)**
January 1	\$25 $(1.005)^4$	\$25.50
April 1	\$25 $(1.005)^3$	\$25.38
July 1	\$25 $(1.005)^2$	\$25.25
October 1	\$25 $(1.005)^1$	\$25.13
Account balance at the end of one year		\$101.26

The chart shows that the first deposit will gain interest through all four compounding periods while the second will earn interest through only three compounding periods. The third and last deposits will earn interest through two and one compounding periods, respectively. The sum of these amounts, \$101.26, is the balance of the account at the end of one year. This sum also represents a finite geometric series where $a_1 = 25.13$, $r = 1.005$, and $n = 4$.

$$25.13 + 25.13(1.005) + 25.13(1.005)^2 + 25.13(1.005)^3$$

Example 6

Real World Application

INVESTMENTS **Hiroshi wants to begin saving money for college. He decides to deposit \$500 at the beginning of each quarter (January 1, April 1, July 1, and October 1) in a savings account that pays an APR of 6% compounded quarterly. The interest for each quarter is posted on the last day of the quarter. Determine Hiroshi's account balance at the end of one year.**

The interest is compounded each quarter. So $n = 4$ and the interest rate per period is 6% ÷ 4 or 1.5%. The common ratio r for the geometric series is then $1 + 0.015$, or 1.015.

The first term a_1 in this series is the account balance at the end of the first quarter. Thus, $a_1 = 500(1.015)$ or 507.5.

Apply the formula for the sum of a geometric series.

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

$$S_4 = \frac{507.5 - 507.5(1.015)^4}{1 - 1.015} \qquad n = 4,\ r = 1.015$$

$$S_4 \approx 2076.13$$

Hiroshi's account balance at the end of one year is \$2076.13.

CONTENTS

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare and contrast** arithmetic and geometric sequences.
2. **Show** that the sequence defined by $a_n = (-3)^{n+1}$ is a geometric sequence.
3. **Explain** why the first term in a geometric sequence must be nonzero.
4. **Find a counterexample** for the statement "The sum of a geometric series cannot be less than its first term."
5. **Determine** whether the given terms form a finite geometric sequence. Write *yes* or *no* and then explain your reasoning.
   a. $3, 6, 18$
   b. $\sqrt{3}, 3, \sqrt{27}$
   c. $x^{-2}, x^{-1}, 1\,;\, x \neq 1$
6. Refer to Example 3.
   a. **Make a table** to represent the situation. In the first row, put the number of years, and in the second row, put the value of the computers.
   b. **Graph** the numbers in the table. Let years be the *x*-coordinate, let value be the *y*-coordinate, and connect the points.
   c. **Describe** the graph found in part **b**.

Guided Practice

Determine the common ratio and find the next three terms of each geometric sequence.

7. $\frac{2}{3}, 4, 24, \ldots$
8. $2, 3, \frac{9}{2}, \ldots$
9. $1.8, -7.2, 28.8, \ldots$

For Exercises 10–14, assume that each sequence or series is geometric.

10. Find the seventh term of the sequence 7, 2.1, 0.63, ….
11. If $r = 2$ and $a_5 = 24$, find the first term of the sequence.
12. Find the first three terms of the sequence for which $a_4 = 2.5$ and $r = 2$.
13. Write a sequence that has two geometric means between 1 and 27.
14. Find the sum of the first nine terms of the series $0.5 - 1 + 2 - \cdots$.

15. **Investment** Mika Rockwell invests in classic cars. He recently bought a 1978 convertible valued at \$20,000. The value of the car is predicted to appreciate at a rate of 3.5% per year. Find the value of the car after 10, 20, and 40 years, assuming that the rate of appreciation remains constant.

EXERCISES

Practice

Determine the common ratio and find the next three terms of each geometric sequence.

16. $10, 2, 0.4, \ldots$
17. $8, -20, 50, \ldots$
18. $\frac{2}{9}, \frac{2}{3}, 2, \ldots$
19. $\frac{3}{4}, \frac{3}{10}, \frac{3}{25}, \ldots$
20. $-7, 3.5, -1.75, \ldots$
21. $3\sqrt{2}, 6, 6\sqrt{2}, \ldots$
22. $9, 3\sqrt{3}, 3, \ldots$
23. $\boldsymbol{i}, -1, -\boldsymbol{i}, \ldots$
24. $t^8, t^5, t^2, \ldots$

www.amc.glencoe.com/self_check_quiz

25. The first term of a geometric sequence is $\frac{a}{b^2}$, and the common ratio is $\frac{b}{a^2}$. Find the next five terms of the geometric sequence.

For Exercises 26–40, assume that each sequence or series is geometric.

26. Find the fifth term of a sequence whose first term is 8 and common ratio is $\frac{3}{2}$.

27. Find the sixth term of the sequence $\frac{1}{2}, -\frac{3}{8}, \frac{9}{32}, \ldots$.

28. Find the seventh term of the sequence 40, 0.4, 0.004, ….

29. Find the ninth term of the sequence $\sqrt{5}, \sqrt{10}, 2\sqrt{5}, \ldots$.

30. If $r = 4$ and $a_6 = 192$, what is the first term of the sequence?

31. If $r = -\sqrt{2}$ and $a_5 = 32\sqrt{2}$, what is the first term of the sequence?

32. Find the first three terms of the sequence for which $a_5 = -6$ and $r = -\frac{1}{3}$.

33. Find the first three terms of the sequence for which $a_5 = 0.32$ and $r = 0.2$.

34. Write a sequence that has three geometric means between 256 and 81.

35. Write a sequence that has two geometric means between −2 and 54.

36. Write a sequence that has one geometric mean between $\frac{4}{7}$ and 7.

37. What is the sum of the first five terms of the series $\frac{5}{3} + 5 + 15 + \cdots$?

38. What is the sum of the first six terms of the series $65 + 13 + 2.6 + \cdots$?

39. Find the sum of the first ten terms of the series $1 - \frac{3}{2} + \frac{9}{4} - \cdots$.

40. Find the sum of the first eight terms of the series $2 + 2\sqrt{3} + 6 + \cdots$.

Applications and Problem Solving

Real World Application

41. **Biology** A cholera bacterium divides every half-hour to produce two complete cholera bacteria.

a. If an initial colony contains a population of b_0 bacteria, write an equation that will determine the number of bacteria present after t hours.

b. Suppose a petri dish contains 30 cholera bacteria. Use the equation from part **a** to determine the number of bacteria present 5 hours later.

c. What assumptions are made in using the formula found in part **a**?

42. **Critical Thinking** Consider the geometric sequence with $a_4 = 4$ and $a_7 = 12$.

a. Find the common ratio and the first term of the sequence.

b. Find the 28th term of the sequence.

43. **Consumerism** High Tech Electronics advertises a weekly installment plan for the purchase of a popular brand of big screen TV. The buyer pays \$5 at the end of the first week, \$5.50 at the end of the second week, \$6.05 at the end of the third week, and so on for one year.

a. What will the payments be at the end of the 10th, 20th, and 40th weeks?

b. Find the total cost of the TV.

c. Why is the cost found in part **b** not entirely accurate?

44. **Statistics** A number x is said to be the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$.

a. Find the harmonic mean of 5 and 8.

b. 8 is the harmonic mean of 20 and another number. What is the number?

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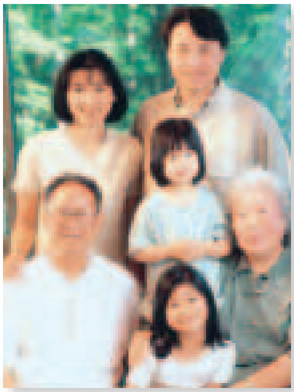
45. **Critical Thinking** In a geometric sequence, $a_1 = -2$ and every subsequent term is defined by $a_n = -3a_{n-1}$, where $n > 1$. Find the nth term in the sequence in terms of n.



46. **Genealogy** Wei-Ling discovers through a research of her Chinese ancestry that one of her fifteenth-generation ancestors was a famous military leader. How many descendants does this ancestor have in the fifteenth-generation, assuming each descendent had an average of 2.5 children?

47. **Personal Finance** Tonisha is about to begin her junior year in high school and is looking ahead to her college career. She estimates that by the time she is ready to enter a university she will need at least \$750 to purchase a computer system that will meet her needs. To avoid purchasing the computer on credit, she opens a savings account earning an APR of 2.4%, compounded monthly, and deposits \$25 at the beginning of each month.

a. Find the balance of the account at the end of the first month.

b. If Tonisha continues this deposit schedule for the next two years, will she have enough money in her account to buy the computer system? Explain.

c. Find the least amount of money Tonisha can deposit each month and still have enough money to purchase the computer.

48. **Critical Thinking** Use algebraic methods to determine which term 6561 is of the geometric sequence $\frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots$.

Mixed Review

49. **Banking** Gloria Castaneda has \$650 in her checking account. She is closing out the account by writing one check for cash against it each week. The first check is for \$20, the second is for \$25, and so on. Each check exceeds the previous one by \$5. In how many weeks will the balance in Ms. Castaneda account be \$0 if there is no service charge? *(Lesson 12-1)*

50. Find the value of $\log_{11} 265$ using the change of base formula. *(Lesson 11-5)*

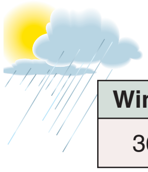
51. Graph the system $xy \geq 2$ and $x - 3y = 2$. *(Lesson 10-8)*

52. Write $3x - 5y + 5 = 0$ in polar form. *(Lesson 9-4)*

53. Write parametric equations of the line $3x + 4y = 5$. *(Lesson 8-6)*

54. If $\csc \theta = 3$ and $0° \leq \theta \leq 90°$, find $\sin \theta$. *(Lesson 7-1)*

55. **Weather** The maximum normal daily temperatures in each season for Lincoln, Nebraska, are given below. Write a sinusoidal function that models the temperatures, using $t = 1$ to represent winter. *(Lesson 6-6)*



Normal Daily Temperatures for Lincoln, Nebraska

Winter	Spring	Summer	Fall
36°	61°	86°	65°

Source: Rand McNally & Company

56. Given $A = 43°$, $b = 20$, and $a = 11$, do these measurements determine one triangle, two triangles, or no triangle? *(Lesson 5-7)*

57. **SAT Practice** **Grid-In** If n and m are integers, and $-(n^2) \leq -\sqrt{49}$ and $m = n + 1$, what is the least possible value of mn?

Extra Practice See p. A48.

CONTENTS

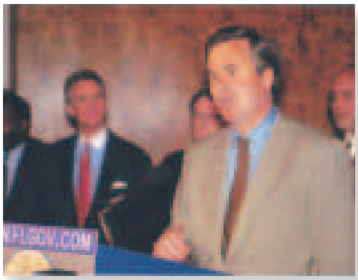
12-3

Infinite Sequences and Series

OBJECTIVES

- Find the limit of the terms of an infinite sequence.
- Find the sum of an infinite geometric series.

ECONOMICS On January 28, 1999, Florida governor Jeb Bush proposed a tax cut that would allow the average family to keep an additional $96. The *marginal propensity to consume (MPC)* is defined as the percentage of a dollar by which consumption increases when income rises by a dollar. Suppose the *MPC* for households and businesses in 1999 was 75%. What would be the total amount of money spent in the economy as a result of just one family's tax savings? *This problem will be solved in Example 5.*


Governor Jeb Bush

Transaction	Expenditure	Terms of Sequence
1	$96(0.75)^1$	72
2	$96(0.75)^2$	54
3	$96(0.75)^3$	40.50
4	$96(0.75)^4$	30.76
5	$96(0.75)^5$	22.78
⋮	⋮	⋮
10	$96(0.75)^{10}$	5.41
⋮	⋮	⋮
100	$96(0.75)^{100}$	3.08×10^{-11}
⋮	⋮	⋮
500	$96(0.75)^{500}$	3.26×10^{-61}
⋮	⋮	⋮
n	$96(0.75)^n$	ar^n

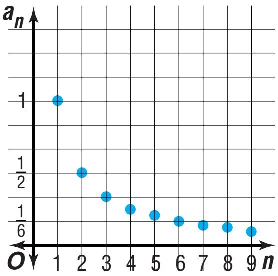
Study the table at the left. Transaction 1 represents the initial expenditure of \$96(0.75) or \$72 by a family. The businesses receiving this money, Transaction 2, would in turn spend 75%, and so on. We can write a geometric sequence to model this situation with $a_1 = 72$ and $r = 0.75$. Thus, the geometric sequence representing this situation is

$$72, 54, 40.50, 30.76, 22.78, \ldots.$$

In theory, the sequence above can have infinitely many terms. Thus, it is called an **infinite sequence.** As n increases, the terms of the sequence decrease and get closer and closer to zero. The terms of the modeling sequence will never actually become zero; however, the terms approach zero as n increases without bound.

Consider the infinite sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots,$ whose nth term, a_n, is $\frac{1}{n}$. Several terms of this sequence are graphed at the right.

Notice that the terms approach a value of 0 as n increases. Zero is called the **limit** of the terms in this sequence.



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This limit can be expressed as follows.

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

∞ is the symbol for infinity.

This is read "the limit of 1 over *n*, as *n* approaches infinity, equals zero."

In fact, when any positive power of *n* appears only in the denominator of a fraction and *n* approaches infinity, the limit equals zero.

$$\lim_{n\to\infty} \frac{1}{n^r} = 0, \text{ for } r > 0$$

If a general expression for the *n*th term of a sequence is known, the limit can usually be found by substituting large values for *n*. Consider the following infinite geometric sequence.

$$7, \frac{7}{4}, \frac{7}{16}, \frac{7}{64}, \frac{7}{256}, \ldots$$

This sequence can be defined by the general expression $a_n = 7\left(\frac{1}{4}\right)^{n-1}$.

$$a_{10} = 7\left(\frac{1}{4}\right)^{10-1} \approx 2.67 \times 10^{-5}$$

$$a_{50} = 7\left(\frac{1}{4}\right)^{50-1} \approx 2.21 \times 10^{-25}$$

$$a_{100} = 7\left(\frac{1}{4}\right)^{100-1} \approx 4.36 \times 10^{-60}$$

Notice that as the value of *n* increases, the value for a_n appears to approach 0, suggesting $\lim\limits_{n\to\infty} 7\left(\frac{1}{4}\right)^{n-1} = 0$.

Example 1 **Estimate the limit of $\frac{9}{5}, \frac{16}{4}, \frac{65}{27}, \ldots, \frac{7n^2+2}{2n^2+3n}, \ldots$.**

The 50th term is $\frac{7(50)^2+2}{2(50)^2+3(50)}$, or about 3.398447.

The 100th term is $\frac{7(100)^2+2}{2(100)^2+3(100)}$, or about 3.448374.

The 500th term is $7\frac{(500)^2+2}{2(500)^2+3(500)}$, or about 3.489535.

The 1000th term is $\frac{7(1000)^2+2}{2(1000)^2+3(1000)}$, or 3.494759.

Notice that as $n \to \infty$, the values appear to approach 3.5, suggesting $\lim\limits_{n\to\infty} \frac{7n^2+2}{2n^2+3n} = 3.5$.

For sequences with more complicated general forms, applications of the following limit theorems, which we will present without proof, can make the limit easier to find.

Theorems for Limits

If the $\lim\limits_{n\to\infty} a_n$ exists, $\lim\limits_{n\to\infty} b_n$ exists, and c is a constant, then the following theorems are true.

Limit of a Sum	$\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$
Limit of a Difference	$\lim\limits_{n\to\infty} (a_n - b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$
Limit of a Product	$\lim\limits_{n\to\infty} a_n \cdot b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$
Limit of a Quotient	$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, where $\lim\limits_{n\to\infty} b_n \neq 0$
Limit of a Constant	$\lim\limits_{n\to\infty} c_n = c$, where $c_n = c$ for each n

The form of the expression for the nth term of a sequence can often be altered to make the limit easier to find.

Example 2 **Find each limit.**

a. $\lim\limits_{n\to\infty} \dfrac{(1 + 3n^2)}{n^2}$

$$\lim_{n\to\infty} \frac{(1 + 3n^2)}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n^2} + 3\right)$$

Rewrite as the sum of two fractions and simplify.

$$= \lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} 3$$

Limit of a Sum

$$= 0 + 3 \text{ or } 3$$

$\lim\limits_{n\to\infty} \dfrac{1}{n^2} = 0$ and $\lim\limits_{n\to\infty} 3 = 3$

Thus, the limit is 3.

Note that the Limit of a Sum theorem only applies here because $\lim\limits_{n\to\infty} \dfrac{1}{n^2}$ and $\lim\limits_{n\to\infty} 3$ each exist.

b. $\lim\limits_{n\to\infty} \dfrac{5n^2 + n - 4}{n^2 + 1}$

The highest power of n in the expression is n^2. Divide each term in the numerator and the denominator by n^2. *Why does doing this produce an equivalent expression?*

$$\lim_{n\to\infty} \frac{5n^2 + n - 4}{n^2 + 1} = \lim_{n\to\infty} \frac{\frac{5n^2}{n^2} + \frac{n}{n^2} - \frac{4}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{5 + \frac{1}{n} - \frac{4}{n^2}}{1 + \frac{1}{n^2}}$$

Simplify.

$$= \frac{\lim\limits_{n\to\infty} 5 + \lim\limits_{n\to\infty} \frac{1}{n} - \lim\limits_{n\to\infty} 4 \cdot \lim\limits_{n\to\infty} \frac{1}{n^2}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n^2}} \qquad \textit{Apply limit theorems.}$$

$$= \frac{5 + 0 - 4 \cdot 0}{1 + 0} \text{ or } 5 \qquad \lim_{n\to\infty} 5 = 5,\ \lim_{n\to\infty} \frac{1}{n} = 0,\ \lim_{n\to\infty} 4 = 4,\ \lim_{n\to\infty} 1 = 1, \text{ and } \lim_{n\to\infty} \frac{1}{n^2} = 0a$$

Thus, the limit is 5.

Limits do not exist for all infinite sequences. If the absolute value of the terms of a sequence becomes arbitrarily great or if the terms do not approach a value, the sequence has no limit. Example 3 illustrates both of these cases.

Example 3 **Find each limit.**

a. $\lim\limits_{n\to\infty} \dfrac{2 + 5n + 4n^2}{2n}$

$$\lim_{n\to\infty} \frac{2 + 5n + 4n^2}{2n} = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{5}{2} + 2n\right) \qquad \textit{Simplify.}$$

Note that $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{5}{2} = \dfrac{5}{2}$, but $2n$ becomes increasingly large as n approaches infinity. Therefore, the sequence has no limit.

b. $\lim\limits_{n\to\infty} \dfrac{(-1)^n n}{8n + 1}$

Begin by rewriting $\dfrac{(-1)^n n}{8n + 1}$ as $(-1)^n \cdot \dfrac{n}{8n + 1}$. Now find $\lim\limits_{n\to\infty} \dfrac{n}{8n + 1}$.

$$\lim_{n\to\infty} \frac{n}{8n + 1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{8n}{n} + \frac{1}{n}} \qquad \textit{Divide the numerator and denominator by } n.$$

$$= \lim_{n\to\infty} \frac{1}{8 + \frac{1}{n}} \qquad \textit{Simplify.}$$

$$= \frac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} 8 + \lim\limits_{n\to\infty} \frac{1}{n}} \qquad \textit{Apply limit theorems.}$$

$$= \frac{1}{8} \qquad \lim_{n\to\infty} 1 = 1,\ \lim_{n\to\infty} 8 = 8, \text{ and } \lim_{n\to\infty} \frac{1}{n} = 0$$

When n is even, $(-1)^n = 1$. When n is odd, $(-1)^n = -1$. Thus, the odd-numbered terms of the sequence described by $\dfrac{(-1)^n n}{8n + 1}$ approach $-\dfrac{1}{8}$, and the even-numbered terms approach $+\dfrac{1}{8}$. Therefore, the sequence has no limit.

An **infinite series** is the indicated sum of the terms of an infinite sequence. Consider the series $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$. Since this is a geometric series, you can find the sum of the first 100 terms by using the formula $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r = \frac{1}{5}$.

$$S_{100} = \frac{\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}}{1 - \frac{1}{5}}$$

$$= \frac{\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}}{\frac{4}{5}}$$

$$= \frac{5}{4}\left[\frac{1}{5} - \frac{1}{5}\left(\frac{1}{5}\right)^{100}\right] \text{ or } \frac{1}{4} - \frac{1}{4}\left(\frac{1}{5}\right)^{100}$$

Since $\left(\frac{1}{5}\right)^{100}$ is very close to 0, S_{100} is nearly equal to $\frac{1}{4}$. No matter how many terms are added, the sum of the infinite series will never exceed $\frac{1}{4}$, and the difference from $\frac{1}{4}$ gets smaller as $n \to \infty$. Thus, $\frac{1}{4}$ is the sum of the infinite series.

Sum of an Infinite Series

If S_n is the sum of the first n terms of a series, and S is a number such that $S - S_n$ approaches zero as n increases without bound, then the sum of the infinite series is S.

$$\lim_{n \to \infty} S_n = S$$

If the sequence of partial sums S_n has a limit, then the corresponding infinite series has a sum, and the nth term a_n of the series approaches 0 as $n \to \infty$. If $\lim_{n \to \infty} a_n \neq 0$, the series has no sum. If $\lim_{n \to \infty} a_n = 0$, the series may or may not have a sum.

The formula for the sum of the first n terms of a geometric series can be written as follows.

$$S_n = a_1 \frac{(1 - r^n)}{1 - r}, r \neq 1$$

Recall that $|r| < 1$ means $-1 < r < 1$.

Suppose $n \to \infty$; that is, the number of terms increases without limit. If $|r| > 1$, r^n increases without limit as $n \to \infty$. However, when $|r| < 1$, r^n approaches 0 as $n \to \infty$. Under this condition, the above formula for S_n approaches a value of $\frac{a_1}{1 - r}$.

Sum of an Infinite Geometric Series

The sum S of an infinite geometric series for which $|r| < 1$ is given by

$$S = \frac{a_1}{1 - r}.$$

Example 4 **Find the sum of the series $21 - 3 + \frac{3}{7} - \cdots$.**

In the series, $a_1 = 21$ and $r = -\frac{1}{7}$. Since $|r| < 1$, $S = \frac{a_1}{1 - r}$.

$$S = \frac{a_1}{1 - r}$$

$$= \frac{21}{1 - \left(-\frac{1}{7}\right)} \qquad a_1 = 21,\ r = -\frac{1}{7}$$

$$= \frac{147}{8} \text{ or } 18\frac{3}{8}$$

The sum of the series is $18\frac{3}{8}$.

In economics, finding the sum of an infinite series is useful in determining the overall effect of economic trends.

Example 5 **ECONOMICS** **Refer to the application at the beginning of the lesson. What would be the total amount of money spent in the economy as a result of just one family's tax savings?**

Real World Application

For the geometric series modeling this situation, $a_1 = 72$ and $r = 0.75$.

Since $|r| < 1$, the sum of the series is equal to $\frac{a_1}{1 - r}$.

$$S = \frac{a_1}{1 - r}$$

$$= \frac{72}{1 - 0.75} \text{ or } 288$$

Therefore, the total amount of money spent is \$288.

You can use what you know about infinite series to write repeating decimals as fractions. The first step is to write the repeating decimal as an infinite geometric series.

Example 6 **Write $0.\overline{762}$ as a fraction.**

$$0.\overline{762} = \frac{762}{1000} + \frac{762}{1{,}000{,}000} + \frac{762}{1{,}000{,}000{,}000} + \cdots$$

In this series, $a_1 = \frac{762}{1000}$ and $r = \frac{1}{1000}$.

(continued on the next page)

$$S = \frac{a_1}{1 - r}$$

$$= \frac{\frac{762}{1000}}{1 - \frac{1}{1000}}$$

$$= \frac{762}{999} \text{ or } \frac{254}{333}$$

Thus, $0.762762 \cdots = \frac{254}{333}$. *Check this with a calculator.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. Consider the sequence given by the general expression $a_n = \frac{n - 1}{n}$.

- **a. Graph** the first ten terms of the sequence with the term number on the x-axis and the value of the term on the y-axis.
- **b. Describe** what happens to the value of a_n as n increases.
- **c. Make a conjecture** based on your observation in part **a** as to the limit of the sequence as n approaches infinity.
- **d. Apply** the techniques presented in the lesson to evaluate $\lim\limits_{n \to \infty} \frac{n - 1}{n}$. How does your answer compare to your conjecture made in part **c**?

2. Consider the infinite geometric sequence given by the general expression r^n.

- **a. Determine** the limit of the sequence for $r = \frac{1}{2}$, $r = \frac{1}{4}$, $r = 1$, $r = 2$, and $r = 5$.
- **b. Write** a general rule for the limit of the sequence, placing restrictions on the value of r.

3. Give an example of an infinite geometric series having no sum.

4. You Decide Tyree and Zonta disagree on whether the infinite sequence described by the general expression $2n - 3$ has a limit. Tyree says that after dividing by the highest-powered term, the expression simplifies to $2 - \frac{3}{n}$, which has a limit of 2 as n approaches infinity. Zonta says that the sequence has no limit. Who is correct? Explain.

Guided Practice

Find each limit, or state that the limit does not exist and explain your reasoning.

5. $\lim\limits_{n \to \infty} \frac{1}{5^n}$

6. $\lim\limits_{n \to \infty} \frac{5 - n^2}{2n}$

7. $\lim\limits_{n \to \infty} \frac{3n - 6}{7n}$

Write each repeating decimal as a fraction.

8. $0.\overline{7}$

9. $5.\overline{126}$

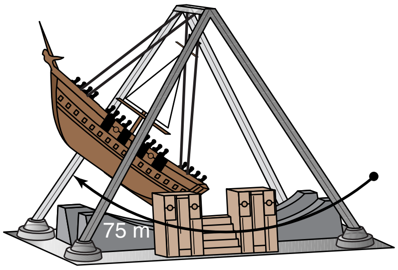
Find the sum of each infinite series, or state that the sum does not exist and explain your reasoning.

10. $-6 + 3 - \frac{3}{2} + \cdots$

11. $\frac{3}{4} + \frac{1}{4} + \frac{1}{12} + \cdots$

12. $\sqrt{3} + 3 + \sqrt{27} + \cdots$

13. Entertainment Pete's Pirate Ride operates like the bob of a pendulum. On its longest swing, the ship travels through an arc 75 meters long. Each successive swing is two-fifths the length of the preceding swing. If the ride is allowed to continue without intervention, what is the total distance the ship will travel before coming to rest?



EXERCISES

Practice

Find each limit, or state that the limit does not exist and explain your reasoning.

14. $\lim\limits_{n \to \infty} \frac{7 - 2n}{5n}$

15. $\lim\limits_{n \to \infty} \frac{n^3 - 2}{n^2}$

16. $\lim\limits_{n \to \infty} \frac{6n^2 + 5}{3n^2}$

17. $\lim\limits_{n \to \infty} \frac{9n^3 + 5n - 2}{2n^3}$

18. $\lim\limits_{n \to \infty} \frac{(3n + 4)(1 - n)}{n^2}$

19. $\lim\limits_{n \to \infty} \frac{8n^2 + 5n + 2}{3 + 2n}$

20. $\lim\limits_{n \to \infty} \frac{4 - 3n + n^2}{2n^3 - 3n^2 + 5}$

21. $\lim\limits_{n \to \infty} \frac{n}{3^n}$

22. $\lim\limits_{n \to \infty} \frac{(-2)^n n}{4 + n}$

23. Find the limit of the sequence described by the general expression $\frac{5n + (-1)^n}{n^2}$, or state that the limit does not exist. Explain your reasoning.

Write each repeating decimal as a fraction.

24. $0.\overline{4}$

25. $0.\overline{51}$

26. $0.\overline{370}$

27. $6.\overline{259}$

28. $0.\overline{15}$

29. $0.\overline{263}$

30. Explain why the sum of the series $0.2 + 0.02 + 0.002 + \cdots$ exists. Then find the sum.

Find the sum of each series, or state that the sum does not exist and explain your reasoning.

31. $16 + 12 + 9 + \cdots$

32. $5 + 7.5 + 11.25 + \cdots$

33. $10 + 5 + 2.5 + \cdots$

34. $6 + 5 + 4 + \cdots$

35. $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \cdots$

36. $-\frac{2}{3} + \frac{1}{9} - \frac{1}{54} + \cdots$

37. $\frac{6}{5} + \frac{4}{5} + \frac{8}{15} + \cdots$

38. $\sqrt{5} + 1 + \frac{\sqrt{5}}{5} + \cdots$

39. $8 - 4\sqrt{3} + 6 - \cdots$

www.amc.glencoe.com/self_check_quiz

Applications and Problem Solving

40. Physics A basketball is dropped from a height of 35 meters and bounces $\frac{2}{5}$ of the distance after each fall.

a. Find the first five terms of the infinite series representing the vertical distance traveled by the ball.

b. What is the total vertical distance the ball travels before coming to rest? (*Hint:* Rewrite the series found in part **a** as the sum of two infinite geometric series.)

41. Critical Thinking Consider the sequence whose nth term is described by $\frac{n^2}{2n+1} - \frac{n^2}{2n-1}$.

a. Explain why $\lim\limits_{n\to\infty}\left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right) \neq \lim\limits_{n\to\infty}\frac{n^2}{2n+1} - \lim\limits_{n\to\infty}\frac{n^2}{2n-1}$.

b. Find $\lim\limits_{n\to\infty}\left(\frac{n^2}{2n+1} - \frac{n^2}{2n-1}\right)$.

42. Engineering Francisco designs a toy with a rotary flywheel that rotates at a maximum speed of 170 revolutions per minute. Suppose the flywheel is operating at its maximum speed for one minute and then the power supply to the toy is turned off. Each subsequent minute thereafter, the flywheel rotates two-fifths as many times as in the preceding minute. How many complete revolutions will the flywheel make before coming to a stop?

43. Critical Thinking Does $\lim\limits_{n\to\infty}\cos\frac{n\pi}{2}$ exist? Explain.

44. Medicine A certain drug developed to fight cancer has a half-life of about 2 hours in the bloodstream. The drug is formulated to be administered in doses of D milligrams every 6 hours. The amount of each dose has yet to be determined.

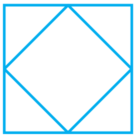
a. What fraction of the first dose will be left in the bloodstream before the second dose is administered?

b. Write a general expression for the geometric series that models the number of milligrams of drug left in the bloodstream after the nth dose.

c. About what amount of medicine is present in the bloodstream for large values of n?

d. A level of more than 350 milligrams of this drug in the bloodstream is considered toxic. Find the largest possible dose that can be given repeatedly over a long period of time without harming the patient.

45. Geometry If the midpoints of a square are joined by straight lines, the new figure will also be a square.

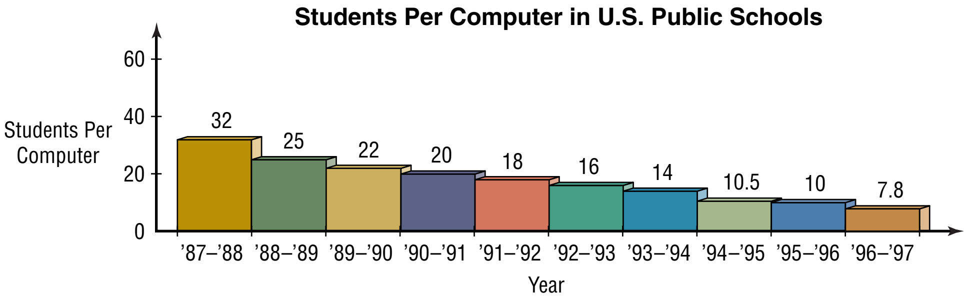


a. If the original square has a perimeter of 20 feet, find the perimeter of the new square. (*Hint:* Use the Pythagorean Theorem.)

b. If this process is continued to form a sequence of "nested" squares, what will be the sum of the perimeters of all the squares?

CONTENTS

46. Technology Since the mid-1980s, the number of computers in schools has steadily increased. The graph below shows the corresponding decline in the student-computer ratio.



Source: *QED's Technology in Public Schools,* 16th Edition

Another publication states that the average number of students per computer in U.S. public schools can be estimated by the sequence model $a_n = 35.812791(0.864605)^n$, *for* $n = 1, 2, 3, \ldots$, with the 1987-1988 school year corresponding to $n = 1$.

a. Find the first ten terms of the model. Round your answers to the nearest tenth.

b. Use the model to estimate the average number of students having to share a computer during the 1995-1996 school year. How does this estimate compare to the actual data given in the graph?

c. Make a prediction as to the average number of students per computer for the 2000-2001 school year.

d. Does this sequence approach a limit? If so, what is the limit?

e. Realistically, will the student computer ratio ever reach this limit? Explain.

Mixed Review

47. The first term of a geometric sequence is -3, and the common ratio is $\frac{2}{3}$. Find the next four terms of the sequence. *(Lesson 12-2)*

48. Find the 16th term of the arithmetic sequence for which $a_1 = 1.5$ and $d = 0.5$. *(Lesson 12-1)*

49. Name the coordinates of the center, foci, and vertices, and the equation of the asymptotes of the hyperbola that has the equation $x^2 - 4y^2 - 12x - 16y = -16$. *(Lesson 10-4)*

50. Graph $r = 6 \cos 3\theta$. *(Lesson 9-2)*

51. Navigation A ship leaving port sails for 125 miles in a direction 20° north of due east. Find the magnitude of the vertical and horizontal components. *(Lesson 8-1)*

52. Use a half-angle identity to find the exact value of $\cos 112.5°$. *(Lesson 7-4)*

53. Graph $y = \cos x$ on the interval $-180° \leq x \leq 360°$. *(Lesson 6-3)*

54. List all possible rational zeros of the function $f(x) = 8x^3 + 3x - 2$. *(Lesson 4-4)*

55. SAT/ACT Practice If $a = 4b + 26$, and b is a positive integer, then a could be divisible by all of the following EXCEPT

A 2 **B** 4 **C** 5 **D** 6 **E** 7

Extra Practice See p. A49.

GRAPHING CALCULATOR EXPLORATION

12-3B Continued Fractions

An Extension of Lesson 12-3

OBJECTIVE

- Explore sequences generated by continued fractions.

An expression of the following form is called a *continued fraction*.

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}$$

By using only a finite number of "decks" and values of a_n and b_n that follow regular patterns, you can often obtain a sequence of terms that approaches a limit, which can be represented by a simple expression. For example, if all of the numbers a_n and b_n are equal to 1, then the continued fraction gives rise to the following sequence.

$$1,\ 1 + \frac{1}{1},\ 1 + \cfrac{1}{1 + \cfrac{1}{1}},\ 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}},\ \ldots$$

It can be shown that the terms of this sequence approach the limit $\frac{1 + \sqrt{5}}{2}$. This number is often called the *golden ratio*.

The golden ratio is closely related to the Fibonacci sequence, which you will learn about in Lesson 12-7.

Now consider the following more general sequence.

$$A,\ A + \frac{1}{A},\ A + \cfrac{1}{A + \cfrac{1}{A}},\ A + \cfrac{1}{A + \cfrac{1}{A + \cfrac{1}{A}}},\ \ldots$$

To help you visualize what this sequence represents, suppose $A = 5$. The sequence becomes $5,\ 5 + \frac{1}{5},\ 5 + \cfrac{1}{5 + \cfrac{1}{5}},\ 5 + \cfrac{1}{5 + \cfrac{1}{5 + \cfrac{1}{5}}},\ \ldots$ or $5,\ \frac{26}{5},\ \frac{135}{26},\ \frac{701}{135},\ \ldots$.

A calculator approximation of this sequence is 5, 5.2, 5.192307692, 5,192592593, … .

Each term of the sequence is the sum of A and the reciprocal of the previous term. The program at the right calculates the value of the nth term of the above sequence for $n \geq 3$ and a specified value of A.

When you run the program it will ask you to input values for A and N.

```
PROGRAM: CFRAC
: Prompt A
: Disp "INPUT TERM"
: Disp "NUMBER N, N ≥ 3"
: Prompt N
: 1 → K
: A + 1/A → C
: Lbl 1
: A + 1/C → C
: K + 1 → K
: If K < N – 1
: Then: Goto 1
: Else: Disp C
```

*inter*NET CONNECTION

Graphing Calculator Programs
To download this graphing calculator program, visit our website at **www.amc.glencoe.com**

CONTENTS

TRY THESE

Enter the program into your calculator and use it for the exercises that follow.

1. What output is given when the program is executed for $A = 1$ and $N = 10$?

2. With $A = 1$, determine the least value of N necessary to obtain an output that agrees with the calculator's nine decimal approximation of $\frac{1 + \sqrt{5}}{2}$.

3. Use algebra to show that the continued fraction $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$ has a value of $\frac{1 + \sqrt{5}}{2}$. (*Hint:* If $x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$, then $x = 1 + \frac{1}{x}$. Solve this last equation for x.)

4. Find the exact value of $3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}$.

5. Execute the program with $A = 3$ and $N = 40$. How does this output compare to the decimal approximation of the expression found in Exercise 4?

6. Find a radical expression for $A + \cfrac{1}{A + \cfrac{1}{A + \cdots}}$.

7. Write a modified version of the program that calculates the nth term of the following sequence for $n \geq 3$.

$$A,\ A + \frac{B}{2A},\ A + \cfrac{B}{2A + \cfrac{B}{2A}},\ A + \cfrac{B}{2A + \cfrac{B}{2A + \cfrac{B}{2A}}},\ \ldots$$

8. Choose several positive integer values for A and B and compare the program output with the decimal approximation of $\sqrt{A^2 + B}$ for several values of n, for $n \geq 3$. Describe your observations.

9. Use algebra to show that for $A > 0$ and $B > 0$, $A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}$ has a value of $\sqrt{A^2 + B}$.

$$\left(\textit{Hint: } \text{If } x = A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}, \text{ then } x + A = 2A + \cfrac{B}{2A + \cfrac{B}{2A + \cdots}}.\right)$$

WHAT DO YOU THINK?

10. If you execute the original program for $A = 1$ and $N = 20$ and then execute it for $A = -1$ and $N = 20$, how will the two outputs compare?

11. What values can you use for A and B in the program for Exercise 7 in order to approximate $\sqrt{15}$?

CONTENTS

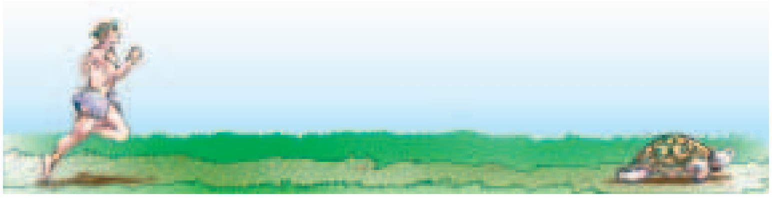
12-4

Convergent and Divergent Series

OBJECTIVE

- Determine whether a series is convergent or divergent.

HISTORY The Greek philosopher Zeno of Elea (c. 490–430 B.C.) proposed several perplexing riddles, or paradoxes. One of Zeno's paradoxes involves a race on a 100-meter track between the mythological Achilles and a tortoise. Zeno claims that even though Achilles can run twice as fast as the tortoise, if the tortoise is given a 10-meter head start, Achilles will never catch him. Suppose Achilles runs 10 meters per second and the tortoise a remarkable 5 meters per second. By the time Achilles has reached the 10-meter mark, the tortoise will be at 15 meters. By the time Achilles reaches the 15-meter mark, the tortoise will be at 17.5 meters, and so on. Thus, Achilles is always behind the tortoise and never catches up.



Is Zeno correct? Let us look at the distance between Achilles and the tortoise after specified amounts of time have passed. Notice that the distance between the two contestants will be zero as n approaches infinity since $\lim\limits_{n\to\infty} \frac{10}{2^n} = 0$.

Time (seconds)	Distance Apart (meters)
0	10
1	$\frac{10}{2} = 5$
$1 + \frac{1}{2} = \frac{3}{2}$	$\frac{10}{4} = 2.5$
$1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$	$\frac{10}{8} = 1.25$
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$ ⋮	$\frac{10}{16} = 0.625$ ⋮
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$	$\frac{10}{2^n}$

To disprove Zeno's conclusion that Achilles will never catch up to the tortoise, we must show that there is a time value for which this 0 difference can be achieved. In other words, we need to show that the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has a sum, or limit. *This problem will be solved in Example 5.*

Starting with a time of 1 second, the partial sums of the time series form the sequence $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{18}, \ldots$. As the number of terms used for the partial sums increases, the value of the partial sums also increases. If this sequence of partial sums approaches a limit, the related infinite series is said to **converge.** If this sequence of partial sums does not have a limit, then the related infinite series is said to **diverge.**

Convergent and Divergent Series	If an infinite series has a sum, or limit, the series is convergent. If a series is not convergent, it is divergent.

Example 1 **Determine whether each arithmetic or geometric series is convergent or divergent.**

There are many series that begin with the first few terms shown in this example. In this chapter, always assume that the expression for the general term is the simplest one possible.

a. $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$

This is a geometric series with $r = -\frac{1}{2}$. Since $|r| < 1$, the series has a limit. Therefore, the series is convergent.

b. $2 + 4 + 8 + 16 + \cdots$

This is a geometric series with $r = 2$. Since $|r| > 1$, the series has no limit. Therefore, the series is divergent.

c. $10 + 8.5 + 7 + 5.5 + \cdots$

This is an arithmetic series with $d = -1.5$. Arithmetic series do not have limits. Therefore, the series is divergent.

When a series is neither arithmetic nor geometric, it is more difficult to determine whether the series is convergent or divergent. Several different techniques can be used. One test for convergence is the **ratio test.** This test can only be used when all terms of a series are positive. The test depends upon the ratio of consecutive terms of a series, which must be expressed in general form.

Ratio Test	Let a_n and a_{n+1} represent two consecutive terms of a series of positive terms. Suppose $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists and that $r = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$. The series is convergent if $r < 1$ and divergent if $r > 1$. If $r = 1$, the test provides no information.

The ratio test is especially useful when the general form for the terms of a series contains powers.

Example 2 **Use the ratio test to determine whether each series is convergent or divergent.**

a. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$

First, find a_n and a_{n+1}. $\quad a_n = \frac{n}{2^n}$ and $a_{n+1} = \frac{n+1}{2^{n+1}}$

Then use the ratio test. $\quad r = \lim\limits_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}$

(continued on the next page)

$$r = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$$

Multiply by the reciprocal of the divisor.

$$r = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n}$$

$\frac{2^n}{2^{n+1}} = \frac{1}{2}$

$$r = \lim_{n\to\infty} \frac{1}{2} \cdot \lim_{n\to\infty} \frac{n+1}{n}$$

Limit of a Product

$$r = \frac{1}{2} \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{1}$$

Divide by the highest power of n and then apply limit theorems.

$$r = \frac{1}{2} \cdot \frac{1+0}{1} \text{ or } \frac{1}{2}$$

Since $r < 1$, the series is convergent.

b. $\frac{\mathbf{1}}{\mathbf{2}} + \frac{\mathbf{2}}{\mathbf{3}} + \frac{\mathbf{3}}{\mathbf{4}} + \frac{\mathbf{4}}{\mathbf{5}} + \cdots$

$$a_n = \frac{n}{n+1} \quad \text{and} \quad a_{n+1} = \frac{n+1}{(n+1)+1} \text{ or } \frac{n+1}{n+2}$$

$$r = \lim_{n\to\infty} \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}}$$

$$r = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n}$$

$\frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2+2n+1}{n^2+2n}$

$$r = \lim_{n\to\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}}$$

Divide by the highest power of n and apply limit theorems.

$$r = \frac{1+0+0}{1+0} \text{ or } 1$$

Since $r = 1$, the test provides no information.

The ratio test is also useful when the general form of the terms of a series contains products of consecutive integers.

Example ③ Use the ratio test to determine whether the series

$$\mathbf{1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \cdots}$$ **is convergent or divergent.**

First find the nth term and $(n + 1)$th term. Then, use the ratio test.

$$a_n = \frac{1}{1 \cdot 2 \cdot \cdots \cdot n} \quad \text{and} \quad a_{n+1} = \frac{1}{1 \cdot 2 \cdot \cdots \cdot (n+1)}$$

$$r = \lim_{n\to\infty} \frac{\frac{1}{1 \cdot 2 \cdot \cdots \cdot (n+1)}}{\frac{1}{1 \cdot 2 \cdot \cdots \cdot n}}$$

$$r = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot (n+1)}$$

Note that $1 \cdot 2 \cdot \cdots \cdot (n+1) = 1 \cdot 2 \cdot \cdots \cdot n \cdot (n+1)$.

$$r = \lim_{n\to\infty} \frac{1}{n+1} \text{ or } 0$$

Simplify and apply limit theorems.

Since $r < 1$, the series is convergent.

When the ratio test does not determine if a series is convergent or divergent, other methods must be used.

Example 4 **Determine whether the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is convergent or divergent.**

Suppose the terms are grouped as follows. Beginning after the second term, the number of terms in each successive group is doubled.

$$(1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

Notice that the first enclosed expression is greater than $\frac{1}{2}$, and the second is equal to $\frac{1}{2}$. Beginning with the third expression, each sum of enclosed terms is greater than $\frac{1}{2}$. Since there are an unlimited number of such expressions, the sum of the series is unlimited. Thus, the series is divergent.

A series can be compared to other series that are known to be convergent or divergent. The following list of series can be used for reference.

Summary of Series for Reference

1. **Convergent:** $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} + \cdots, |r| < 1$
2. **Divergent:** $a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} + \cdots, |r| > 1$
3. **Divergent:** $a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots$
4. **Divergent:** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots$ *This series is known as the harmonic series.*
5. **Convergent:** $1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, p > 1$

If a series has all positive terms, the **comparison test** can be used to determine whether the series is convergent or divergent.

Comparison Test

- A series of positive terms is convergent if, for $n > 1$, each term of the series is equal to or less than the value of the corresponding term of some convergent series of positive terms.
- A series of positive terms is divergent if, for $n > 1$, each term of the series is equal to or greater than the value of the corresponding term of some divergent series of positive terms.

Example 5 **Use the comparison test to determine whether the following series are convergent or divergent.**

a. $\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \cdots$

The general term of this series is $\frac{4}{2n + 3}$. The general term of the divergent series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is $\frac{1}{n}$. Since for all $n > 1$, $\frac{4}{2n + 3} > \frac{1}{n}$, the series $\frac{4}{5} + \frac{4}{7} + \frac{4}{9} + \frac{4}{11} + \cdots$ is also divergent.

b. $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

The general term of the series is $\frac{1}{(2n-1)^2}$. The general term of the convergent series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is $\frac{1}{n^2}$. Since $\frac{1}{(2n-1)^2} \leq \frac{1}{n^2}$ for all n, the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$ is also convergent.

With a better understanding of convergent and divergent infinite series, we are now ready to tackle Zeno's paradox.

Example 6 Real World Application

HISTORY **Refer to the application at the beginning of the lesson. To disprove Zeno's conclusion that Achilles will never catch up to the tortoise, we must show that the infinite time series $1 + 0.5 + 0.25 + \cdots$ has a limit.**

To show that the series $1 + 0.5 + 0.25 + \cdots$ has a limit, we need to show that the series is convergent.

The general term of this series is $\frac{1}{2^n}$. Try using the ratio test for convergence of a series.

$$a_n = \frac{1}{2^n} \quad \text{and} \quad a_{n+1} = \frac{1}{2^{n+1}}$$

$$r = \frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}$$

$$= \frac{1}{2} \qquad \frac{1}{2^{n+1}} \cdot \frac{2^n}{1} = \frac{1}{2}$$

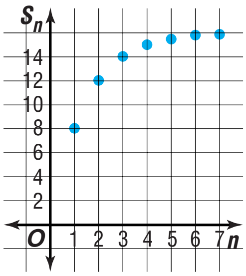
Since $r < 1$, the series converges and therefore has a sum. Thus, there is a time value for which the distance between Achilles and the tortoise will be zero. *You will determine how long it takes him to do so in Exercise 34.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. a. **Write** an example, of an infinite geometric series in which $|r| > 1$.
   b. **Determine** the 25th, 50th, and 100th terms of your series.
   c. **Identify** the sum of the first 25, 50, and 100 terms of your series.
   d. **Explain** why this type of infinite geometric series does not converge.
2. **Estimate** the sum S_n of the series whose partial sums are graphed at the right.



3. Consider the infinite series $\frac{1}{3} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \frac{4^2}{3^4} + \cdots$.

a. **Sketch** a graph of the first eight partial sums of this series.

b. **Make** a conjecture based on the graph found in part **a** as to whether the series is convergent or divergent.

c. **Determine** a general term for this series.

d. **Write a convincing argument** using the general term found in part **c** to support the conjecture you made in part **b**.

4. ***Math Journal*** **Make a list** of the methods presented in this lesson and in the previous lesson for determining convergence or divergence of an infinite series. Be sure to indicate any restrictions on a method's use. Then number your list as to the order in which these methods should be applied.

Guided Practice

Use the ratio test to determine whether each series is *convergent* or *divergent.*

5. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots$

6. $\frac{3}{4} + \frac{7}{8} + \frac{11}{12} + \frac{15}{16} + \cdots$

7. Use the comparison test to determine whether the series $\frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots$ is *convergent* or *divergent.*

Determine whether each series is *convergent* or *divergent.*

8. $\frac{1}{4} + \frac{5}{16} + \frac{3}{8} + \frac{7}{16} + \cdots$

9. $\frac{1}{2 + 1^2} + \frac{1}{2 + 2^2} + \frac{1}{2 + 3^2} + \cdots$

10. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots$

11. $4 + 3 + \frac{9}{4} + \cdots$

12. Ecology An underground storage container is leaking a toxic chemical. One year after the leak began, the chemical had spread 1500 meters from its source. After two years, the chemical had spread 900 meters more, and by the end of the third year, it had reached an additional 540 meters.

a. If this pattern continues, how far will the spill have spread from its source after 10 years?

b. Will the spill ever reach the grounds of a school located 4000 meters away from the source? Explain.

EXERCISES

Practice

Use the ratio test to determine whether each series is *convergent* or *divergent.*

13. $\frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \frac{4}{81} + \cdots$

14. $\frac{2}{5} + \frac{4}{10} + \frac{8}{15} + \cdots$

15. $2 + \frac{4}{2^2} + \frac{8}{3^2} + \frac{16}{4^2} + \cdots$

16. $\frac{2}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{2}{4 \cdot 5} + \cdots$

17. $1 + \frac{3}{1 \cdot 2 \cdot 3} + \frac{5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots$

18. $5 + \frac{5^2}{1 \cdot 2} + \frac{5^3}{1 \cdot 2 \cdot 3} + \cdots$

19. Use the ratio test to determine whether the series $\frac{2 \cdot 4}{2} + \frac{4 \cdot 6}{4} + \frac{6 \cdot 8}{8} + \frac{8 \cdot 10}{16} + \cdots$ is convergent or divergent.

Use the comparison test to determine whether each series is *convergent* or *divergent.*

20. $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$

21. $\frac{1}{2} + \frac{1}{9} + \frac{1}{28} + \frac{1}{65} + \cdots$

22. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

23. $\frac{5}{3} + \frac{5}{4} + 1 + \frac{5}{6} + \cdots$

24. Use the comparison test to determine whether the series $\frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17} \cdots$ is *convergent* or *divergent*.

Determine whether each series is *convergent* or *divergent.*

25. $\frac{1}{2} - \frac{3}{8} + \frac{9}{32} - \cdots$

26. $3 + \frac{5}{3} + \frac{7}{5} + \cdots$

27. $\frac{1}{5 + 1^2} + \frac{1}{5 + 2^2} + \frac{1}{5 + 3^2} + \cdots$

28. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$

29. $\frac{4\pi}{3} + \frac{5\pi}{6} + \frac{\pi}{3} + \cdots$

30. $\frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \frac{7}{32} + \cdots$

Applications and Problem Solving

31. Economics The MagicSoft software company has a proposal to the city council of Alva, Florida, to relocate there. The proposal claims that the company will generate \$3.3 million for the local economy by the \$1 million in salaries that will be paid. The city council estimates that 70% of the salaries will be spent in the local community, and 70% of that money will again be spent in the community, and so on.

a. According to the city council's estimates, is the claim made by MagicSoft accurate? Explain.

b. What is the correct estimate of the amount generated to the local economy?

32. Critical Thinking Give an example of a series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ that diverges, but when its terms are squared, the resulting series $a_1{}^2 + a_2{}^2 + a_3{}^2 + \cdots + a_n{}^2 + \cdots$ converges.

33. Cellular Growth Leticia Cox is a biochemist. She is testing two different types of drugs that induce cell growth. She has selected two cultures of 1000 cells each. To culture A, she administers a drug that raises the number of cells by 200 each day and every day thereafter. Culture B gets a drug that increases cell growth by 8% each day and everyday thereafter.

a. Assuming no cells die, how many cells will have grown in each culture by the end of the seventh day?

b. At the end of one month's time, which drug will prove to be more effective in promoting cell growth? Explain.

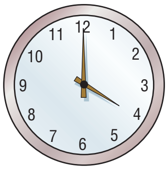
34. Critical Thinking Refer to Example 6 of this lesson. The sequence of partial sums, $S_1, S_2, S_3, \ldots, S_n, \ldots$, for the time series is $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$.

a. Find a general expression for the nth term of this sequence.

b. To determine how long it takes for Achilles to catch-up to the tortoise, find the sum of the infinite time series. (*Hint:* Recall from the definition of the sum S of an infinite series that $\lim_{n \to \infty} S_n = S$.)

CONTENTS

35. Clocks The hour and minute hands of a clock travel around its face at different speeds, but at certain times of the day, the two hands coincide. In addition to noon and midnight, the hands also coincide at times occurring between the hours. According the figure at the right, it is 4:00.



a. When the minute hand points to 4, what fraction of the distance between 4 and 5 will the hour hand have traveled?

b. When the minute hand reaches the hour hand's new position, what additional fraction will the hour hand have traveled?

c. List the next two terms of this series representing the distance traveled by the hour hand as the minute hand "chases" its position.

d. At what time between the hours of 4 and 5 o'clock will the two hands coincide?

Mixed Review

36. Evaluate $\lim_{n \to \infty} \frac{4n^2 + 5}{3n^2 - 2n}$. *(Lesson 12-3)*

37. Find the ninth term of the geometric sequence $\sqrt{2}, 2, 2\sqrt{2}, \ldots$. *(Lesson 12-2)*

38. Form an arithmetic sequence that has five arithmetic means between -11 and 19. *(Lesson 12-1)*

39. Solve $45.9 = e^{0.075t}$ *(Lesson 11-6)*

40. Navigation A submarine sonar is tracking a ship. The path of the ship is recorded as $6 = 12r \cos(\theta - 30°)$. Find the linear equation of the path of the ship. *(Lesson 9-4)*

41. Find an ordered pair that represents $\overrightarrow{AB}$ for $A(8, -3)$ and $B(5, -1)$. *(Lesson 8-2)*

42. SAT/ACT Practice How many numbers from 1 to 200 inclusive are equal to the cube of an integer?

A one **B** two **C** three **D** four **E** five

MID-CHAPTER QUIZ

1. Find the 19th term in the sequence for which $a_1 = 11$ and $d = -2$. (Lesson 12-1)

2. Find S_{20} for the arithmetic series for which $a_1 = -14$ and $d = 6$. (Lesson 12-1)

3. Form a sequence that has two geometric means between 56 and 189. (Lesson 12-2)

4. Find the sum of the first eight terms of the series $3 - 6 + 12 - \cdots$. (Lesson 12-2)

5. Find $\lim_{n \to \infty} \frac{n^2 + 2n - 5}{n^2 - 1}$ or explain why the limit does not exist. (Lesson 12-3)

6. Recreation A bungee jumper rebounds 55% of the height jumped. If a bungee jump is made using a cord that stretches 250 feet, find the total distance traveled by the jumper before coming to rest. (Lesson 12-3)

7. Find the sum of the following series.
$\frac{1}{25} + \frac{1}{250} + \frac{1}{2500} + \cdots$. (Lesson 12-3)

Determine whether each series is *convergent* or *divergent*. (Lesson 12-4)

8. $\frac{1}{10} + \frac{2}{100} + \frac{6}{1000} + \frac{24}{10{,}000} + \cdots$

9. $\frac{6}{5} + \frac{2}{5} + \frac{2}{15} + \cdots$

10. Finance Ms. Fuentes invests \$500 quarterly (January 1, April 1, July 1, and October 1) in a retirement account that pays an APR of 12% compounded quarterly. Interest for each quarter is posted on the last day of the quarter. Determine the value of her investment at the end of the year. (Lesson 12-2)

12-5

Sigma Notation and the *n*th Term

OBJECTIVE

- Use sigma notation.

MANUFACTURING Manufacturers are required by the Environmental Protection Agency to meet certain emission standards. If these standards are not met by a preassigned date, the manufacturer is fined. To encourage swift compliance, the fine increases a specified amount each day until the manufacturer is able to pass inspection. Suppose a manufacturing plant is charged \$2000 for not meeting its January 1st deadline. The next day it is charged \$2500, the next day \$3000, and so on, until it passes inspection on January 21st. What is the total amount of fines owed by the manufacturing plant? *This problem will be solved in Example 2.*

In mathematics, the uppercase Greek letter sigma, Σ, is often used to indicate a sum or series. A series like the one indicated above may be written using **sigma notation.**

$$\begin{array}{r}\textit{maximum value of } n \longrightarrow \\ \textit{starting value of } n \longrightarrow\end{array} \sum_{n=1}^{k} a_n \longleftarrow \textit{expression for general term}$$

The variable *n* used with the sigma notation is called the **index of summation.**

Other variables besides n may be used for the index of summation.

Sigma Notation of a Series

For any sequence a_1, a_2, a_3, …, the sum of the first *k* terms may be written $\sum_{n=1}^{k} a_n$, which is read "the summation from $n = 1$ to k of a_n." Thus,

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k, \text{ where } k \text{ is an integer value.}$$

Example 1

Write each expression in expanded form and then find the sum.

a. $\sum_{n=1}^{4} (n^2 - 3)$

First, write the expression in expanded form.

$$\sum_{n=1}^{4} (n^2 - 3) = \overset{n\,=\,1}{(1^2 - 3)} + \overset{n\,=\,2}{(2^2 - 3)} + \overset{n\,=\,3}{(3^2 - 3)} + \overset{n\,=\,4}{(4^2 - 3)}$$

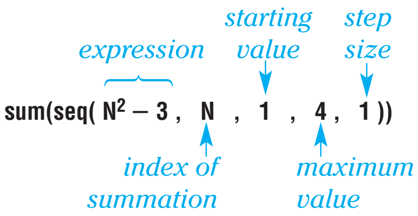
Now find the sum.

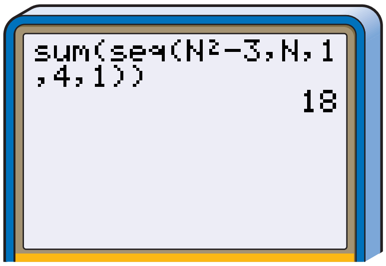
Method 1 Simplify the expanded form.

$$(1^2 - 3) + (2^2 - 3) + (3^2 - 3) + (4^2 - 3) = -2 + 1 + 6 + 13 \text{ or } 18$$

Method 2 Use a graphing calculator.

You can combine **sum(** with **seq(** to obtain the sum of a finite sequence. In the **LIST** menu, select the **MATH** option to locate the **sum(** command. The **seq(** command can be found in the **OPS** option of the **LIST** menu.





```
sum(seq(N²-3,N,1
,4,1))
                18
```

b. $\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1}$

$$\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1} = \overset{n=1}{5\left(-\frac{2}{7}\right)^{1-1}} + \overset{n=2}{5\left(-\frac{2}{7}\right)^{2-1}} + \overset{n=3}{5\left(-\frac{2}{7}\right)^{3-1}} + \overset{n=4}{5\left(-\frac{2}{7}\right)^{4-1}} + \cdots$$

$$= 5 + \frac{10}{7} + \frac{20}{49} + \frac{40}{343} + \cdots$$

This is an infinite geometric series. Use the formula $S = \frac{a_1}{1-r}$.

$$S = \frac{5}{1 - \left(-\frac{2}{7}\right)} \qquad a_1 = 5,\ r = -\frac{2}{7}$$

$$S = \frac{35}{9}$$

Therefore, $\sum_{n=1}^{\infty} 5\left(-\frac{2}{7}\right)^{n-1} = \frac{35}{9}$.

A series in expanded form can be written using sigma notation if a general formula can be written for the *n*th term of the series.

Example 2 **MANUFACTURING** **Refer to the application at the beginning of this lesson.**

a. How much is the company fined on the 20th day?

b. What is the total amount in fines owed by the manufacturing plant?

c. Represent this sum using sigma notation.

a. Since this sequence is arithmetic, we can use the formula for the *n*th term of an arithmetic sequence to find the amount of the fine charged on the 20th day.

$$a_n = a_1 + (n - 1)d$$
$$a_{20} = 2000 + (20 - 1)500 \qquad a_1 = 2000,\ n = 20,\text{ and } d = 500$$
$$= 11{,}500$$

The fine on the 20th day will be $11,500.

b. To determine the total amount owed in fines, we can use the formula for the sum of an arithmetic series. The plant will not be charged for the day it passes inspection, so it is assessed 20 days in fines.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ &= \frac{20}{2}(2000 + 11{,}500) \qquad \textit{n = 20, } a_1 \textit{ = 2000, and } a_{20} \textit{ = 11,500} \\ &= 135{,}000 \end{aligned}$$

The plant must pay a total of $135,000 in fines.

c. To determine the fine for the nth day, we can again use the formula for the nth term of an arithmetic sequence.

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ &= 2000 + (n - 1)500 \qquad a_1 \textit{ = 2000 and } d \textit{ = 500} \\ &= 2000 + 500n - 500 \\ &= 500n + 1500 \end{aligned}$$

The fine on the nth day is $\$500n + \$1500$.

Since $2000 is the fine on the first day and $12,000 is the fine on the 20th day, the index of summation goes from $n = 1$ to $n = 20$.

Therefore, $2000 + 2500 + 3000 + \cdots + 11{,}500 = \sum_{n=1}^{20} (500n + 1500)$ or $\sum_{n=1}^{20} 500(n + 3)$.

When using sigma notation, it is not always necessary that the sum start with the index equal to 1.

Example 3 **Express the series $15 + 24 + 35 + 48 + \cdots + 143$ using sigma notation.**

Notice that each term is 1 less than a perfect square. Thus, the nth term of the series is $n^2 - 1$. Since $4^2 - 1 = 15$ and $12^2 - 1 = 143$, the index of summation goes from $n = 4$ to $n = 12$.

Therefore, $15 + 24 + 35 + 48 + \cdots + 143 = \sum_{n=4}^{12} (n^2 - 1)$.

As you have seen, not all sequences are arithmetic or geometric. Some important sequences are generated by products of consecutive integers. The product $n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$ is called ***n* factorial** and symbolized $n!$.

n Factorial

The expression $n!$ (n factorial) is defined as follows for n, an integer greater than zero.

$$n! = n(n - 1)(n - 2) \cdots 1$$

By definition 0! = 1.

Factorial notation can be used to express the general form of a series.

Example ④ **Express the series $\frac{2}{2} - \frac{4}{6} + \frac{6}{24} - \frac{8}{120} + \frac{10}{720}$ using sigma notation.**

The sequence representing the numerators is 2, 4, 6, 8, 10. This is an arithmetic sequence with a common difference of 2. Thus the nth term can be represented by $2n$.

Because the series has alternating signs, one factor for the general term of the series is $(-1)^{n+1}$. Thus, when n is odd, the terms are positive, and when n is even, the terms are negative.

The sequence representing the denominators is 2, 6, 24, 120, 720. This sequence is generated by factorials.

$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$
$6! = 720$

Therefore, $\frac{2}{2} - \frac{4}{6} + \frac{6}{24} - \frac{8}{120} + \frac{10}{720} = \sum_{n=1}^{5} \frac{(-1)^{n+1} 2n}{(n+1)!}$.

You can check this answer by substituting values of n into the general term.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Find a counterexample** to the following statement: "The summation notion used to represent a series is unique."
2. In Example 4 of this lesson, the alternating signs of the series were represented by a factor of $(-1)^{n+1}$.
   - a. **Write** a different factor that could have been used in the general form of the nth term of the series.
   - b. **Determine** a factor that could be used if the alternating signs of the series began with a negative first term.
3. Consider the series $\sum_{j=2}^{10} (-2j + 1)$.
   - a. **Identify** the number of terms in this series.
   - b. **Write** a formula that determines the number of terms t in a finite series if the index of summation has a minimum value of a and a maximum value of b.
   - c. **Use** the formula in part **b** to identify the number of terms in the series $\sum_{k=-2}^{3} \frac{1}{k+3}$.
   - d. **Verify** your answer in part **c** by writing the series $\sum_{k=-2}^{3} \frac{1}{k+3}$ in expanded form.

Guided Practice

Write each expression in expanded form and then find the sum.

4. $\sum_{n=1}^{6} (n-3)$
5. $\sum_{k=2}^{5} 4k$
6. $\sum_{a=0}^{4} \frac{1}{2^a}$
7. $\sum_{p=0}^{\infty} 5\left(\frac{3}{4}\right)^p$

Express each series using sigma notation.

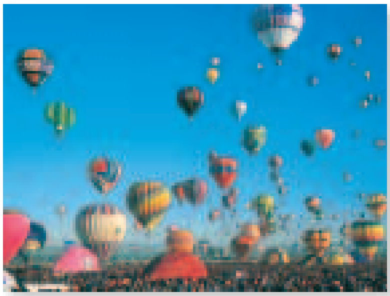
8. $5 + 10 + 15 + 20 + 25$

9. $2 + 4 + 10 + 28$

10. $2 - 4 - 10 - 16$

11. $\frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \cdots$

12. $-3 + 9 - 27 + \cdots$



13. Aviation Each October Albuquerque, New Mexico, hosts the Balloon Fiesta. In 1998, 873 hot air balloons participated in the opening day festivities. One of these balloons rose 389 feet after 1 minute. Because the air in the balloon was not reheated, each succeeding minute the balloon rose 63% as far as it did the previous minute.

a. Use sigma notation to represent the height of the balloon above the ground after one hour. Then calculate the total height of the balloon after one hour to the nearest foot.

b. What was the maximum height achieved by this balloon?

EXERCISES

Practice

Write each expression in expanded form and then find the sum.

14. $\sum_{n=1}^{4} (2n - 7)$

15. $\sum_{a=2}^{5} 5a$

16. $\sum_{b=3}^{8} (6 - 4b)$

17. $\sum_{k=2}^{6} (k + k^2)$

18. $\sum_{n=5}^{8} \frac{n}{n - 4}$

19. $\sum_{j=4}^{8} 2^j$

20. $\sum_{m=0}^{3} 3^{m-1}$

21. $\sum_{r=1}^{3} \left(\frac{1}{2} + 4^r\right)$

22. $\sum_{i=3}^{5} (0.5)^{-i}$

23. $\sum_{k=3}^{7} k!$

24. $\sum_{p=0}^{\infty} 4(0.75)^p$

25. $\sum_{n=1}^{\infty} 4\left(\frac{2}{5}\right)^n$

26. Write $\sum_{n=2}^{5} n + \mathbf{i}^n$ in expanded form. Then find the sum.

Express each series using sigma notation.

27. $6 + 9 + 12 + 15$

28. $1 + 4 + 16 + \cdots + 256$

29. $8 + 10 + 12 + \cdots + 24$

30. $-8 + 4 - 2 + 1$

31. $10 + 50 + 250 + 1250$

32. $13 + 9 + 5 + 1$

33. $\frac{1}{9} + \frac{1}{14} + \frac{1}{19} + \cdots + \frac{1}{49}$

34. $\frac{2}{3} + \frac{4}{5} + \frac{8}{7} + \frac{16}{9} + \cdots$

35. $4 - 9 + 16 - 25 + \cdots$

36. $5 + 5 + \frac{5}{2} + \frac{5}{6} + \frac{5}{24} + \cdots$

37. $-32 + 16 - 8 + 4 - \cdots$

38. $2 + \frac{6}{2} + \frac{24}{3} + \frac{120}{4} + \cdots$

39. $\frac{1}{5} + \frac{2}{7} + \frac{3}{11} + \frac{4}{19} + \frac{5}{35} + \cdots$

40. $\frac{3}{9 \cdot 2} + \frac{8}{27 \cdot 6} + \frac{15}{81 \cdot 24} + \cdots$

41. Express the series $\frac{\sqrt{2}}{3} + \frac{2}{6} + \frac{\sqrt{8}}{18} + \frac{4}{72} + \frac{\sqrt{32}}{360} + \cdots$ using sigma notation.

Simplify. Assume that *n* and *m* are positive integers, $a > b$, and $a > 2$.

42. $\frac{(a - 2)!}{a!}$

43. $\frac{(a + 1)!}{(a - 2)!}$

44. $\frac{(a + b)!}{(a + b - 1)!}$

Graphing Calculator

45. Use a graphing calculator to find the sum of $\sum_{n=1}^{100} \frac{8n^3 - 2n^2 + 5}{n^4}$. Round to the nearest hundredth.

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CONTENTS

Applications and Problem Solving

46. Advertising A popular shoe manufacturer is planning to market a new style of tennis shoe in a city of 500,000 people. Using a prominent professional athlete as their spokesperson, the company's ad agency hopes to induce 35% of the people to buy the product. The ad agency estimates that these satisfied customers will convince 35% of 35% of 500,000 to buy a pair of shoes, and those will persuade 35% of 35% of 35% of 500,000, and so on.

a. Model this situation using sigma notation.

b. Find the total number of people that will buy the product as a result of the advertising campaign.

c. What percentage of the population is this?

d. What important assumption does the advertising agency make in proposing the figure found in part **b** to the shoe manufacturer?

47. Critical Thinking Solve each equation for x.

a. $\sum_{n=1}^{6} (x - 3n) = -3$ **b.** $\sum_{n=0}^{5} n(n - x) = 25$

48. Critical Thinking Determine whether each equation is *true* or *false*. Explain your answer.

a. $\sum_{k=3}^{7} 3^k + \sum_{b=7}^{9} 3^b = \sum_{a=3}^{9} 3^a$ **b.** $\sum_{n=2}^{8} (2n - 3) = \sum_{m=3}^{9} (2m - 5)$

c. $2\sum_{n=3}^{7} n^2 = \sum_{n=3}^{7} 2n^2$ **d.** $\sum_{k=1}^{10} (5 + k) = \sum_{p=0}^{9} (4 + p)$

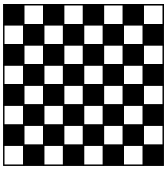
49. Word Play An *anagram* is a word or phrase that is made by rearranging the letters of another word or phrase. Consider the word "SILENT."

a. How many different arrangements of the letters in this word are possible? Write this number as a factorial. (*Hint:* First solve a simpler problem to see a pattern, such as how many different arrangements are there of just 2 letters? 3 letters?)

b. If a friend gives you a hint and tells you that an anagram of this word starts with "L," how many different arrangements still remain?

c. Your friend gives you one more hint. The last letter in the anagram is "N." Determine how many possible arrangements remain and then determine the anagram your friend is suggesting.

50. Chess A standard chess board contains 64 small black or white squares. These squares make up many other larger squares of various sizes.



a. How many 8×8 squares are there on a standard 8×8 chessboard? How many 7×7 squares?

b. Continue this list until you have accounted for all 8 sizes of squares.

c. Use sigma notation to represent the total number of squares found on an 8×8 chessboard. Then calculate this sum.

Mixed Review

51. Use the comparison test to determine whether the series $\frac{3}{3} + \frac{3}{4} + \frac{3}{5} + \frac{3}{6} + \cdots$ is convergent or divergent. *(Lesson 12-4)*

52. Chemistry A vacuum pump removes 20% of the air in a sealed jar on each stroke of its piston. The jar contains 21 liters of air before the pump starts. After how many strokes will only 42% of the air remain? *(Lesson 12-3)*

53. Find the first four terms of the geometric sequence for which $a_5 = 32\sqrt{2}$ and $r = -\sqrt{2}$. *(Lesson 12-2)*

54. Evaluate $\log_{10}0.001$. *(Lesson 11-4)*

55. Write the standard form of the equation of the circle that passes through points at (0, 9), (−7, 2), and (0, −5). *(Lesson 10-2)*

56. Simplify $\left(\sqrt{2} + \boldsymbol{i}\right)\left(4\sqrt{2} + \boldsymbol{i}\right)$. *(Lesson 9-5)*

57. Sports Find the initial vertical and horizontal velocities of a javelin thrown with an initial velocity of 59 feet per second at an angle of 63° with the horizontal. *(Lesson 8-7)*

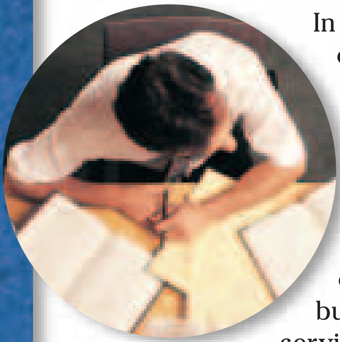
58. Find the equation of the line that bisects the obtuse angle formed by the graphs of $2x - 3y + 9 = 0$ and $x + 4y + 4 = 0$. *(Lesson 7-7)*

59. SAT/ACT Practice If $\frac{5 + m}{9 + m} = \frac{2}{3}$, then $m = ?$

A 8 **B** 6 **C** 5 **D** 3 **E** 2

CAREER CHOICES

Operations Research Analyst



In the changing economy of today, it is difficult to start and maintain a successful business. A business operator needs to be sure that the income from the business exceeds the expenses. Sometimes, businesses need the services of an operations research analyst. If you enjoy mathematics and solving tough problems, then you may want to consider a career as an operations research analyst.

In this occupation, you would gather many types of data about a business and analyze that data using mathematics and statistics. Examples of ways you might assist a business are: help a retail store determine the best store layout, help a bank in processing deposits more efficiently, or help a business set prices. Most operations research analysts work for private industry, private consulting firms, or the government.

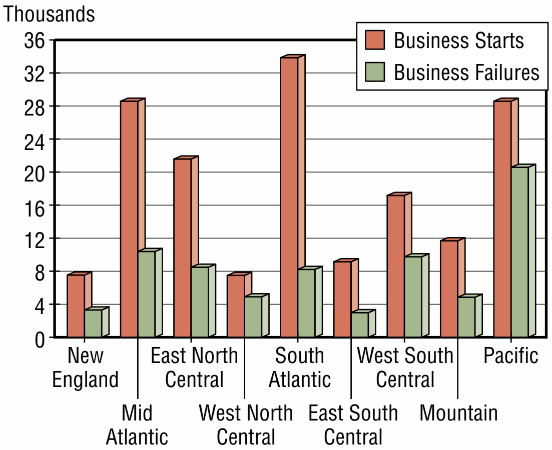
CAREER OVERVIEW

Degree Preferred:
bachelor's degree in applied mathematics

Related Courses:
mathematics, statistics, computer science, English

Outlook:
faster than average through the year 2006

1997 Business Starts and Failures



interNET CONNECTION For more information on careers in operations research, visit: **www.amc.glencoe.com**

CONTENTS

12-6

The Binomial Theorem

OBJECTIVE

- Use the Binomial Theorem to expand binomials.

FAMILY On November 19, 1997, Bobbie McCaughey, an Iowa seamstress, gave birth by Caesarian section to seven babies. The birth of the septuplets was the first of its kind in the United States since 1985. The babies, born after just 30 weeks of pregnancy, weighed from 2 pounds 5 ounces to 3 pounds 4 ounces. *A problem related to this will be solved in Example 2.*

Recall that a binomial, such as $x + y$, is an algebraic expression involving the sum of two unlike terms, in this case x and y. Just as there are patterns in sequences and series, there are numerical patterns in the expansion of powers of binomials. Let's examine the expansion of $(x + y)^n$ for $n = 0$ to $n = 5$. You already know a few of these expansions and the others can be obtained using algebraic properties.

$$
\begin{aligned}
(x+y)^0 &= 1x^0y^0\\
(x+y)^1 &= 1x^1y^0 + 1x^0y^1\\
(x+y)^2 &= 1x^2y^0 + 2x^1y^1 + 1x^0y^2\\
(x+y)^3 &= 1x^3y^0 + 3x^2y^1 + 3x^1y^2 + 1x^0y^3\\
(x+y)^4 &= 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4\\
(x+y)^5 &= 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5
\end{aligned}
$$

Patterns in the Binomial Expansion of $(x + y)^n$

The following patterns can be observed in the expansion of $(x + y)^n$.

1. The expansion of $(x + y)^n$ has $n + 1$ terms.
2. The first term is x^n, and the last term is y^n.
3. In successive terms, the exponent of x decreases by 1, and the exponent of y increases by 1.
4. The degree of each term (the sum of the exponents of the variables) is n.
5. In any term, if the coefficient is multiplied by the exponent of x and the product is divided by the number of that term, the result is the coefficient of the following term.
6. The coefficients are symmetric. That is, the first term and the last term have the same coefficient. The second term and the second from the last term have the same coefficient, and so on.

If just the coefficients of these expansions are extracted and arranged in a triangular array, they form a pattern called **Pascal's triangle.** As you examine the triangle shown below, note that if two consecutive numbers in any row are added, the sum is a number in the following row. These three numbers also form a triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

This triangle may be extended indefinitely.

Example 1 **Use Pascal's triangle to expand each binomial.**

a. $(x + y)^6$

First, write the series without the coefficients. Recall that the expression should have 6 + 1 or 7 terms, with the first term being x^6 and the last term being y^6. Also note that the exponents of x should decrease from 6 to 0 while the exponents of y should increase from 0 to 6, while the degree of each term is 6.

$$x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6 \quad y^0 = 1,\ x^0 = 1$$

Then, use the numbers in the seventh row of Pascal's triangle as the coefficients of the terms. *Why is the seventh row used instead of the sixth row?*

$$(x + y)^6 = \overset{\overset{1}{\downarrow}}{x^6} + \overset{\overset{6}{\downarrow}}{6x^5y} + \overset{\overset{15}{\downarrow}}{15x^4y^2} + \overset{\overset{20}{\downarrow}}{20x^3y^3} + \overset{\overset{15}{\downarrow}}{15x^2y^4} + \overset{\overset{6}{\downarrow}}{6xy^5} + \overset{\overset{1}{\downarrow}}{y^6}$$

b. $(3x + 2y)^7$

Extend Pascal's triangle to the eighth row.

1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Then, write the expression and simplify each term. Replace each x with $3x$ and y with $2y$.

$$\begin{aligned}(3x + 2y)^7 &= (3x)^7 + 7(3x)^6(2y) + 21(3x)^5(2y)^2 + 35(3x)^4(2y)^3 + 35(3x)^3(2y)^4 + \\ &\quad 21(3x)^2(2y)^5 + 7(3x)(2y)^6 + (2y)^7 \\ &= 2187x^7 + 10{,}206x^6y + 20{,}412x^5y^2 + 22{,}680x^4y^3 + 15{,}120x^3y^4 + \\ &\quad 6048x^2y^5 + 1344xy^6 + 128y^7\end{aligned}$$

You can use Pascal's triangle to solve real-world problems in which there are only two outcomes for each event. For example, you can determine the distribution of answers on true-false tests, the combinations of heads and tails when tossing a coin, or the possible sequences of boys and girls in a family.

Example 2 Real World Application

FAMILY **Refer to the application at the beginning of the lesson. Of the seven children born to the McCaughey's, at least three were boys. How many of the possible groups of boys and girls have at least three boys?**

Let g represent girls and b represent boys. To find the number of possible groups, expand $(g + b)^7$. Use the eighth row of Pascal's triangle for the expansion.

$$g^7 + 7g^6b + 21g^5b^2 + 35g^4b^3 + 35g^3b^4 + 21g^2b^5 + 7gb^6 + b^7$$

To have at least three boys means that there could be 3, 4, 5, 6, or 7 boys. The total number of ways to have at least three boys is the same as the sum of the coefficients of g^4b^3, g^3b^4, g^2b^5, gb^6, and b^7. This sum is 35 + 35 + 21 + 7 + 1 or 99.

Thus, there are 99 possible groups of boys and girls in which there are at least three boys.

The general expansion of $(x + y)^n$ can also be determined by the **Binomial Theorem.**

Binomial Theorem

If n is a positive integer, then the following is true.

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}y^3 + \cdots + y^n$$

Example **3** **Use the Binomial Theorem to expand $(2x - y)^6$.**

The expansion will have seven terms. Find the first four terms using the sequence 1, $\frac{6}{1}$ or 6, $\frac{6 \cdot 5}{1 \cdot 2}$ or 15, $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$ or 20. Then use symmetry to find the remaining terms, 15, 6, and 1.

$$\begin{aligned}(2x - y)^6 &= (2x)^6 + 6(2x)^5(-y) + 15(2x)^4(-y)^2 + 20(2x)^3(-y)^3 + 15(2x)^2(-y)^4 + \\ &\quad 6(2x)(-y)^5 + (-y)^6 \\ &= 64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6\end{aligned}$$

An equivalent form of the Binomial Theorem uses both sigma and factorial notation. It is written as follows, where n is a positive integer and r is a positive integer or zero.

$$(x + y)^n = \sum_{r=0}^{n} \frac{n!}{r!(n - r)!} x^{n-r}y^r$$

You can use this form of the Binomial Theorem to find individual terms of an expansion.

Example **4** **Find the fifth term of $(4a + 3b)^7$.**

$$(4a + 3b)^7 = \sum_{r=0}^{7} \frac{7!}{r!(7 - r)!} (4a)^{7-r}(3b)^r$$

To find the fifth term, evaluate the general term for $r = 4$. *Since r increases from 0 to n, r is one less than the number of the term.*

$$\begin{aligned}\frac{7!}{r!(7 - r)!}(4a)^{7-r}(3b)^r &= \frac{7!}{4!(7 - 4)!}(4a)^{7-4}(3b)^4 \\ &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!\,3!}(4a)^3(3b)^4 \\ &= 181{,}440a^3b^4\end{aligned}$$

The fifth term of $(4a + 3b)^7$ is $181{,}440a^3b^4$.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. a. **Calculate** the sum of the numbers in each row of Pascal's triangle for $n = 0$ to 5.

   b. **Write** an expression to represent the sum of the numbers in the nth row of Pascal's triangle.

2. Examine the expansion of $(x - y)^n$ for $n = 3$, 4, and 5.

- **a.** **Identify** the sign of the second term for each expansion.
- **b.** **Identify** the sign of the third term for each expansion.
- **c.** **Explain** how to determine the sign of a term without writing out the entire expansion.

3. **Restate** what is meant by the observation that each term in a binomial expansion has degree n.

4. If ax^7y^b is a term from the expansion of $(x + y)^{12}$, **describe** how to determine its coefficient a and missing exponent b without writing the entire expansion.

Guided Practice

5. Use Pascal's triangle to expand $(c + d)^5$.

Use the Binomial Theorem to expand each binomial.

6. $(a + 3)^6$ **7.** $(5 - y)^3$ **8.** $(3p - 2q)^4$

Find the designated term of each binomial expansion.

9. 6th term of $(a - b)^7$

10. 4th term of $\left(x + \sqrt{3}\right)^9$

11. **Coins** A coin is flipped five times. Find the number of possible sets of heads and tails that have each of the following.

- **a.** 0 heads
- **b.** 2 heads
- **c.** at least 4 heads
- **d.** at most 3 heads

EXERCISES

Practice

Use Pascal's triangle to expand each binomial.

12. $(a + b)^8$ **13.** $(n - 4)^6$ **14.** $(3c - d)^4$

15. Expand $(2 + a)^9$ using Pascal's triangle.

Use the Binomial Theorem to expand each binomial.

16. $(d + 2)^7$ **17.** $(3 - x)^5$ **18.** $(4a + b)^4$

19. $(2x - 3y)^3$ **20.** $\left(3m + \sqrt{2}\right)^4$ **21.** $\left(\sqrt{c} - 1\right)^6$

22. $\left(\frac{1}{2}n + 2\right)^5$ **23.** $\left(3a + \frac{2}{3}b\right)^4$ **24.** $(p^2 + q)^8$

25. Expand $(xy - 2z^3)^6$ using the Binomial Theorem.

Find the designated term of each binomial expansion.

26. 5th term of $(x + y)^9$ **27.** 4th term of $\left(a - \sqrt{2}\right)^8$

28. 4th term of $(2a - b)^7$ **29.** 7th term of $(3c + 2d)^9$

30. 8th term of $\left(\frac{1}{2}x - y\right)^{10}$ **31.** 6th term of $(2p - 3q)^{11}$

32. Find the middle term of the expansion of $\left(\sqrt{x} + \sqrt{y}\right)^8$.

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CONTENTS

Applications and Problem Solving

33. **Business** A company decides to form a recycling committee to find a more efficient means of recycling waste paper. The committee is to be composed of eight employees. Of these eight employees, at least four women are to be on the committee. How many of the possible groups of men and women have at least four women?

34. **Critical Thinking** Describe a strategy that uses the Binomial Theorem to expand $(a + b + c)^{12}$.

35. **Education** Rafael is taking a test that contains a section of 12 true-false questions.
    - a. How many of the possible groups of answers to these questions have exactly 8 correct answers of false?
    - b. How many of the possible groups of answers to these questions have at least 6 correct answers of true?

36. **Critical Thinking** Find a term in the expansion of $\left(3x^2 - \frac{1}{4x}\right)^6$ that does not contain the variable x.

37. **Numerical Analysis** Before the invention of modern calculators and computers, mathematicians searched for ways to shorten lengthy calculations such as $(1.01)^4$.
    - a. Express 1.01 as a binomial.
    - b. Use the binomial representation of 1.01 found in part **a** and the Binomial Theorem to calculate the value of $(1.01)^4$ to eight decimal places.
    - c. Use a calculator to estimate $(1.01)^4$ to eight decimal places. Compare this value to the value found in part **b**.

Mixed Review

38. Write $\sum_{k=2}^{7} 5 - 2k$ in expanded form and then find the sum. *(Lesson 12-5)*

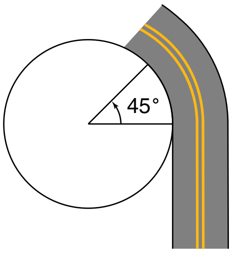
39. Use the ratio test to determine whether the series $2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots$ is *convergent* or *divergent*. *(Lesson 12-4)*

40. Find the sum of $\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$ or explain why one does not exist. *(Lesson 12-3)*

41. **Finance** A bank offers a home mortgage for an annual interest rate of 8%. If a family decides to mortgage a \$150,000 home over 30 years with this bank, what will the monthly payment for the principal and interest on their mortgage be? *(Lesson 11-2)*

42. Write $\overrightarrow{MK}$ as the sum of unit vectors if $M(-2, 6, 3)$ and $K(4, 8, -2)$. *(Lesson 8-3)*

43. **Construction** A highway curve, in the shape of an arc of a circle, is 0.25 mile. The direction of the highway changes 45° from one end of the curve to the other. Find the radius of the circle in feet that the curve follows. *(Lesson 6-1)*



44. **SAT/ACT Practice** If b is a prime integer such that $3b > 10 > \frac{5}{6}b$, which of the following is a possible value of b?

A 2 **B** 3 **C** 4 **D** 11 **E** 13

12-7

Special Sequences and Series

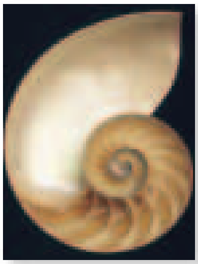
OBJECTIVES

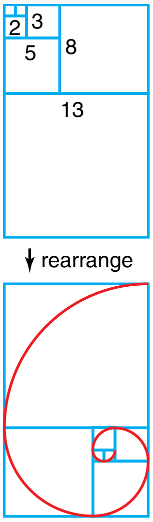
- Approximate e^x, trigonometric values, and logarithms of negative numbers by using series.
- Use Euler's Formula to write the exponential form of a complex number.

NATURE An important sequence found in nature can be seen in the spiral of a Nautilus shell. To see this sequence follow procedure below.

- Begin by placing two small squares with side length 1 next to each other.
- Below both of these, draw a square with side length 2.
- Now draw a square that touches both a unit square and the 2-square. This square will have sides 3 units long.
- Add another square that touches both the 2-square and the 3-square. This square will have sides of 5 units.
- Continue this pattern around the picture so that each new square has sides that are as long as the sum of the latest two square's sides.

The side lengths of these squares form what is known as the **Fibonacci sequence:** 1, 1, 2, 3, 5, 8, 13, …. A spiral that closely models the Nautilus spiral can be drawn by first rearranging the squares so that the unit squares are in the interior and then connecting quarter circles, each of whose radius is a side of a new square. This spiral is known as the *Fibonacci spiral. A problem related to the Fibonacci sequence will be solved in Example 1.*





The Fibonacci sequence describes many patterns of numbers found in nature. This sequence was presented by an Italian mathematician named Leonardo de Pisa, also known as Fibonacci (pronounced *fih buh NACH ee*), in 1201. The first two numbers in the sequence are 1, that is, $a_1 = 1$ and $a_2 = 1$. As you have seen in the example above, adding the two previous terms generates each additional term in the sequence.

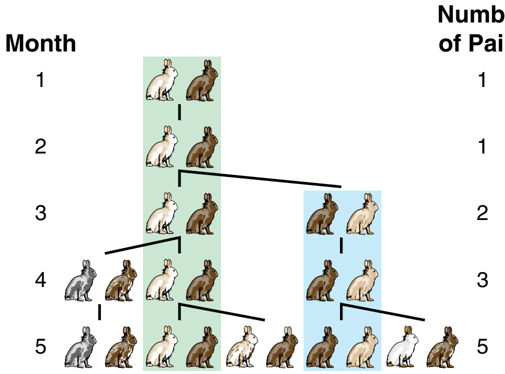
The original problem that Fibonacci investigated in 1202 involved the reproductive habits of rabbits under ideal conditions.

Example 1

NATURE **Suppose a newly born pair of rabbits are allowed to breed in a controlled environment. How many rabbit pairs will there be after one year if the following assumptions are made?**

- **A male and female rabbit can mate at the age of one month.**
- **At the end of its second month, a female rabbit can produce another pair of rabbits (one male, one female).**
- **The rabbits never die.**
- **The female always produces one new pair every month from the second month on.**

Based on these assumptions, at the end of the first month, there will be one pair of rabbits. And at the end of the second month, the female produces a new pair, so now there are two pairs of rabbits.



The following table shows the pattern.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Number of Pairs	1	1	2	3	5	8	13	21	34	55	89	144

There will be 144 pairs of rabbits during the twelfth month.

Each term in the sequence is the sum of the two previous terms.

Another important series is the series that is used to define the irrational number e. The Swiss mathematician Leonhard Euler (pronounced OY ler), published a work in which he developed this irrational number. It has been suggested that in his honor the number is called e, the Euler number. The number can be expressed as the sum of the following infinite series.

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots + \frac{1}{n!} + \cdots$$

The Binomial Theorem can be used to derive the series for e as follows. Let k be any positive integer and apply the Binomial Theorem to $\left(1 + \frac{1}{k}\right)^k$.

$$\left(1 + \frac{1}{k}\right)^k = 1 + k\left(\frac{1}{k}\right) + \frac{k(k-1)}{2!}\left(\frac{1}{k}\right)^2 + \frac{k(k-1)(k-2)}{3!}\left(\frac{1}{k}\right)^3 + \cdots$$

$$+ \frac{k(k-1)\,(k-2)\cdots 1}{k!}\left(\frac{1}{k}\right)^k$$

$$= 1 + 1 + \frac{1\left(1 - \frac{1}{k}\right)}{2!} + \frac{1\left(1 - \frac{1}{k}\right)\left(1 - \frac{2}{k}\right)}{3!} + \cdots + \frac{1\left(1 - \frac{1}{k}\right)\left(1 - \frac{2}{k}\right)\cdots\frac{1}{k}}{k!}$$

Then, find the limit of $\left(1 + \frac{1}{k}\right)^k$ as k increases without bound.

Recall that $\lim_{k\to\infty} \frac{1}{k} = 0$.

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

As $k \to \infty$, the number of terms in the sum becomes infinite.

Thus, e can be defined as follows.

$$e = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k \quad \text{or} \quad e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

The value of e^x can be approximated by using the following series. This series is often called the **exponential series.**

Exponential Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Example 2 **Use the first five terms of the exponential series and a calculator to approximate the value of $e^{2.03}$ to the nearest hundredth.**

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{2.03} \approx 1 + 2.03 + \frac{(2.03)^2}{2!} + \frac{(2.03)^3}{3!} + \frac{(2.03)^4}{4!}$$

$$\approx 1 + 2.03 + 2.06045 + 1.394237833 + 0.7075757004$$

$$\approx 7.19$$

Euler's name is associated with a number of important mathematical relationships. Among them is the relationship between the exponential series and a series called the **trigonometric series.** The trigonometric series for cos x and sin x are given below.

Trigonometric Series

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

These two series are convergent for all values of x. By replacing x with any angle measure expressed in radians and carrying out the computations, approximate values of the trigonometric functions can be found to any desired degree of accuracy.

CONTENTS

Example 3 **Use the first five terms of the trigonometric series to approximate the value of $\cos \frac{\pi}{3}$ to four decimal places.**

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

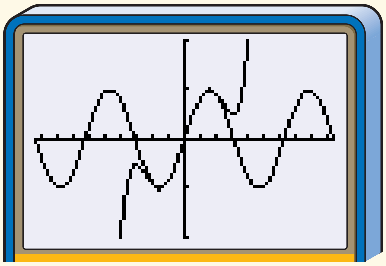
$$\cos \frac{\pi}{3} \approx 1 - \frac{(1.0472)^2}{2!} + \frac{(1.0472)^4}{4!} - \frac{(1.0472)^6}{6!} + \frac{(1.0472)^8}{8!}$$ *$x = \frac{\pi}{3}$ or about 1.0472*

$$\cos \frac{\pi}{3} \approx 1 - 0.54831 + 0.05011 - 0.00183 + 0.00004$$

$$\cos \frac{\pi}{3} \approx 0.5004$$ *Compare this result to the actual value.*

GRAPHING CALCULATOR EXPLORATION

In this Exploration you will examine polynomial functions that can be used to approximate $\sin x$. The graph below shows the graphs of $f(x) = \sin x$ and $g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ on the same screen.



[−3π, 3π] scl:1 by [−2, 2] scl:1

TRY THESE

1. Use TRACE to help you write an inequality describing the x-values for which the graphs seem very close together.
2. In absolute value, what are the greatest and least differences between the values of $f(x)$ and $g(x)$ for the values of x described by the inequality you wrote in Exercise 1?
3. Repeat Exercises 1 and 2 using $h(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ instead of $g(x)$.
4. Repeat Exercises 1 and 2 using $k(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$.

WHAT DO YOU THINK?

5. Are the intervals for which you get good approximations for $\sin x$ larger or smaller for polynomials that have more terms?
6. What term should be added to $k(x)$ to obtain a polynomial with six terms that gives good approximations to $\sin x$?

Another very important formula is derived by replacing x by $\boldsymbol{i}\alpha$ in the exponential series, where $\boldsymbol{i}$ is the imaginary unit and α is the measure of an angle in radians.

$$e^{\boldsymbol{i}\alpha} = 1 + \boldsymbol{i}\alpha + \frac{(\boldsymbol{i}\alpha)^2}{2!} + \frac{(\boldsymbol{i}\alpha)^3}{3!} + \frac{(\boldsymbol{i}\alpha)^4}{4!} + \cdots$$

$$e^{\boldsymbol{i}\alpha} = 1 + \boldsymbol{i}\alpha - \frac{\alpha^2}{2!} - \boldsymbol{i}\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \cdots$$ *$\boldsymbol{i}^2 = -1$, $\boldsymbol{i}^3 = -\boldsymbol{i}$, $\boldsymbol{i}^4 = 1$*

Group the terms according to whether they contain the factor $\boldsymbol{i}$.

$$e^{\boldsymbol{i}\alpha} = \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots\right) + \boldsymbol{i}\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots\right)$$

Notice that the real part is exactly $\cos \alpha$ and the imaginary part is exactly $\sin \alpha$. This relationship is called **Euler's Formula.**

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Euler's Formula	$e^{i\alpha} = \cos\alpha + \boldsymbol{i}\sin\alpha$

If $-\boldsymbol{i}\alpha$ had been substituted for x rather than $\boldsymbol{i}\alpha$, the result would have been $e^{-\boldsymbol{i}\alpha} = \cos\alpha - \boldsymbol{i}\sin\alpha$.

Euler's Formula can be used to write a complex number, $a + b\boldsymbol{i}$, in its exponential form, $re^{\boldsymbol{i}\theta}$.

$$\begin{aligned} a + b\boldsymbol{i} &= r(\cos\theta + \boldsymbol{i}\sin\theta) \\ &= re^{\boldsymbol{i}\theta} \end{aligned}$$

Example ④ **Write $1 + \sqrt{3}\boldsymbol{i}$ in exponential form.**

Write the polar form of $1 + \sqrt{3}\boldsymbol{i}$. Recall that $a + b\boldsymbol{i} = r(\cos\theta + \boldsymbol{i}\sin\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = \text{Arctan}\,\frac{b}{a}$ when $a > 0$.

$$r = \sqrt{(1)^2 + (\sqrt{3})^2} \text{ or } 2, \text{ and } \theta = \text{Arctan}\,\frac{\sqrt{3}}{1} \text{ or } \frac{\pi}{3} \qquad a = 1 \text{ and } b = \sqrt{3}$$

$$\begin{aligned} 1 + \sqrt{3}\boldsymbol{i} &= 2\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right) \\ &= 2e^{\boldsymbol{i}\frac{\pi}{3}} \end{aligned}$$

Thus, the exponential form of $1 + \sqrt{3}\boldsymbol{i}$ is $2e^{\boldsymbol{i}\frac{\pi}{3}}$.

Look Back
Refer to Lesson 9-6 to review the polar form of complex numbers.

The equations for $e^{\boldsymbol{i}\alpha}$ and $e^{-\boldsymbol{i}\alpha}$ can be used to derive the exponential values of $\sin\alpha$ and $\cos\alpha$.

$$\begin{aligned} e^{\boldsymbol{i}\alpha} - e^{-\boldsymbol{i}\alpha} &= (\cos\alpha + \boldsymbol{i}\sin\alpha) - (\cos\alpha - \boldsymbol{i}\sin\alpha) \\ e^{\boldsymbol{i}\alpha} - e^{-\boldsymbol{i}\alpha} &= 2\boldsymbol{i}\sin\alpha \\ \sin\alpha &= \frac{e^{\boldsymbol{i}\alpha} - e^{-\boldsymbol{i}\alpha}}{2\boldsymbol{i}} \end{aligned}$$

$$\begin{aligned} e^{\boldsymbol{i}\alpha} + e^{-\boldsymbol{i}\alpha} &= (\cos\alpha + \boldsymbol{i}\sin\alpha) + (\cos\alpha - \boldsymbol{i}\sin\alpha) \\ e^{\boldsymbol{i}\alpha} + e^{-\boldsymbol{i}\alpha} &= 2\cos\alpha \\ \cos\alpha &= \frac{e^{\boldsymbol{i}\alpha} + e^{-\boldsymbol{i}\alpha}}{2} \end{aligned}$$

From your study of logarithms, you know that there is no real number that is the logarithm of a negative number. However, you can use a special case of Euler's Formula to find a complex number that is the natural logarithm of a negative number.

$$\begin{aligned} e^{\boldsymbol{i}\alpha} &= \cos\alpha + \boldsymbol{i}\sin\alpha \\ e^{\boldsymbol{i}\pi} &= \cos\pi + \boldsymbol{i}\sin\pi && \text{Let } \alpha = \pi. \\ e^{\boldsymbol{i}\pi} &= -1 + \boldsymbol{i}(0) \\ e^{\boldsymbol{i}\pi} &= -1 && \text{So } e^{\boldsymbol{i}\pi} + 1 = 0. \end{aligned}$$

If you take the natural logarithm of both sides of $e^{i\pi} = -1$, you can obtain a value for $\ln(-1)$.

$$\begin{aligned} \ln e^{i\pi} &= \ln(-1) \\ i\pi &= \ln(-1) \end{aligned}$$

Thus, the natural logarithm of a negative number $-k$, for $k > 0$, can be defined using $\ln(-k) = \ln(-1)k$ or $\ln(-1) + \ln k$, a complex number.

Example ⑤ **Evaluate ln (−270).**

$$\begin{aligned} \ln(-270) &= \ln(-1) + \ln(270) \\ &\approx i\pi + 5.5984 \end{aligned}$$ *Use a calculator to compute ln (270).*

Thus, $\ln(-270) \approx i\pi + 5.5984$. *The logarithm is a complex number.*

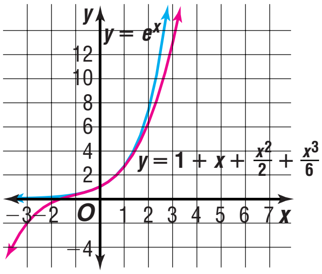
CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why in Example 2 of this lesson the exponential series gives an approximation of 7.19 for $e^{2.03}$, while a calculator gives an approximation of 7.61.

2. **Estimate** for what values of x the series $1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ gives a good approximation of the function e^x using the graph at the right.



3. **List** at least two reasons why Fibonacci's rabbit problem in Example 1 is not realistic.

4. **Write** a recursive formula for the terms of the Fibonacci sequence.

Guided Practice

Find each value to four decimal places.

5. $\ln(-7)$

6. $\ln(-0.379)$

Use the first five terms of the exponential series and a calculator to approximate each value to the nearest hundredth.

7. $e^{0.8}$

8. $e^{1.36}$

9. Use the first five terms of the trigonometric series to approximate the value of $\sin \pi$ to four decimal places. Then, compare the approximation to the actual value.

Write each complex number in exponential form.

10. $\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right)$

11. $-1 + \sqrt{3}i$

CONTENTS

12. **Investment** The Cyberbank advertises an Advantage Plus savings account with a 6% interest rate compounded continuously. Morgan is considering opening up an Advantage Plus account, because she needs to double the amount she will deposit over 5 years.

a. If Morgan deposits P dollars into the account, approximate the return on her investment after 5 years using three terms of the exponential series. (*Hint:* The formula for continuous compounding interest is $A = Pe^{rt}$.)

b. Will Morgan double her money in the desired amount of time? Explain.

c. Check the approximation found in part **a** using a calculator. How do the two answers compare?

EXERCISES

Practice

Find each value to four decimal places.

13. $\ln(-4)$
14. $\ln(-3.1)$
15. $\ln(-0.25)$
16. $\ln(-0.033)$
17. $\ln(-238)$
18. $\ln(-1207)$

Use the first five terms of the exponential series and a calculator to approximate each value to the nearest hundredth.

19. $e^{1.1}$
20. $e^{-0.2}$
21. $e^{4.2}$
22. $e^{0.55}$
23. $e^{3.5}$
24. $e^{2.73}$

Use the first five terms of the trigonometric series to approximate the value of each function to four decimal places. Then, compare the approximation to the actual value.

25. $\cos \pi$
26. $\sin \frac{\pi}{4}$
27. $\cos \frac{\pi}{6}$

28. Approximate the value of $\sin \frac{\pi}{2}$ to four decimal places using the first five terms of the trigonometric series.

Write each complex number in exponential form.

29. $5\left(\cos \frac{5\pi}{3} + \boldsymbol{i} \sin \frac{5\pi}{3}\right)$
30. $\boldsymbol{i}$
31. $1 + \boldsymbol{i}$
32. $\sqrt{3} + \boldsymbol{i}$
33. $-\sqrt{2} + \sqrt{2}\boldsymbol{i}$
34. $-4\sqrt{3} - 4\boldsymbol{i}$

35. Write the expression $3 + 3\boldsymbol{i}$ in exponential form.

Applications and Problem Solving

36. **Research** Explore the meaning of the term *transcendental number.*

37. **Critical Thinking** Show that for all real numbers x, $\sin x = \frac{e^{\boldsymbol{i}x} - e^{-\boldsymbol{i}x}}{2\boldsymbol{i}}$ and $\cos x = \frac{e^{\boldsymbol{i}x} + e^{-\boldsymbol{i}x}}{2}$.

38. **Research** Investigate and write a one-page paper on the occurrence of Fibonacci numbers in plants.

www.amc.glencoe.com/self_check_quiz

CONTENTS

39. Critical Thinking Examine Pascal's triangle. What relationship can you find between Pascal's triangle and the Fibonacci sequence?

interNET CONNECTION

Research For more information about the Fibonacci sequence, visit: **www.amc.glencoe.com**

40. Number Theory Consider the Fibonacci sequence 1, 1, 2, 3, …, $F_{n-2} + F_{n-1}$.

a. Find $\frac{F_n}{F_{n-1}}$ for the second through eleventh terms of the Fibonacci sequence.

b. Is this sequence of ratios arithmetic, geometric, or neither?

c. Sketch a graph of the terms found in part **a**. Let n be the x-coordinate and $\frac{F_n}{F_{n-1}}$ be the y-coordinate, and connect the points.

d. Based on the graph found in part c, does this sequence appear to approach a limit? If so, use the last term found to approximate this limit to three decimal places.

e. The *golden ratio* has a value of approximately 1.61804. How does the limit of the sequence $\frac{F_n}{F_{n-1}}$ compare to the golden ratio?

f. Research the term *golden ratio*. Write several paragraphs on the history of the golden ratio and describe its application to art and architecture.

41. Investment When Cleavon turned 5 years old, his grandmother decided it was time to start saving for his college education. She deposited \$5000 in a special account earning 5% interest compounded continuously. By the time Cleavon begins college at the age of 18, his grandmother estimates that her grandson will need \$40,000 for college tuition.

a. Approximate Cleavon's savings account balance on his 18th birthday using five terms of the exponential series.

b. Will the account have sufficient funds for Cleavon's college tuition by the time he is ready to start college? Explain.

c. Use a calculator to compute how long it will take the account to accumulate \$40,000. How old would Cleavon be by this time?

d. To have at least \$40,000 when Cleavon is 18 years old, how much should his grandmother have invested when he was 5 years old to the nearest dollar?

42. Critical Thinking Find a pattern, if one exists, for the following types of Fibonacci numbers.

a. even numbers

b. multiples of 3

Mixed Review

43. Use the Binomial Theorem to expand $(2x + y)^6$. *(Lesson 12-6)*

44. Express the series $2 + 4 + 8 + \cdots + 64$ using sigma notation. *(Lesson 12-5)*

45. **Number Theory** If you tear a piece of paper that is 0.005 centimeter thick in half, and place the two pieces on top of each other, the height of the pile of paper is 0.01 centimeter. Let's call this the second pile. If you tear these two pieces of paper in half, the third pile will have four pieces of paper in it. *(Lesson 12-2)*

a. How high is the third pile? the fourth pile?

b. Write a formula to determine how high the nth pile is.

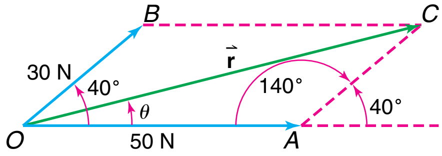
c. Use the formula to determine in theory how high the 10th pile would be and how high the 100th pile would be.

46. Evaluate $\frac{8^{\frac{2}{3}}}{8^{\frac{1}{3}}}$. *(Lesson 11-1)*

47. Write an equation of the parabola in general form that has a horizontal axis and passes through points at $(0, 0)$, $(2, -1)$, and $(4, -4)$. *(Lesson 10-5)*

48. Find the quotient of $16\left(\cos\frac{\pi}{8} + \boldsymbol{i}\sin\frac{\pi}{8}\right)$ divided by $4\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$. *(Lesson 9-7)*

49. **Physics** Two forces, one of 30 N and the other of 50 N, act on an object. If the angle between the forces is 40°, find the magnitude and the direction of the resultant force. *(Lesson 8-5)*



50. **Carpentry** Carpenters use circular sanders to smooth rough surfaces, such as wood or plaster. The disk of a sander has a radius of 6 inches and is rotating at a speed of 5 revolutions per second. *(Lesson 6-2)*

a. Find the angular velocity of the sander disk in radians per second.

b. What is the linear velocity of a point on the edge of the sander disk in feet per second?

51. **Education** You may answer up to 30 questions on your final exam in history class. It consists of multiple-choice and essay questions. Two 48-minute class periods have been set aside for taking the test. It will take you 1 minute to answer each multiple-choice question and 12 minutes for each essay question. Correct answers of multiple-choice questions earn 5 points and correct essay answers earn 20 points. If you are confident that you will answer all of the questions you attempt correctly, how many of each type of questions should you answer to receive the highest score? *(Lesson 2-7)*

52. **SAT/ACT Practice** In triangle ABC, if $BD = 5$, what is the length of $\overline{BC}$?

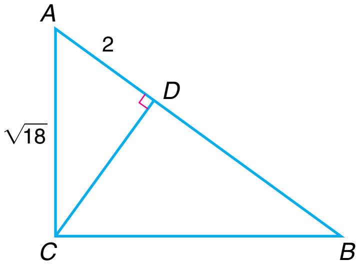
A 3

B 5

C $\sqrt{39}$

D $\sqrt{70}$

E $\sqrt{126}$



Extra Practice See p. A50.

CONTENTS

12-8

Sequences and Iteration

OBJECTIVES

- Iterate functions using real and complex numbers.

ECOLOGY The population of grizzly bears on the high Rocky Mountain Front near Choteau, Montana, has a growth factor of 1.75. The maximum population of bears that can be sustained in the area is 500 bears, and the current population is 240. Write an equation to model the population. Use the equation to find the population of bears at the end of fifteen years. *This problem will be solved in Example 1.*

The population of a species in a defined area changes over time. Changes in the availability of food, good or bad weather conditions, the amount of hunting allowed, disease, and the presence or absence of predators can all affect the population of a species. We can use a mathematical equation to model the changes in a population.

One such model is the *Verhulst population model*. The model uses the recursive formula $p_{n+1} = p_n + rp_n(1 - p_n)$, where n represents the number of time periods that have passed, p_n represents the percent of the maximum sustainable population that exists at time n, and r is a growth factor.

Example 1 **ECOLOGY** **Refer to the application above.**

a. Write an equation to model the population.

For this problem, the Verhulst population model can be used to represent the changes in the population of a species.

$$p_{n+1} = p_n + rp_n(1 - p_n)$$
$$p_{n+1} = p_n + 1.75p_n(1 - p_n) \quad r = 1.75$$

b. Find the population of bears at the end of fifteen years.

The initial percent of the maximum sustainable population can be represented by the ratio $p_0 = \frac{240}{500}$ or 0.48. This means that the current population is 48% of the maximum sustainable population.

Now, we can find the first few iterates as follows.

$p_1 = 0.48 + 1.75(0.48)\ (1 - 0.48)$ or 0.9168 — $n = 1, p_0 = 0.48, r = 1.75$
$(0.9168)\ (500) \approx 458$ bears

$p_2 = 0.9168 + 1.75(0.9168)(1 - 0.9168)$ or 1.0503 — $n = 2, p_1 = 0.9168, r = 1.75$
$(1.0503)(500) \approx 525$ bears

$p_3 = 1.0503 + 1.75(1.0503)(1 - 1.0503)$ or 0.9578 — $n = 3, p_2 = 1.0503, r = 1.75$
$(0.9578)(500) \approx 479$ bears

(continued on the next page)

The table shows the remainder of the values for the first fifteen years.

n	4	5	6	7	8	9	10	11	12	13	14	15
number of bears	514	489	508	494	505	497	503	498	501	499	501	499

At the end of the first 15 years, there could be 499 bears.

Notice that to determine the percent of the maximum sustainable population for a year, you must use the percent from the previous year. In Lesson 1-2, you learned that this process of composing a function with itself repeatedly is called *iteration*. Each output is called an *iterate*. To iterate a function $f(x)$, find the function value $f(x_0)$ of the initial value x_0. The second iterate is the value of the function performed on the output, that is $f(f(x_0))$.

Example Real World Application

2 **BANKING** **Selina Anthony has a savings account that has an annual yield of 6.3%. Find the balance of the account after each of the first three years if her initial balance is $4210.**

The balance of a savings account, using simple interest compounded at the end of a period of time, can be found by iterating the function $p_{n+1} = p_n + rp_n$, where p_n is the principal after n periods of time and r is the interest rate for a period of time.

$$p_1 = p_0 + rp_0$$

$$p_1 = 4210 + (0.063)(4210) \qquad p_0 = 4210,\ r = 0.063$$

$$p_1 = \$4475.23$$

$$p_2 = 4475.23 + (0.063)(4475.23) \qquad p_1 = 4475.23,\ r = 0.063$$

$$p_2 \approx \$4757.17$$

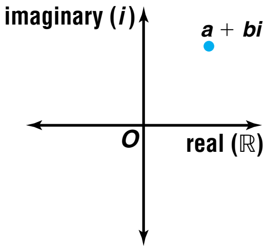
$$p_3 = 4757.17 + (0.063)(4757.17) \qquad p_2 = 4757.17,\ r = 0.063$$

$$p_3 \approx \$5056.87$$

The balance in Ms. Anthony's account at the end of the three years is $5056.87.

Look Back
You can review graphing numbers in the complex plane in Lesson 9-6

Remember that every complex number has a real part and an imaginary part. The complex number $a + bi$ has been graphed on the complex plane at the right. The horizontal axis of the complex plane represents the real part of the number, and the vertical axis represents the imaginary part.



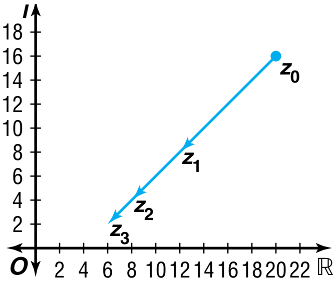
Functions can be iterated using the complex numbers as the domain.

Example 3 **Find the first three iterates of the function $f(z) = 0.5z + 2$, if the initial value is $20 + 16i$.**

f(z) is used to denote a function on the complex number plane.

$z_0 = 20 + 16i$
$z_1 = 0.5(20 + 16i) + 2$ or $12 + 8i$
$z_2 = 0.5(12 + 8i) + 2$ or $8 + 4i$
$z_3 = 0.5(8 + 4i) + 2$ or $6 + 2i$

We can graph this sequence of iterates on the complex plane. The graph at the right shows the **orbit,** or sequence of successive iterates, of the initial value of $z_0 = 20 + 16i$ from Example 3.



Example 4 **Consider the function $f(z) = z^2 + c$, where c and z are complex numbers.**

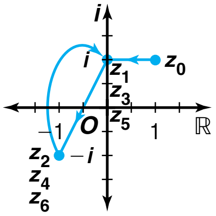
a. Find the first six iterates of the function for $z_0 = 1 + i$ and $c = -i$.

b. Plot the orbit of the initial point at $1 + i$ under iteration of the function $f(z) = z^2 - i$ for six iterations.

c. Describe the long-term behavior of the function under iteration.

a.
$$z_1 = (1 + i)^2 - i = i$$
$$z_2 = (i)^2 - i = -1 - i$$
$$z_3 = (-1 - i)^2 - i = i$$
$$z_4 = (i)^2 - i = -1 - i$$
$$z_5 = (-1 - i)^2 - i = i$$
$$z_6 = (i)^2 - i = -1 - i$$

b.

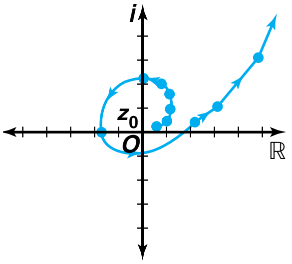


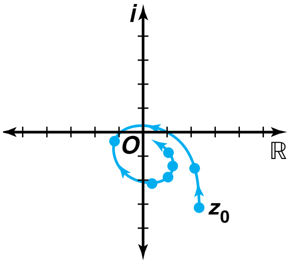
c. The iterates repeat every two iterations.

For many centuries, people have used science and mathematics to better understand the world. However, Euclidean geometry, the familiar geometry of points, lines, and planes, is not adequate to describe the natural world. Objects like coastlines, clouds, and mountain ranges can be more easily approximated by a new type of geometry. This type of geometry is called **fractal geometry.** The function $f(z) = z^2 + c$, where c and z are complex numbers is central to the study of fractal geometry.

At the heart of fractal geometry are the *Julia sets.* In Chapter 9, you learned that Julia sets involve graphing the behavior of a function that is iterated in the complex plane. As a function is iterated, the iterates either escape or are held prisoner. If the sequence of iterates remains within a finite distance from $0 + 0\boldsymbol{i}$, the initial point is called a **prisoner point.** If the sequence of iterates does not remain within a finite distance of $0 + 0\boldsymbol{i}$, the point is called an **escaping point.**

Suppose the function $f(z) = z^2 + c$ is iterated for two different initial values. If the first one is an escaping point and the second one is a prisoner point, the graphs of the orbits may look like those shown below.





Example 5 **Find the first four iterates of the function $f(z) = z^2 + c$ where $c = 0 + 0i$ for initial values whose moduli are in the regions $|z_0| < 1$, $|z_0| = 1$, and $|z_0| > 1$. Graph the orbits and describe the results.**

Recall from Lesson 9-6 that the modulus of a complex number is the distance that the graph of the complex number is from the origin. So $|a + b\boldsymbol{i}| = \sqrt{a^2 + b^2}$.

Choose an initial value in each of the intervals. For $|z_0| < 1$, we will use $z_0 = 0.5 + 0.5\boldsymbol{i}$; for $|z_0| = 1$, $0.5 + 0.5\sqrt{3}\boldsymbol{i}$; and for $|z_0| > 1$, $0.75 + 0.75\boldsymbol{i}$.

For $|z_0| < 1$

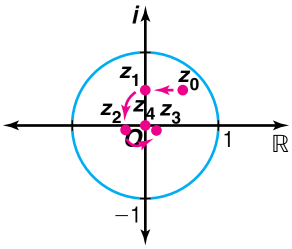
$z_0 = 0.5 + 0.5\boldsymbol{i}$

$z_1 = (0.5 + 0.5\boldsymbol{i})^2$ or $0.5\boldsymbol{i}$

$z_2 = (0.5\boldsymbol{i})^2$ or -0.25

$z_3 = (-0.25)^2$ or 0.0625

$z_4 = 0.0625^2$ or 0.0039



For $|z_0| = 1$

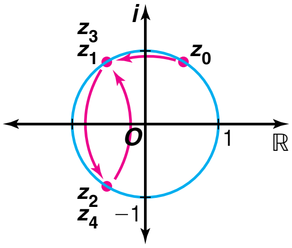
$z_0 = 0.5 + 0.5\sqrt{3}\boldsymbol{i}$

$z_1 = -0.5 + 0.5\sqrt{3}\boldsymbol{i}$

$z_2 = -0.5 - 0.5\sqrt{3}\boldsymbol{i}$

$z_3 = -0.5 + 0.5\sqrt{3}\boldsymbol{i}$

$z_4 = -0.5 - 0.5\sqrt{3}\boldsymbol{i}$



For $|z_0| > 1$

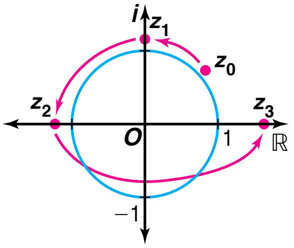
$z_0 = 0.75 + 0.75\boldsymbol{i}$

$z_1 = 1.125\boldsymbol{i}$

$z_2 = -1.2656$

$z_3 = 1.602$

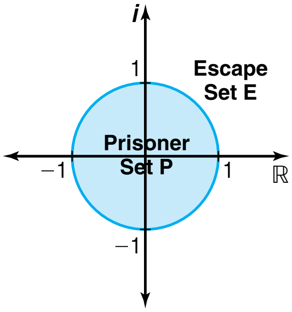
$z_4 = 2.566$



In the first case, where $|z_0| < 1$, the iterates will approach 0, since each successive square will be smaller than the one before. Thus, the orbit in this case approaches 0, a fixed point. In the second case, where $|z_0| = 1$, the iterates are orbiting around on the unit circle. When $|z_0| > 1$, the iterates approach infinity, since each square will be greater than the one before.

The orbits in Example 5 demonstrate that the iterative behavior of various initial values in different regions behave in different ways. An initial value whose graph is inside the unit circle is a prisoner point. A point chosen on the unit circle stays on the unit circle, and a point outside of the unit circle escapes. Other functions of the form $f(z) = z^2 + c$ when iterated over the complex plane also have regions in which the points behave this way; however, they are usually not circles.

All of the initial points for a function on the complex plane are split into three sets, those that escape (called the *escape set* E), those that do not escape (called the *prisoner set* P) and the boundary between the escape set and the prisoner set (the Julia set). The escape set, the prisoner set, and the Julia set for the function in Example 5 are graphed at the right. The Julia set in Example 5 is the unit circle.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** how iteration and composition functions are related.
2. **Describe** the orbit of a complex number under iteration.
3. **Explain** how a Julia set of a function is related to the prisoner set of the function.

Guided Practice

Find the first four iterates of each function using the given initial value. If necessary, round answers to the nearest hundredth.

4. $f(x) = x^2; x_0 = -1$
5. $f(x) = 2x - 5; x_0 = 2$

Find the first three iterates of the function $f(z) = 0.6z + 2i$ for each initial value.

6. $z_0 = 6\boldsymbol{i}$
7. $z_0 = 25 + 40\boldsymbol{i}$

Find the first three iterates of the function $f(z) = z^2 + c$ for each given value of c and each initial value.

8. $c = 1 + 2\boldsymbol{i}; z_0 = 0$
9. $c = 2 - 3\boldsymbol{i}; z_0 = 1 + 2\boldsymbol{i}$

10. **Biology** The Verhulst population model describing the population of antelope in an area is $p_{n+1} = p_n + 1.75p_n(1 - p_n)$. The maximum population sustainable in the area is 40, and the current population is 24. Find the population of antelope after each of the first ten years.

EXERCISES

Practice

Find the first four iterates of each function using the given initial value. If necessary, round answers to the nearest hundredth.

11. $f(x) = 3x - 7; x_0 = 4$ **12.** $f(x) = x^2; x_0 = -2$ **13.** $f(x) = (x - 5)^2; x_0 = 4$

14. $f(x) = x^2 - 1; x_0 = -1$ **15.** $f(x) = 2x^2 - x; x_0 = 0.1$

16. Find the first ten iterates of $f(x) = \frac{2}{x}$ for each initial value.
- **a.** $x_0 = 1$ **b.** $x_0 = 4$ **c.** $x_0 = 7$
- **d.** What do you observe about the iterates of this function?

Find the first three iterates of the function $f(z) = 2z + (3 - 2i)$ for each initial value.

17. $z_0 = 5\boldsymbol{i}$ **18.** $z_0 = 4$ **19.** $z_0 = 1 + 2\boldsymbol{i}$

20. $z_0 = -1 - 2\boldsymbol{i}$ **21.** $z_0 = 6 + 2\boldsymbol{i}$ **22.** $z_0 = 0.3 - \boldsymbol{i}$

23. Find the first three iterates of the function $f(z) = 3z - 2\boldsymbol{i}$ for $z_0 = \frac{1}{3} + \frac{2}{3}\boldsymbol{i}$.

Find the first three iterates of the function $f(z) = z^2 + c$ for each given value of c and each initial value.

24. $c = -1; z_0 = 0 - \boldsymbol{i}$ **25.** $c = 1 - 3\boldsymbol{i}; z_0 = \boldsymbol{i}$

26. $c = 3 + 2\boldsymbol{i}; z_0 = 1$ **27.** $c = -4\boldsymbol{i}; z_0 = 1 + \boldsymbol{i}$

28. $c = 0; z_0 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\boldsymbol{i}$ **29.** $c = 2 + 3\boldsymbol{i}; z_0 = 1 - \boldsymbol{i}$

Applications and Problem Solving

30. Banking Amelia has a savings account that has an annual yield of 5.2%. Find the balance of the account after each of the first five years if her initial balance is \$2000.

31. Ecology The population of elk on the Bridger Range in the Rocky Mountains of western Montana has a growth factor of 2.5. The population in 1984 was 10% of the maximum population sustainable. What percent of the maximum sustainable population should be present in 2002?

32. Critical Thinking If $f(z) = z^2 + c$ is iterated with an initial value of $2 + 3\boldsymbol{i}$ and $z_1 = -1 + 15\boldsymbol{i}$, find c.

33. Critical Thinking In Exercise 16, find an initial value that produces iterates that all have the same value.

34. Research Investigate the applications of fractal geometry to agriculture. How is fractal geometry being used in this field? Write several paragraphs about your findings.

35. Critical Thinking
- **a.** Use a calculator to find $\sqrt{2}$, $\sqrt{\sqrt{2}}$, $\sqrt{\sqrt{\sqrt{2}}}$, and $\sqrt{\sqrt{\sqrt{\sqrt{2}}}}$.
- **b.** Define an iterative function $f(z)$ that models the situation described in part a.
- **c.** Determine the limit of $f(z)$ as the number of iterations approaches infinity.
- **d.** Determine the limit of $f(z)$ as the number of iterations approaches infinity for integral values of $z_0 > 0$.

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Mixed Review

36. Write $2\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right)$ in exponential form. *(Lesson 12-7)*

37. Find the fifth term in the binomial expansion of $(2a - 3b)^8$. *(Lesson 12-6)*

38. State whether the series $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$ is *convergent* or *divergent*. Explain your reasoning. *(Lesson 12-4)*

39. Nuclear Power A nuclear cooling tower has an eccentricity of $\frac{7}{5}$. At its narrowest point the cooling tower is 130 feet wide. Determine the equation of the hyperbola used to generate the hyperboloid of the cooling tower. *(Lesson 10-4)*

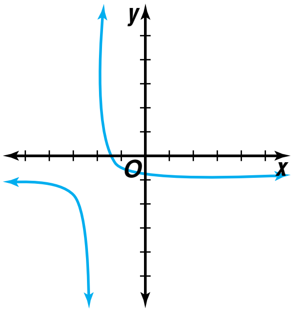
40. Optics A beam of light strikes a diamond at an angle of 42°. What is the angle of refraction if the index of refraction of a diamond is 2.42 and the index of refraction of air is 1.00? (Use Snell's Law, $n_1 \sin I = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium it is entering, I is the angle of incidence, and r is the angle of refraction.) *(Lesson 7-5)*

41. Air Traffic Safety The traffic pattern for airplanes into San Diego's airport is over the heart of downtown. Therefore, there are restrictions on the heights of new construction. The owner of an office building wishes to erect a microwave tower on top of the building. According to the architect's design, the angle of elevation from a point on the ground to the top of the 40-foot tower is 56°. The angle of elevation from the ground point to the top of the building is 42°. *(Lesson 5-4)*

a. Draw a sketch of this situation.

b. The maximum allowed height of any structure is 100 feet. Will the city allow the building of this tower? Explain.

42. Determine whether the graph has infinite discontinuity, jump discontinuity, or point discontinuity, or it is continuous. *(Lesson 3-5)*



43. Find the maximum and minimum values of the function $f(x, y) = 2x + 8y + 10$ for the polygonal convex set determined by the following system of inequalities. *(Lesson 2-6)*

$x \geq 3$
$x \leq 8$
$5 \leq y \leq 9$
$x + y \leq 14$

44. SAT/ACT Practice Two students took a science test and received different scores between 10 and 100. If H equals the higher score and L equals the lower score, and the difference between the two scores equals the average of the two scores, what is the value of $\frac{H}{L}$?

A $\frac{3}{2}$ **B** 2 **C** $\frac{5}{2}$ **D** 3

E It cannot be determined from the information given.

Extra Practice See p. A50.

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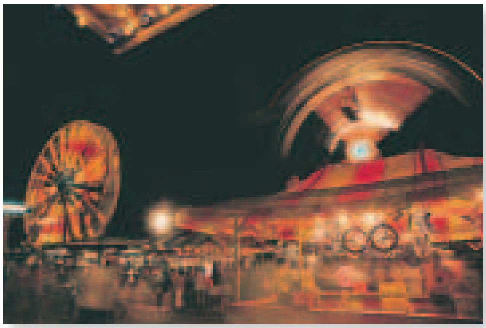
12-9

Mathematical Induction

OBJECTIVE

- Use mathematical induction to prove the validity of mathematical statements.



BUSINESS Felipe and Emily work a booth on the midway at the state fair. Before the fair opens, they discover that their cash supply contains only \$5 and \$10 bills. Rene volunteers to go and get a supply of \$1s, but before leaving, Rene remarks that she could have given change for any amount greater than \$4 had their cash supply contained only \$2 and \$5 bills. Felipe replies, "Even if we had \$2 bills, you would still need \$1 bills." Rene disagrees and says that she can prove that she is correct. *This problem will be solved in Example 4.*

A method of proof called **mathematical induction** can be used to prove certain conjectures and formulas. Mathematical induction depends on a recursive process that works much like an unending line of dominoes arranged so that if any one domino falls the next one will also fall.

Suppose the first domino is knocked over.	Condition 1
The first will knock down the second.	Condition 2
The second will knock down the third.	Condition 3
The third will knock down the fourth.	Condition 4
⋮	⋮

Thus, the whole line of dominos will eventually fall.

Mathematical induction operates in a similar manner. If a statement S_k implies the truth of S_{k+1} and the statement S_1 is true, then the chain reaction follows like an infinite set of tumbling dominoes.

S_1 is true.	Condition 1
S_1 implies that S_2 is true.	Condition 2
S_2 implies that S_3 is true.	Condition 3
S_3 implies that S_4 is true.	Condition 4
⋮	⋮

In general, the following steps are used to prove a conjecture by mathematical induction.

Proof by Mathematical Induction

1. First, verify that the conjecture S_n is valid for the first possible case, usually $n = 1$. *This is called the anchor step.*
2. Then, assume that S_n is valid for $n = k$, and use this assumption to prove that it is also valid for $n = k + 1$. *This is called the induction step.*

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Thus, since S_n is valid for $n = 1$ (or any other first case), it is valid for $n = 2$. Since it is valid for $n = 2$, it is valid for $n = 3$, and so on, indefinitely.

Mathematical induction can be used to prove summation formulas.

Example ① **Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.**

Here S_n is defined as $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

1. First, verify that S_n is valid for $n = 1$.

Since the first positive integer is 1 and $\frac{1(1+1)}{2} = 1$, the formula is valid for $n = 1$.

2. Then assume that S_n is valid for $n = k$.

$S_k \Rightarrow 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ *Replace n with k.*

Next, prove that it is also valid for $n = k + 1$.

$S_{k+1} \Rightarrow 1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$ *Add (k + 1) to both sides of S_k.*

We can simplify the right side by adding $\frac{k(k+1)}{2} + (k+1)$.

$$S_{k+1} \Rightarrow 1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1) + 2(k+1)}{2}$$

(k + 1) is a common factor

$$= \frac{(k+1)(k+2)}{2}$$

If $k + 1$ is substituted into the original formula $\left(\frac{n(n+1)}{2}\right)$, the same result is obtained.

$$\frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Thus, if the formula is valid for $n = k$, it is also valid for $n = k + 1$. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. That is, the formula for the sum of the first n positive integers holds.

Mathematical induction is also used to prove properties of divisibility. Recall that an integer p is *divisible* by an integer q if $p = qr$ for some integer r.

Example ② **Prove that $4^n - 1$ is divisible by 3 for all positive integers n.**

Using the definition of divisibility, we can state the conjecture as follows:

Notice that here S_n does not represent a sum but a conjecture.

$S_n \Rightarrow 4^n - 1 = 3r$ for some integer r

1. First verify that S_n is valid for $n = 1$.

$S_1 \Rightarrow 4^1 - 1 = 3$. Since 3 is divisible by 3, S_n is valid for $n = 1$.

2. Then assume that S_n is valid for $n = k$ and use this assumption to prove that it is also valid for $n = k + 1$.

$S_k \Rightarrow 4^k - 1 = 3r$ for some integer r *Assume S_k is true.*

$S_{k+1} \Rightarrow 4^{k+1} - 1 = 3t$ for some integer t *Show that S_{k+1} must follow.*

(continued on the next page)

CONTENTS

For this proof, rewrite the left-hand side of S_k so that it matches the left-hand side of S_{k+1}.

$4^k - 1 = 3r$	*S_k*
$4(4^k - 1) = 4(3r)$	*Multiply each side by 4.*
$4^{k+1} - 4 = 12r$	*Simplify.*
$4^{k+1} - 1 = 12r + 3$	*Add 3 to each side.*
$4^{k+1} - 1 = 3(4r + 3)$	*Factor.*

Let $t = 4r + 3$, an integer. Then $4^{k+1} - 1 = 3t$

We have shown that if S_k is valid, then S_{k+1} is also valid. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. Hence, $4^n - 1$ is divisible by 3 for all positive integers n.

There is no "fixed" way of completing Step 2 for a proof by mathematical induction. Often, each problem has its own special characteristics that require a different technique to complete the proof. You may have to multiply the numerator and denominator of an expression by the same quantity, factor or expand an expression in a special way, or see an important relationship with the distributive property.

Example ③ **Prove that $6^n - 2^n$ is divisible by 4 for all positive integers n.**

Begin by restating the conjecture using the definition of divisibility.

S_n ➡ $6^n - 2^n = 4r$ for some integer r

1. Verify that S_n is valid for $n = 1$

S_1 ➡ $6^1 - 2^1 = 4$. Since 4 is divisible by 4, S_n is valid for $n = 1$.

2. Assume that S_n is valid for $n = k$ and use this to prove that it is also valid for $n = k + 1$.

S_k ➡ $6^k - 2^k = 4r$ for some integer r	*Assume S_k is true*
S_{k+1} ➡ $6^{k+1} - 2^{k+1} = 4t$ for some integer t	*Show that S_{k+1} must follow.*

Begin by rewriting S_k as $6^k = 2^k + 4r$. Now rewrite the left-hand side of this expression so that it matches the left-hand side of S_{k+1}.

$6^k = 2^k + 4r$	*S_k*
$6^k \cdot 6 = (2^k + 4r)(2 + 4)$	*Multiply each side by a quantity equal to 6.*
$6^{k+1} = 2^{k+1} + 4(2^k) + 8r + 16r$	*Distributive Property*
$6^{k+1} - 2^{k+1} = 2^{k+1} + 4(2^k) + 8r + 16r - 2^{k+1}$	*Subtract 2^{k+1} from each side.*
$6^{k+1} - 2^{k+1} = 4 \cdot 2^k + 24r$	*Simplify.*
$6^{k+1} - 2^{k+1} = 4(2^k + 6r)$	*Factor.*

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Let $t = 2^k + 6r$, an integer. Then $6^{k+1} - 2^{k+1} = 4t$.

We have shown that if S_k is valid, then S_{k+1} is also valid. Since S_n is valid for $n = 1$, it is also valid for $n = 2$, $n = 3$, and so on. Hence, $6^n - 2^n$ is divisible by 4 for all positive integers n.

Let's look at another proof that requires some creative thinking to complete.`

Example 4

Real World Application

BUSINESS Refer to the application at the beginning of the lesson. Use mathematical induction to prove that Emily can give change in \$2s and \$5s for amounts of \$4, \$5, \$6, ..., \$*n*.

Let n = the dollar amount for which a customer might need change.
Let a = the number of \$2 bills used to make the exchange.
Let b = the number of \$5 bills used to make the exchange.

$S \Rightarrow n = 2a + 5b$

Notice that the first possible case for n is not always 1.

1. First, verify that S_n is valid for $n = 4$. In other words, are there values for a and b such that $4 = 2a + 5b$?

   Yes. If $a = 2$ and $b = 0$, then $4 = 2(2) + 5(0)$, so S_n is valid for the first case.

2. Then, assume that S_n is valid for $n = k$ and prove that it is also valid for $n = k + 1$.

   $$\begin{aligned} S_k &\Rightarrow k = 2a + 5b \\ S_{k+1} &\Rightarrow k + 1 = 2a + 5b + 1 && \textit{Add 1 to each side.} \\ & = 2a + 5b + 6 - 5 && 1 = 6 - 5 \\ & = 2(a + 3) + 5(b - 1) && 2a + 6 = 2(a + 3);\ 5b - 5 = 5(b - 1) \end{aligned}$$

   Notice that this expression is true for all $a \geq 0$ and $b \geq 1$, since the expression $b - 1$ must be nonnegative. Therefore, we must also consider the case where $b = 0$.

   If $b = 0$, then $n = 2a + 5(0) = 2a$. Assuming S_n to be true for $n = k$, we must show that it is also true for $k + 1$.

   $$\begin{aligned} S_k &\Rightarrow k = 2a \\ S_{k+1} &\Rightarrow k + 1 = 2a + 1 \\ & = 2(a - 2) + 4 + 1 && 2a = 2(a - 2) + 4 \\ & = 2(a - 2) + 1 \cdot 5 && 4 + 1 = 1 \cdot 5 \end{aligned}$$

   Notice that this expression is only true for all $a \geq 2$, since the expression $a - 2$ must be nonnegative. However, we have already considered the case where $a = 0$.

   Thus, it can be concluded that since the conjecture is true for $k = 4$, it also valid for $n = k + 1$. Therefore, Emily can make change for amounts of \$4, \$5, \$6, ... \$$n$ using only \$2 and \$5 bills.

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CHECK FOR UNDERSTANDING

Communicating Mathematics

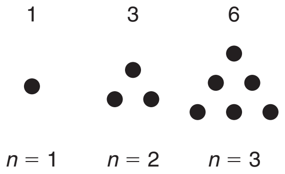
Read and study the lesson to answer each question.

1. **Explain** why it is necessary to prove the $n + 1$ case in the process of mathematical induction.
2. **Describe** a method you might use to show that a conjecture is false.
3. Consider the series $3 + 5 + 7 + 9 + \cdots + 2n + 1$.
   - a. **Write a conjecture** for the general formula S_n for the sum of the first n terms.
   - b. Verify S_n for $n = 1$, 2, and 3.
   - c. Write S_k and S_{k+1}.
4. **Restate** the conjecture that $8^n - 1$ is divisible by 7 for all positive integers n using the definition of divisibility.
5. *Math Journal* The "domino effect" presented in this lesson is just one way to illustrate the principle behind mathematical induction. **Write** a paragraph describing another situation in real life that illustrates this principle.

Guided Practice

Use mathematical induction to prove that each proposition is valid for all positive integral values of n.

6. $3 + 5 + 7 + \cdots + (2n + 1) = n(n + 2)$
7. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$
8. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$
9. $3^n - 1$ is divisible by 2.



10. **Number Theory** The ancient Greeks were very interested in number patterns. Triangular numbers are numbers that can be represented by a triangular array of dots, with n dots on each side. The first three triangular numbers are 1, 3, and 6.
    - a. Find the next five triangular numbers.
    - b. Write a general formula for the nth term of this sequence.
    - c. Prove that the sum of first n triangular numbers can be found using the formula $\frac{n(n + 1)(n + 2)}{6}$.

EXERCISES

Practice

For Exercises 11-19, use mathematical induction to prove that each proposition is valid for all positive integral values of n.

11. $1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$
12. $1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}$
13. $-\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \cdots - \frac{1}{2^n} = \frac{1}{2^n} - 1$
14. $1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}$
15. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}$

www.amc.glencoe.com/self_check_quiz

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16. $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$

17. $7^n + 5$ is divisible by 6

18. $8^n - 1$ is divisible by 7

19. $5^n - 2^n$ is divisible by 3

20. Prove S_n → $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$.

21. Prove S_n → $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$.

22. Prove S_n → $2^{2n+1} + 3^{2n+1}$ is divisible by 5.

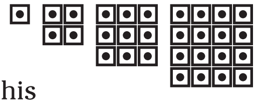
Applications and Problem Solving

23. Historical Proof In 1730, Abraham De Moivre proposed the following theorem for finding the power of a complex number written in polar form.

$$[r(\cos \theta + \boldsymbol{i} \sin \theta)]^n = r^n(\cos n\theta + \boldsymbol{i} \sin n\theta)$$

Prove that De Moivre's Theorem is valid for any positive integer n.

24. Number Patterns Consider the pattern of dots shown below.



a. Find the next figure in this pattern.

b. Write a sequence to represent the number of dots added to the previous figure to create the next figure in this pattern. The first term in your sequence should be 1.

c. Find the general form for the nth term of the sequence found in part **b**.

d. Determine a formula that will calculate the total number of dots for the nth figure in this dot pattern.

e. Prove that the formula found in part d is correct using mathematical induction.

25. Critical Thinking Prove that $n^2 + 5n$ is divisible by 2 for all positive integral values of n.

26. Club Activites Melissa is the activities director for her school's science club and needs to coordinate a "get to know you" activity for the group's first meeting. The activity must last at least 40 minutes but no more than 60 minutes. Melissa has chosen an activity that will require each participant to interact with every other participant in attendance only once and estimates that if her directions are followed, each interaction should take approximately 30 seconds.

a. If n people participate in Melissa's activity, develop a formula to calculate the total number of such interactions that should take place.

b. Prove that the formula found in part a is valid for all positive integral values of n.

c. If 15 people participate, will Melissa's activity meet the guidelines provided to her? Explain.

27. **Critical Thinking** Use mathematical induction to prove that the Binomial Theorem is valid for all positive integral values of n.

28. **Number Theory** Consider the following statement: $0.99999 \cdots = 1$.

a. Write $0.\overline{9}$ as an infinite series.

b. Write an expression for the nth term of this series.

c. Write a formula for the sum of the first n terms of this series.

d. Prove that the formula you found in part **c** is valid for all positive integral values of n using mathematical induction.

e. Use the formula found in part **c** to prove that $0.99999 \cdots = 1$. (*Hint:* Use what you know about the limits of infinite sequences.)

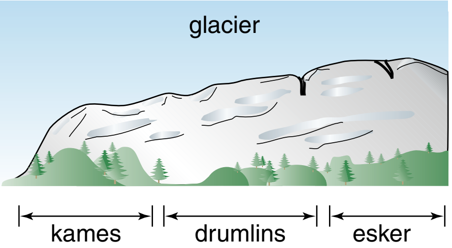
Mixed Review

29. Find the first three iterates of the function $f(z) = 2z + \boldsymbol{i}$ for $z_0 = 4 - \boldsymbol{i}$. *(Lesson 12-8)*

30. Find d for the arithmetic sequence for which $a_1 = -6$ and $a_{29} = 64$. *(Lesson 12-1)*

31. Write the standard form of the equation $25x^2 + 4y^2 - 100x - 40y + 100 = 0$. Then identify the conic section this equation represents. *(Lesson 10-6)*

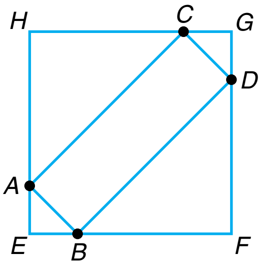
32. **Geology** A *drumlin* is an elliptical streamlined hill whose shape can be expressed by the equation $r = \ell \cos k\theta$ for $-\frac{\pi}{2k} \leq \theta \leq \frac{\pi}{2k}$, where ℓ is the length of the drumlin and $k > 1$ is a parameter that is the ratio of the length to the width. Find the area in square centimeters, $A = \frac{\ell^2\pi}{4k}$, of a drumlin modeled by the equation $r = 250 \cos 4\theta$. *(Lesson 9-3)*



33. Write an equation of a sine function with amplitude $\frac{3}{4}$ and period π. *(Lesson 6-4)*

34. Find the values of x in the interval $0° \leq x \leq 360°$ for which $x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$. *(Lesson 5-5)*

35. **SAT Practice** The area of $EFGH$ is 25 square units. Points A, B, C, and D are on the square. $ABCD$ is a rectangle, but not a square. Calculate the perimeter of $ABCD$ if the distance from E to A is 1 and the distance from E to B is 1.



A 64 units
B $10\sqrt{2}$ units
C 10 units
D 8 units
E $5\sqrt{2}$ units

Extra Practice See p. A50.

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CHAPTER 12 STUDY GUIDE AND ASSESSMENT

VOCABULARY

- arithmetic mean (p. 760)
- arithmetic sequence (p. 759)
- arithmetic series (p. 761)
- Binomial Theorem (p. 803)
- common difference (p. 759)
- common ratio (p. 766)
- comparison test (p. 789)
- convergent series (p. 786)
- divergent series (p. 786)
- escaping point (p. 818)
- Euler's Formula (p. 809)
- exponential series (p. 808)
- Fibonacci sequence (p. 806)
- fractal geometry (p. 817)
- geometric mean (p. 768)
- geometric sequence (p. 766)
- geometric series (p. 769)
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- infinite sequence (p. 774)
- infinite series (p. 778)
- limit (p. 774)
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- recursive formula (p. 760)
- sequence (p. 759)
- sigma notation (p. 794)
- term (p. 759)
- trigonometric series (p. 808)

UNDERSTANDING AND USING THE VOCABULARY

Choose the letter of the term that best matches each statement or phrase.

1. each succeeding term is formulated from one or more previous terms

2. ratio of successive terms in a geometric sequence

3. used to determine convergence

4. can have infinitely many terms

5. $e^{ix} = \cos \alpha + \boldsymbol{i} \sin \alpha$

6. the terms between any two nonconsecutive terms of a geometric sequence

7. indicated sum of the terms of an arithmetic sequence

8. $n! = n(n-1)(n-2) \cdot \cdots \cdot 1$

9. used to demonstrate the validity of a conjecture based on the truth of a first case, the assumption of truth of a *k*th case, and the demonstration of truth for the $(k + 1)$th case

10. an infinite series with a sum or limit

a. term
b. mathematical induction
c. arithmetic series
d. recursive formula
e. *n* factorial
f. geometric mean
g. sigma notation
h. convergent
i. common ratio
j. infinite sequence
k. Euler's Formula
l. prisoner point
m. ratio test
n. limit

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SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

REVIEW EXERCISES

Lesson 12-1 Find the nth term and arithmetic means of an arithmetic sequence.

Find the 35th term in the arithmetic sequence $-5, -1, 3, \ldots$.
Begin by finding the common difference d.
$d = -1 - (-5)$ or 4
Use the formula for the nth term.
$a_n = a_1 + (n - 1)d$
$a_{35} = -5 + (35 - 1)(4)$ or 131

11. Find the next four terms of the arithmetic sequence 3, 4.3, 5.6, ….

12. Find the 20th term of the arithmetic sequence for which $a_1 = 5$ and $d = -3$.

13. Form an arithemetic sequence that has three arithmetic means between 6 and -4.

Lesson 12-1 Find the sum of n terms of an arithmetic series.

The sum S_n of the first n terms of an arithmetic series is given by
$S_n = \frac{n}{2}(a_1 + a_n)$.

14. What is the sum of the first 14 terms in the arithmetic series $-30 - 23 - 16 - \cdots$?

15. Find n for the arithmetic series for which $a_1 = 2$, $d = 1.4$, and $S_n = 250.2$.

Lesson 12-2 Find the nth term and geometric means of a geometric sequence.

Find an approximation for the 12th term of the sequence $-8, 4, -2, 1, \ldots$.
First, find the common ratio.
$a_2 \div a_1 = 4 \div (-8)$ or -0.5
Use the formula for the nth term.
$a_{12} = -8(-0.5)^{12-1}$ *$a_n = a_1r^{n-1}$*
$= -8(-0.5)^{11}$ or about 0.004

16. Find the next three terms of the geometric sequence 49, 7, 1, ….

17. Find the 15th term of the geometric sequence for which $a_1 = 2.2$ and $r = 2$.

18. If $r = 0.2$ and $a_7 = 8$, what is the first term of the geometric sequence?

19. Write a geometric sequence that has three geometric means between 0.2 and 125.

Lesson 12-2 Find the sum of n terms of a geometric series.

Find the sum of the first 12 terms of the geometric series $4 + 10 + 25 + 62.5 + \cdots$.
First find the common ratio.
$a_2 \div a_1 = 10 \div 4$ or 2.5
Now use the formula for the sum of a finite geometric series.

$$S_n = \frac{a_1 - a_1r^n}{1 - r}$$

$$S_{12} = \frac{4 - 4(2.5)^{12}}{1 - 2.5} \quad n = 12,\ a_1 = 4,\ r = 2.5$$

$$S_{12} \approx 158{,}943.05 \quad \text{Use a calculator.}$$

20. What is the sum of the first nine terms of the geometric series $1.2 - 2.4 + 4.8 - \cdots$?

21. Find the sum of the first eight terms of the geometric series $4 + 4\sqrt{2} + 8 + \cdots$.

OBJECTIVES AND EXAMPLES

Lesson 12-3 Find the limit of the terms and the sum of an infinite geometric series.

Find $\lim\limits_{n\to\infty} \frac{2n^2+5}{3n^2}$.

$$\lim_{n\to\infty} \frac{2n^2+5}{3n^2} = \lim_{n\to\infty}\left(\frac{2}{3} + \frac{5}{3n^2}\right)$$

$$= \lim_{n\to\infty} \frac{2}{3} + \lim_{n\to\infty} \frac{5}{3} \cdot \lim_{n\to\infty} \frac{1}{n^2}$$

$$= \frac{2}{3} + \frac{5}{3} \cdot 0$$

Thus, the limit is $\frac{2}{3}$.

REVIEW EXERCISES

Find each limit, or state that the limit does not exist and explain your reasoning.

22. $\lim\limits_{n\to\infty} \frac{3n}{4n+1}$

23. $\lim\limits_{n\to\infty} \frac{6n-3}{n}$

24. $\lim\limits_{n\to\infty} \frac{2^n\, n^3}{3n^3}$

25. $\lim\limits_{n\to\infty} \frac{4n^3-3n}{n^4-4n^3}$

26. Write $5.\overline{123}$ as a fraction.

27. Find the sum of the infinite series $1260 + 504 + 201.6 + 80.64 + \cdots$, or state that the sum does not exist and explain your reasoning.

Lesson 12-4 Determine whether a series is convergent or divergent.

Use the ratio test to determine whether the series $3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!}$ is convergent or divergent.

The nth term a_n of this series has a general form of $\frac{3^n}{n!}$ and $a_{n+1} = \frac{3^{n+1}}{(n+1)!}$. Find $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$.

$$r = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}$$

$$r = \lim_{n\to\infty} \left[\frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}\right]$$

$$r = \lim_{n\to\infty} \frac{3}{n+1} \text{ or } 0$$

Since $r < 0$, the series is convergent.

28. Use the ratio test to determine whether the series $\frac{1}{5} + \frac{2^2}{5^2} + \frac{3^2}{5^3} + \frac{4^2}{5^4} + \cdots$ is *convergent* or *divergent*.

29. Use the comparison test determine whether the series $\frac{6}{1} + \frac{7}{2} + \frac{8}{3} + \frac{9}{4} + \cdots$ is *convergent* or *divergent*.

30. Determine whether the series $2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7} + \cdots$ is *convergent* or *divergent*.

Lesson 12-5 Use sigma notation.

Write $\sum\limits_{n=1}^{3} (n^2-1)$ in expanded form and then find the sum.

$$\sum_{n=1}^{3} (n^2-1) = (1^2-1) + (2^2-1) + (3^2-1)$$

$$= 0 + 3 + 8 \text{ or } 11$$

Write each expression in expanded form and then find the sum.

31. $\sum\limits_{a=5}^{9} (3a-3)$

32. $\sum\limits_{k=1}^{\infty} (0.4)^k$

Express each series using sigma notation

33. $-1 + 1 + 3 + 5 + \cdots$

34. $2 + 5 + 10 + 17 + \cdots + 82$

CONTENTS

OBJECTIVES AND EXAMPLES

REVIEW EXERCISES

Lesson 12-6 Use the Binomial Theorem to expand binomials.

Find the fourth term of $(2x - y)^6$.

$$(2x - y)^6 = \sum_{r=0}^{6} \frac{6!}{r!(6 - r)!}(2x)^{6-r}(-y)^r$$

To find the fourth term, evaluate the general term for $r = 3$.

$$\frac{6!}{3!(6 - 3)!}(2x)^{6-3}(-y)^3$$

$$= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3! \cdot 3!}(2x)^3(-y^3) \text{ or } -160x^3y^3$$

Use the Binomial Theorem to expand each binomial.

35. $(a - 4)^6$

36. $(2r + 3s)^4$

Find the designated term of each binomial expansion.

37. 5th term of $(x - 2)^{10}$

38. 3rd term of $(4m + 1)^8$

39. 8th term of $(x + 3y)^{10}$

40. 6th term of $(2c - d)^{12}$

Lesson 12-7 Use Euler's Formula to write the exponential form of a complex number.

Write $\sqrt{3} - \boldsymbol{i}$ in exponential form.

Write the polar form of $\sqrt{3} - \boldsymbol{i}$.

$r = \sqrt{(\sqrt{3})^2 + (-1)^2}$ or 2, and

$\theta = \text{Arctan } \frac{-1}{\sqrt{3}}$ or $\frac{5\pi}{6}$

$$\sqrt{3} - \boldsymbol{i} = 2\left(\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}\right) = 2e^{\boldsymbol{i}\frac{5\pi}{6}}$$

Write each expression or complex number in exponential form.

41. $2\left(\cos \frac{3\pi}{4} + \boldsymbol{i} \sin \frac{3\pi}{4}\right)$

42. $4\boldsymbol{i}$

43. $2 - 2\boldsymbol{i}$

44. $3\sqrt{3} + 3\boldsymbol{i}$

Lesson 12-8 Iterate functions using real and complex numbers.

Find the first three iterates of the function $f(z) = 2z$ if the initial value is $3 - \boldsymbol{i}$.

$z_0 = 3 - \boldsymbol{i}$

$z_1 = 2(3 - \boldsymbol{i})$ or $6 - 2\boldsymbol{i}$

$z_2 = 2(6 - 2\boldsymbol{i})$ or $12 - 4\boldsymbol{i}$

$z_3 = 2(12 - 4\boldsymbol{i})$ or $24 - 8\boldsymbol{i}$

Find the first four iterates of each function using the given initial value. If necessary, round your answers to the nearest hundredth.

45. $f(x) = 6 - 3x,\ x_0 = 2$

46. $f(x) = x^2 + 4,\ x_0 = -3$

Find the first three iterates of the function $f(z) = 0.5z + (4 - 2i)$ for each initial value.

47. $z_0 = 4\boldsymbol{i}$

48. $z_0 = -8$

49. $z_0 = -4 + 6\boldsymbol{i}$

50. $z_0 = 12 - 8\boldsymbol{i}$

Lesson 12-9 Use mathematical induction to prove the validity of mathematical statements.

Proof by mathematical induction:

1. First, verify that the conjecture S_n is valid for the first possible case, usually $n = 1$.
2. Then, assume that S_n is valid for $n = k$ and use this assumption to prove that it is also valid for $n = k + 1$.

Use mathematical induction to prove that each proposition is valid for all positive integral values of *n*.

51. $1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$

52. $3 + 8 + 15 + \cdots + n(n - 2) = \frac{n(n + 1)(2n + 7)}{6}$

53. $9^n - 4^n$ is divisible by 5.

CONTENTS

APPLICATIONS AND PROBLEM SOLVING

54. Physics If an object starting at rest falls in a vacuum near the surface of Earth, it will fall 16 feet during the first second, 48 feet during the next second, 80 feet during the third second, and so on. *(Lesson 12-1)*

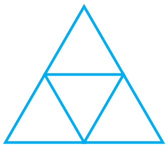
a. How far will the object fall during the twelfth second?

b. How far will the object have fallen after twelve seconds?

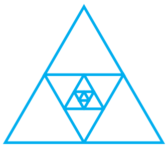
55. Budgets A major corporation plans to cut the budget on one of its projects by 3 percent each year. If the current budget for the project to be cut is $160 million, what will the budget for that project be in 10 years? *(Lesson 12-2)*

56. Geometry If the midpoints of the sides of an equilateral triangle are joined by straight lines, the new figure will also be an equilateral triangle. *(Lesson 12-3)*

a. If the original triangle has a perimeter of 6 units, find the perimeter of the new triangle.



b. If this process is continued to form a sequence of “nested” triangles, what will be the sum of the perimeters of all the triangles?



ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. A sequence has a common difference of 3.
   - **a.** Is this sequence arithmetic or geometric? Explain.
   - **b.** Form a sequence that has this common difference. Write a recursive formula for your sequence.
2. Write a general expression for an infinite sequence that has no limit. Explain your reasoning.

PORTFOLIO

Explain the difference between a convergent and a divergent series. Give an example of each type of series and show why it is that type of series.

Additional Assessment See p. A67 for Chapter 12 practice test.

Unit 4 interNET Project

THE UNITED STATES CENSUS BUREAU

That’s a lot of people!

- Use the Internet to find the population of the United States from at least 1900 through 2000. Write a sequence using the population for each ten-year interval, for example, 1900, 1910, and so on.
- Write a formula for an arithmetic sequence that provides a reasonable model for the population sequence.
- Write a formula for a geometric sequence that provides a reasonable model for the population sequence.
- Use your models to predict the U.S. population for the year 2050. Write a one-page paper comparing the arithmetic and geometric sequences you used to model the population data. Discuss which formula you think best models the data.

CONTENTS

Percent Problems

A few common words and phrases used in percent problems, along with their translations into mathematical expressions, are listed below.

what ➡ x (variable)

of ➡ × (multiply)

is ➡ = (equals)

What percent of A is B? ➡ $\frac{x}{100} \cdot A = B$

What is A percent more than B? ➡ $x = B + \frac{A}{100} \cdot B$

What is A percent less than B? ➡ $x = B - \frac{A}{100} \cdot B$

percent of change (increase or decrease) ➡ $\frac{\text{amount of change}}{\text{original amount}} \times 100$

THE PRINCETON REVIEW

TEST-TAKING TIP

With common percents, like 10%, 25%, or 50%, it is faster to use the fraction equivalents and mental math than a calculator.

ACT EXAMPLE

1. If c is positive, what percent of $3c$ is 9?

A $\frac{c}{100}\%$ **B** $\frac{300}{c}\%$ **C** $\frac{9}{c}\%$ **D** 3% **E** $\frac{c}{3}\%$

HINT Use variables just as you would use numbers.

Solution Start by translating the question into an equation.

$$\begin{array}{ccccc} \underbrace{\text{What percent}} & \text{of} & 3c & \text{is} & 9? \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ \frac{x}{100} & \cdot & 3c & = & 9 \end{array}$$

Now solve the equation for x, not c.

$\frac{x}{100} = \frac{9}{3c}$ *Divide each side by 3c.*

$\frac{x}{100} = \frac{3}{c}$ *Simplify.*

$x = \frac{300}{c}$ *Solve for x.*

The answer is choice **B**.

Alternate Solution "Plug in" 3 for c. The question becomes "what percent of 9 is 9?" The answer is 100%, so check each expression choice to see if it is equal to 100% when $c = 3$.

Choice A: $\frac{c}{100}\% = \frac{3}{100}\%$

Choice B: $\frac{300}{c}\% = 100\%$

Choice **B** is correct.

SAT EXAMPLE

2. An electronics store offers a 25% discount on all televisions during a sale week. How much must a customer pay for a television marked at $240?

A $60 **B** $300 **C** $230.40

D $180 **E** $215

HINT A discount is a decrease in the price of an item. So, the question asked is "What is 25% less than 240?"

Solution Start by translating this question into an equation.

What is 25% less than 240? ➡ $x = 240 - \frac{25}{100} \cdot 240$

Now simplify the right-hand side of the equation.

$$x = 240 - \frac{1}{4} \cdot 240 \text{ or } 180$$

Choice **D** is correct.

Alternate Solution If there is a 25% discount, a customer will pay (100 − 25)% or 75% of the marked price.

What is 75% of the marked price? ➡ $x = 0.75 \cdot 240 \text{ or } 180$

The answer is choice **D**.

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SAT AND ACT PRACTICE

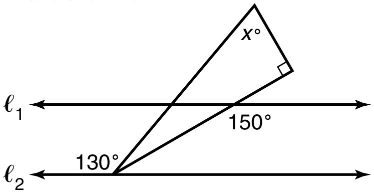
After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. Shanika has a collection of 80 tapes. If 40% of her records are jazz tapes and the rest are blues tapes, how many blues tapes does she have?

A 32 **B** 40 **C** 42 **D** 48 **E** 50

2. If ℓ_1 is parallel to ℓ_2 in the figure below, what is the value of x?



A 20 **B** 50 **C** 70 **D** 80 **E** 90

3. There are k gallons of gasoline available to fill a tank. After d gallons have been pumped, then, in terms of k and d, what percent of the gasoline has been pumped?

A $\frac{100d}{k}\%$ **B** $\frac{k}{100d}\%$ **C** $\frac{100k}{d}\%$

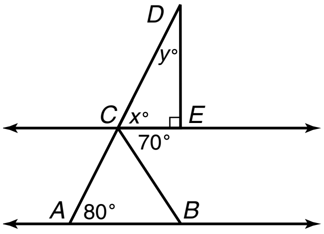
D $\frac{k}{100(k-d)}\%$ **E** $\frac{100(k-d)}{k}\%$

4. In 1985, Andrei had a collection of 48 baseball caps. Since then he has given away 13 caps, purchased 17 new caps, and traded 6 of his caps to Pierre for 8 of Pierre's caps. Since 1985, what has been the net percent increase in Andrei's collection?

A 6% **B** $12\frac{1}{2}\%$ **C** $16\frac{2}{3}\%$

D 25% **E** $28\frac{1}{2}\%$

5. In the figure below, $AB = AC$ and AD is a line segment. What is the value of $x - y$?



Note: Figure is NOT drawn to scale.

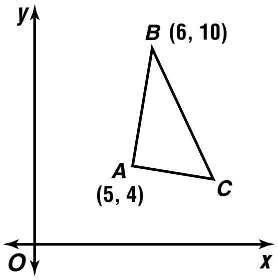
A 10 **B** 20 **C** 30 **D** 70 **E** 90

6. At the beginning of 2000, the population of Rockville was 204,000, and the population of Springfield was 216,000. If the population of each city increased by exactly 20% in 2000, how many more people lived in Springfield than in Rockville at the end of 2000?

A 9,600 **B** 10,000 **C** 12,000

D 14,400 **E** 20,000

7. In the figure, the slope of $\overline{AC}$ is $-\frac{1}{6}$, and $m\angle C = 30°$. What is the length of $\overline{BC}$?



A $\sqrt{37}$

B $\sqrt{111}$

C 2

D $2\sqrt{37}$

E It cannot be determined from the information given.

8. If $x + 6 > 0$ and $1 - 2x > -1$, then x could equal each of the following EXCEPT ?

A -6 **B** -4 **C** -2 **D** 0 **E** $\frac{1}{2}$

9. The percent increase from 99 to 100 is which of the following?

A greater than 1

B 1

C less than 1, but more than $\frac{1}{2}$

D less than $\frac{1}{2}$, but more than 0

E 0

10. **Grid-In** One fifth of the cars in a parking lot are blue, and $\frac{1}{2}$ of the blue cars are convertibles. If $\frac{1}{4}$ of the convertibles in the parking lot are blue, then what percent of the cars in the lot are neither blue nor convertibles?

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