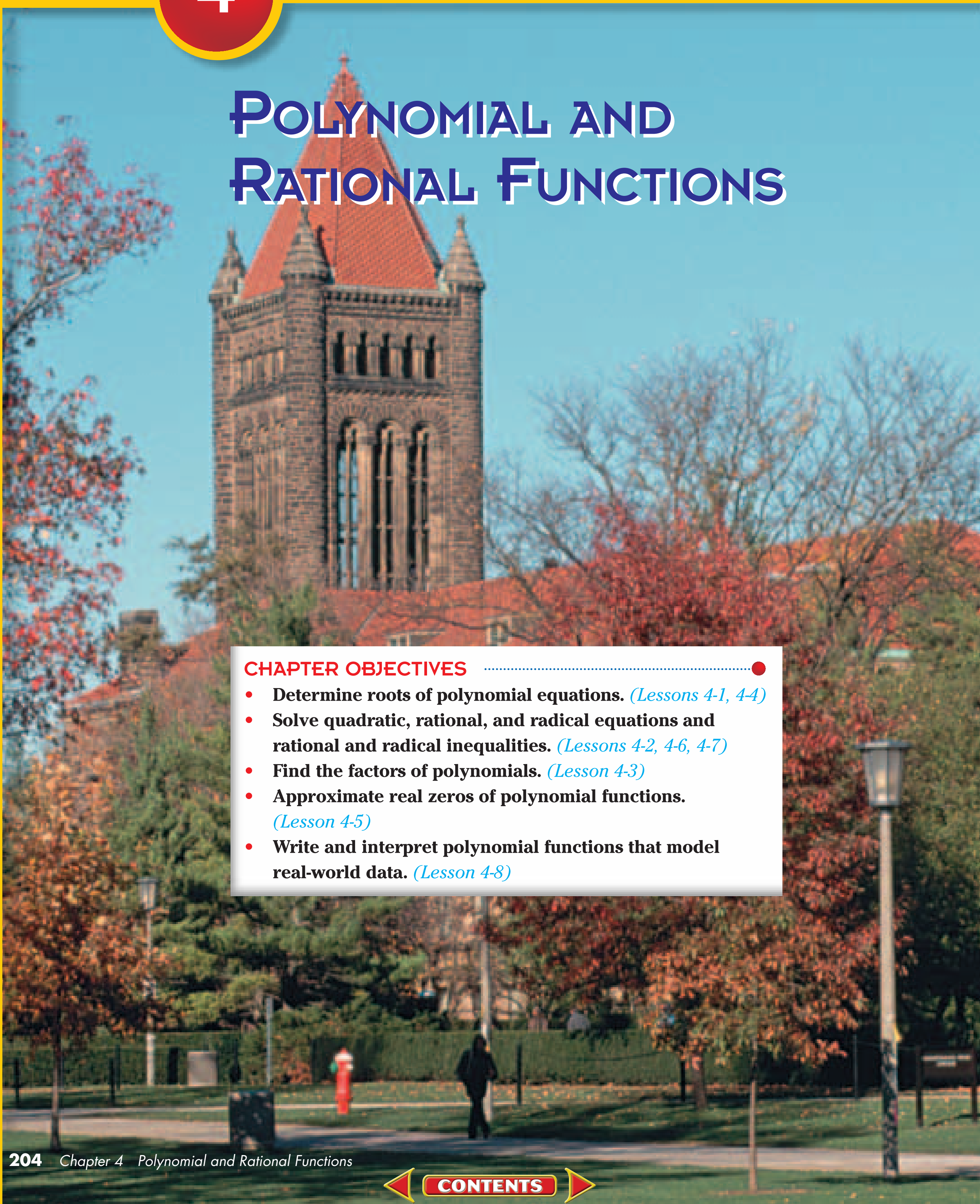


POLYNOMIAL AND RATIONAL FUNCTIONS

CHAPTER OBJECTIVES

- **Determine roots of polynomial equations.** *(Lessons 4-1, 4-4)*
- **Solve quadratic, rational, and radical equations and rational and radical inequalities.** *(Lessons 4-2, 4-6, 4-7)*
- **Find the factors of polynomials.** *(Lesson 4-3)*
- **Approximate real zeros of polynomial functions.** *(Lesson 4-5)*
- **Write and interpret polynomial functions that model real-world data.** *(Lesson 4-8)*

4-1

Polynomial Functions

OBJECTIVES

- Determine roots of polynomial equations.
- Apply the Fundamental Theorem of Algebra.

Real World Application

INVESTMENTS Many grandparents invest in the stock market for their grandchildren's college fund. Eighteen years ago, Della Brooks purchased \$1000 worth of merchandising stocks at the birth of her first grandchild Owen. Ten years ago, she purchased \$500 worth of transportation stocks, and five years ago, she purchased \$250 worth of technology stocks. The stocks will be used to help pay for Owen's college education. If the stocks appreciate at an average annual rate of 12.25%, determine the current value of the college fund. *This problem will be solved in Example 1.*

Appreciation is the increase in value of an item over a period of time. The formula for compound interest can be used to find the value of Owen's college fund after appreciation. The formula is $A = P(1 + r)^t$, where P is the original amount of money invested, r is the interest rate or rate of return (written as a decimal), and t is the time invested (in years).

Example 1 Real World Application

INVESTMENTS **The value of Owen's college fund is the sum of the current values of his grandmother's investments.**

a. Write a function in one variable that models the value of the college fund for any rate of return.

b. Use the function to determine the current value of the college fund for an average annual rate of 12.25%.

a. Let x represent $1 + r$ and $T(x)$ represent the total current value of the three stocks. The times invested, which are the exponents of x, are 18, 10, and 5, respectively.

Total	=	*merchandising*	+	*transportation*	+	*technology*
$T(x)$	=	$1000x^{18}$	+	$500x^{10}$	+	$250x^5$

b. Since $r = 0.1225$, $x = 1 + 0.1225$ or 1.1225. Now evaluate $T(x)$ for $x = 1.1225$.

$$T(x) = 1000x^{18} + 500x^{10} + 250x^5$$
$$T(1.1225) = 1000(1.1225)^{18} + 500(1.1225)^{10} + 250(1.1225)^5$$
$$T(1.1225) \approx 10{,}038.33$$

The present value of Owen's college fund is about \$10,038.33.

The expression $1000x^{18} + 500x^{10} + 250x^5$ is a **polynomial in one variable.**

Polynomial in One Variable

A polynomial in one variable, x, is an expression of the form $a_0x^n + a_1x^{n-1} + \dots + a_{n-2}x^2 + a_{n-1}x + a_n$. The coefficients $a_0, a_1, a_2, \dots, a_n$ represent complex numbers (real or imaginary), a_0 is not zero, and n represents a nonnegative integer.

The **degree** of a polynomial in one variable is the greatest exponent of its variable. The coefficient of the variable with the greatest exponent is called the **leading coefficient.** For the expression $1000x^{18} + 500x^{10} + 250x^5$, 18 is the degree, and 1000 is the leading coefficient.

If a function is defined by a polynomial in one variable with real coefficients, like $T(x) = 1000x^{18} + 500x^{10} + 250x^5$, then it is a **polynomial function.** If $f(x)$ is a polynomial function, the values of x for which $f(x) = 0$ are called the **zeros** of the function. *If the function is graphed, these zeros are also the x-intercepts of the graph.*

Example 2 **Consider the polynomial function $f(x) = x^3 - 6x^2 + 10x - 8$.**

a. State the degree and leading coefficient of the polynomial.

b. Determine whether 4 is a zero of $f(x)$.

a. $x^3 - 6x^2 + 10x - 8$ has a degree of 3 and a leading coefficient of 1.

b. Evaluate $f(x) = x^3 - 6x^2 + 10x - 8$ for $x = 4$. That is, find $f(4)$.

$f(4) = 4^3 - 6(4^2) + 10(4) - 8 \quad x = 4$
$f(4) = 64 - 96 + 40 - 8$
$f(4) = 0$

Since $f(4) = 0$, 4 is a zero of $f(x) = x^3 - 6x^2 + 10x - 8$.

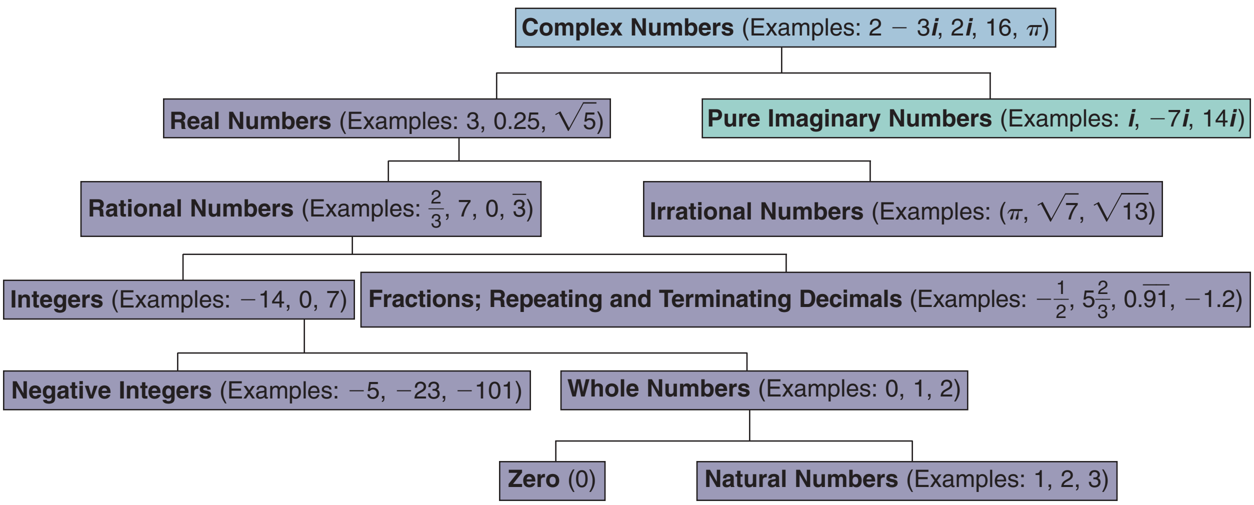
Graphing Calculator Tip

To find a value of a polynomial for a given value of *x*, enter the polynomial in the **Y=** list. Then use the **1:value** option in the **CALC** menu.

Since 4 is a zero of $f(x) = x^3 - 6x^2 + 10x - 8$, it is also a solution for the **polynomial equation** $x^3 - 6x^2 + 10x - 8 = 0$. The solution for a polynomial equation is called a **root.** The words *zero* and *root* are often used interchangeably, but technically, you find the *zero of a function* and the *root of an equation.*

A root or zero may also be an **imaginary number** such as $3\boldsymbol{i}$. By definition, the imaginary unit $\boldsymbol{i}$ equals $\sqrt{-1}$. Since $\boldsymbol{i} = \sqrt{-1}$, $\boldsymbol{i}^2 = -1$. It also follows that $\boldsymbol{i}^3 = \boldsymbol{i}^2 \times \boldsymbol{i}$ or $-\boldsymbol{i}$ and $\boldsymbol{i}^4 = \boldsymbol{i}^2 \times \boldsymbol{i}^2$ or 1.

The imaginary numbers combined with the real numbers compose the set of **complex numbers.** A complex number is any number of the form $a + b\boldsymbol{i}$ where a and b are real numbers. If $b = 0$, then the complex number is a real number. If $a = 0$ and $b \neq 0$, then the complex number is called a **pure imaginary number.**



One of the most important theorems in mathematics is the **Fundamental Theorem of Algebra.**

Fundamental Theorem of Algebra

Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.

A corollary to the Fundamental Theorem of Algebra states that the degree of a polynomial indicates the number of possible roots of a polynomial equation.

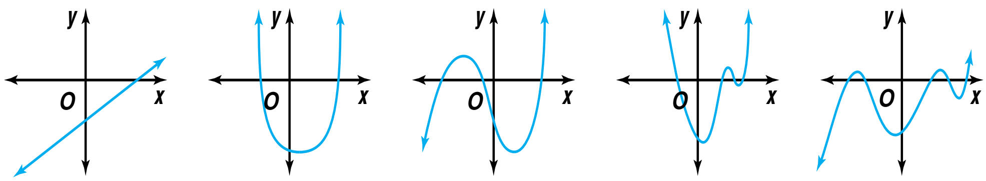
Corollary to the Fundamental Theorem of Algebra

Every polynomial $P(x)$ of degree n $(n > 0)$ can be written as the product of a constant k $(k \neq 0)$ and n linear factors.

$$P(x) = k(x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n)$$

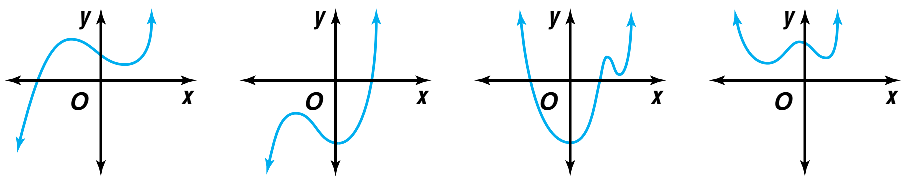
Thus, a polynomial equation of degree n has exactly n complex roots, namely $r_1, r_2, r_3, \ldots, r_n$.

The general shapes of the graphs of polynomial functions with positive leading coefficients and degree greater than 0 are shown below. These graphs also show the *maximum* number of times the graph of each type of polynomial may cross the x-axis.



Degree 1 **Degree 2** **Degree 3** **Degree 4** **Degree 5**

Since the x-axis only represents real numbers, imaginary roots cannot be determined by using a graph. The graphs below have the general shape of a third-degree function and a fourth-degree function. In these graphs, the third-degree function only crosses the x-axis once, and the fourth-degree function crosses the x-axis twice or not at all.



Degree 3, 1 x-intercept **Degree 3, 1 x-intercept** **Degree 4, 2 x-intercepts** **Degree 4, 0 x-intercepts**

The graph of a polynomial function with odd degree must cross the x-axis at least once. The graph of a function with even degree may or may not cross the x-axis. If it does, it will cross an even number of times. Each x-intercept represents a real root of the corresponding polynomial equation.

If you know the roots of a polynomial equation, you can use the corollary to the Fundamental Theorem of Algebra to find the polynomial equation. That is, if a and b are roots of the equation, the equation must be $(x - a)(x - b) = 0$.

Examples

3 **a. Write a polynomial equation of least degree with roots 2, $4i$, and $-4i$.**

b. Does the equation have an odd or even degree? How many times does the graph of the related function cross the x-axis?

a. If $x = 2$, then $x - 2$ is a factor of the polynomial. Likewise, if $x = 4\boldsymbol{i}$ and $x = -4\boldsymbol{i}$, then $x - 4\boldsymbol{i}$ and $x - (-4\boldsymbol{i})$ are factors of the polynomial. Therefore, the linear factors for the polynomial are $x - 2$, $x - 4\boldsymbol{i}$, and $x + 4\boldsymbol{i}$. Now find the products of these factors.

$$(x - 2)(x - 4\boldsymbol{i})(x + 4\boldsymbol{i}) = 0$$

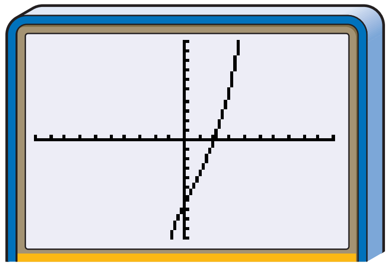
$$(x - 2)(x^2 - 16\boldsymbol{i}^2) = 0$$

$$(x - 2)(x^2 + 16) = 0 \quad -16\boldsymbol{i}^2 = -16(-1) \text{ or } 16$$

$$x^3 - 2x^2 + 16x - 32 = 0$$

A polynomial equation with roots 2, $4\boldsymbol{i}$, and $-4\boldsymbol{i}$ is $x^3 - 2x^2 + 16x - 32 = 0$.

b. The degree of this equation is 3. Thus, the equation has an odd degree since 3 is an odd number. Since two of the roots are imaginary, the graph will only cross the x-axis once. The graphing calculator image at the right verifies these conclusions.



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4 **State the number of complex roots of the equation $9x^4 - 35x^2 - 4 = 0$. Then find the roots and graph the related function.**

The polynomial has a degree of 4, so there are 4 complex roots.

Factor the equation to find the roots.

$$9x^4 - 35x^2 - 4 = 0$$

$$(9x^2 + 1)(x^2 - 4) = 0$$

$$(9x^2 + 1)(x + 2)(x - 2) = 0$$

To find each root, set each factor equal to zero.

$$9x^2 + 1 = 0$$

$$x^2 = -\frac{1}{9} \quad \textit{Solve for } x^2.$$

$$x = \pm\sqrt{\frac{1}{9} \cdot (-1)} \quad \textit{Take the square root of each side.}$$

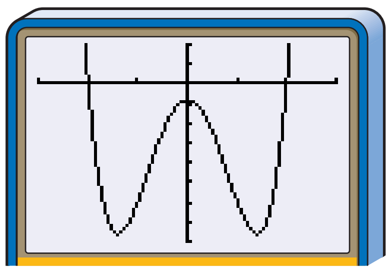
$$x = \pm\frac{1}{3}\sqrt{-1} \text{ or } \pm\frac{1}{3}\boldsymbol{i}$$

$$x + 2 = 0 \qquad x - 2 = 0$$

$$x = -2 \qquad x = 2$$

The roots are $\pm\frac{1}{3}\boldsymbol{i}$, -2, and 2.

Use a table of values or a graphing calculator to graph the function. The x-intercepts are -2 and 2.



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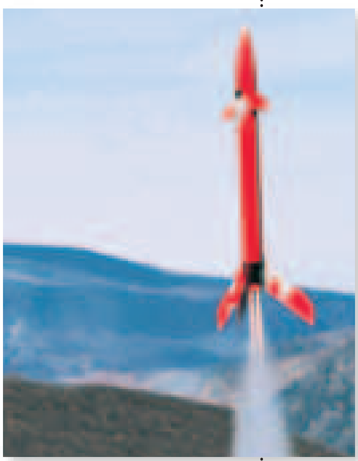
The function has an even degree and has 2 real zeros.

Example 5 **METEOROLOGY** **A meteorologist sends a temperature probe on a small weather rocket through a cloud layer. The launch pad for the rocket is 2 feet off the ground. The height of the rocket after launching is modeled by the equation $h = -16t^2 + 232t + 2$, where h is the height of the rocket in feet and t is the elapsed time in seconds.**

Real World Application



a. When will the rocket be 114 feet above the ground?

b. Verify your answer using a graph.

a.

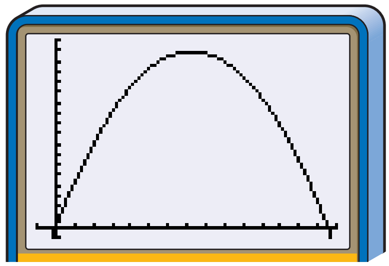
$$\begin{aligned} h &= -16t^2 + 232t + 2 & \\ 114 &= -16t^2 + 232t + 2 & &\textit{Replace h with 114.} \\ 0 &= -16t^2 + 232t - 112 & &\textit{Subtract 114 from each side.} \\ 0 &= -8(2t^2 - 29t + 14) & &\textit{Factor.} \\ 0 &= -8(2t - 1)(t - 14) & &\textit{Factor.} \end{aligned}$$

$$2t - 1 = 0 \quad \text{or} \quad t - 14 = 0$$

$$t = \frac{1}{2} \qquad\qquad 14 = t$$

The weather rocket will be 114 feet above the ground after $\frac{1}{2}$ second and again after 14 seconds.

b. To verify the answer, graph $h(t) = -16t^2 + 232t - 112$. The graph appears to verify this solution.



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CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** several sentences about the relationship between zeros and roots.
2. **Explain** why zeros of a function are also the x-intercepts of its graph.
3. **Define** a complex number and tell under what conditions it will be a pure imaginary number. Write two examples and two **nonexamples** of a pure imaginary numbers.
4. **Sketch** the general graph of a sixth-degree function.

Guided Practice

State the degree and leading coefficient of each polynomial.

5. $a^3 + 6a + 14$

6. $5m^2 + 8m^5 - 2$

Determine whether each number is a root of $x^3 - 5x^2 - 3x - 18 = 0$. Explain.

7. 5

8. 6

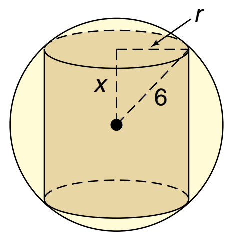
Write a polynomial equation of least degree for each set of roots. Does the equation have an odd or even degree? How many times does the graph of the related function cross the x-axis?

9. $-5, 7$ **10.** $6, 2\boldsymbol{i}, -2\boldsymbol{i}, \boldsymbol{i}, -\boldsymbol{i}$

State the number of complex roots of each equation. Then find the roots and graph the related functions.

11. $x^2 - 14x + 49 = 0$ **12.** $a^3 + 2a^2 - 8a = 0$ **13.** $t^4 - 1 = 0$

14. Geometry A cylinder is inscribed in a sphere with a radius of 6 units as shown.



a. Write a function that models the volume of the cylinder in terms of x. (*Hint:* The volume of a cylinder equals $\pi r^2 h$.)

b. Write this function as a polynomial function.

c. Find the volume of the cylinder if $x = 4$.

EXERCISES

Practice

State the degree and leading coefficient of each polynomial.

15. $5t^4 + t^3 - 7$ **16.** $3x^7 - 4x^5 + x^3$ **17.** $9a^2 + 5a^3 - 10$

18. $14b - 25b^5$ **19.** $p^5 + 7p^3 - p^6$ **20.** $14y + 30 + y^2$

21. Determine if $x^3 + 3x + \sqrt{5}$ is a polynomial in one variable. Explain.

22. Is $\frac{1}{a} + a^2$ a polynomial in one variable? Explain.

Determine whether each number is a root of $a^4 - 13a^2 + 12a = 0$. Explain.

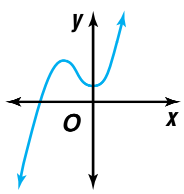
23. 0 **24.** -1 **25.** 1 **26.** -4 **27.** -3 **28.** 3

29. Is -2 a root of $b^4 - 3b^2 - 2b + 4 = 0$?

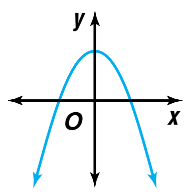
30. Is -1 a root of $x^4 - 4x^3 - x^2 + 4x = 0$?

31. Each graph represents a polynomial function. State the number of complex zeros and the number of real zeros of each function.

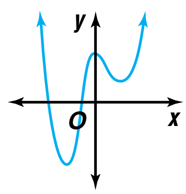
a.



b.



c.



Write a polynomial equation of least degree for each set of roots. Does the equation have an odd or even degree? How many times does the graph of the related function cross the x-axis?

32. $-2, 3$ **33.** $-1, 1, 5$ **34.** $-2, -0.5, 4$

35. $-3, -2\boldsymbol{i}, 2\boldsymbol{i}$ **36.** $-5\boldsymbol{i}, -\boldsymbol{i}, \boldsymbol{i}, 5\boldsymbol{i}$ **37.** $-1, 1, 4, -4, 5$

38. Write a polynomial equation of least degree whose roots are -1, 1, 3, and -3.

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CONTENTS

State the number of complex roots of each equation. Then find the roots and graph the related function.

39. $x + 8 = 0$

40. $a^2 - 81 = 0$

41. $b^2 + 36 = 0$

42. $t^3 + 2t^2 - 4t - 8 = 0$

43. $n^3 - 9n = 0$

44. $6c^3 - 3c^2 - 45c = 0$

45. $a^4 + a^2 - 2 = 0$

46. $x^4 - 10x^2 + 9 = 0$

47. $4m^4 + 17m^2 + 4 = 0$

48. Solve $(u + 1)(u^2 - 1) = 0$ and graph the related polynomial function.

49. Sketch a fourth-degree equation for each situation.

a. no x-intercept
b. one x-intercept
c. two x-intercepts
d. three x-intercepts
e. four x-intercepts
f. five x-intercepts

Graphing Calculator

50. Use a graphing calculator to graph $f(x) = x^4 - 2x^2 + 1$.

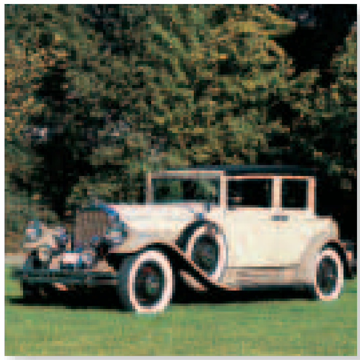
a. What is the maximum number of x-intercepts possible for this function?

b. How many x-intercepts are there? Name the intercept(s).

c. Why are there fewer x-intercepts than the maximum number? (*Hint:* The factored form of the polynomial is $(x^2 - 1)^2$.)

Applications and Problem Solving

51. Classic Cars Sonia Orta invests in vintage automobiles. Three years ago, she purchased a 1953 Corvette roadster for \$99,000. Two years ago, she purchased a 1929 Pierce-Arrow Model 125 for \$55,000. A year ago she purchased a 1909 Cadillac Model Thirty for \$65,000.



a. Let x represent 1 plus the average rate of appreciation. Write a function in terms of x that models the value of the automobiles.

b. If the automobiles appreciate at an average annual rate of 15%, find the current value of the three automobiles.

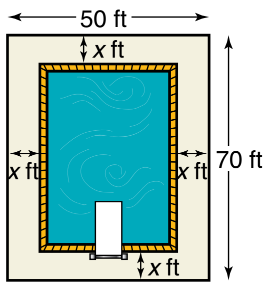
52. Critical Thinking One of the zeros of a polynomial function is 1. After translating the graph of the function left 2 units, 1 is a zero of the new function. What do you know about the original function?

53. Aeronautics At liftoff, the space shuttle *Discovery* has a constant acceleration, a, of 16.4 feet per second squared. The function $d(t) = \frac{1}{2}at^2$ can be used to determine the distance from Earth for each time interval, t, after takeoff.

a. Find its distance from Earth after 30 seconds, 1 minute, and 2 minutes.

b. Study the pattern of answers to part a. If the time the space shuttle is in flight doubles, how does the distance from Earth change? Explain.

54. Construction The Santa Fe Recreation Department has a 50-foot by 70-foot area for construction of a new public swimming pool. The pool will be surrounded by a concrete sidewalk of constant width. Because of water restrictions, the pool can have a maximum area of 2400 square feet. What should be the width of the sidewalk that surrounds the pool?



55. **Marketing** Each week, Marino's Pizzeria sells an average of 160 large supreme pizzas for \$16 each. Next week, the pizzeria plans to run a sale on these large supreme pizzas. The owner estimates that for each 40¢ decrease in the price, the store will sell approximately 16 more large pizzas. If the owner wants to sell \$4,000 worth of the large supreme pizzas next week, determine the sale price.

56. **Critical Thinking** If B and C are the real roots of $x^2 + Bx + C = 0$, where $B \neq 0$ and $C \neq 0$, find the values of B and C.

Mixed Review

57. Create a function in the form $y = f(x)$ that has a vertical asymptote at $x = -2$ and $x = 0$, and a hole at $x = 2$. *(Lesson 3-7)*

58. **Construction** Selena wishes to build a pen for her animals. He has 52 yards of fencing and wants to build a rectangular pen. *(Lesson 3-6)*

a. Find a model for the area of the pen as a function of the length and width of the rectangle.

b. What are the dimensions that would produce the maximum area?

59. Describe how the graphs of $y = 2x^3$ and $y = 2x^3 + 1$ are related. *(Lesson 3-2)*

60. Find the coordinates of P' if $P(4, 9)$ and P' are symmetric with respect to $M(-1, 9)$. *(Lesson 3-1)*

61. Find the determinant for $\begin{bmatrix} -15 & 5 \\ -9 & 3 \end{bmatrix}$. Tell whether an inverse exists for the matrix. *(Lesson 2-5)*

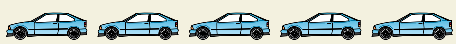
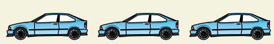
62. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -9 & 2 \\ 5 & 7 & -6 \end{bmatrix}$, find AB. *(Lesson 2-3)*

63. Graph $x + 4y < 9$. *(Lesson 1-8)*

64. The slope of $\overleftrightarrow{AB}$ is 0.6. The slope of $\overleftrightarrow{CD}$ is $\frac{3}{5}$. State whether the lines are *parallel, perpendicular,* or *neither.* Explain. *(Lesson 1-5)*

65. Find $[f \circ g](x)$ and $[g \circ f](x)$ for the functions $f(x) = x^2 - 4$ and $g(x) = \frac{1}{2}x + 6$. *(Lesson 1-2)*

66. **SAT Practice** In 2003, Bob's Quality Cars sold 270 more cars than in 2004. How many cars does each  represent?

Year	Cars Sold by Bob's Quality Cars
2003	
2004	

A 135
B 125
C 100
D 90
E 85

Extra Practice See p. A32.

4-2 Quadratic Equations

OBJECTIVES

- Solve quadratic equations.
- Use the discriminant to describe the roots of quadratic equations.

Real World Application

BASEBALL On September 8, 1998, Mark McGwire of the St. Louis Cardinals broke the home-run record with his 62nd home run of the year. He went on to hit 70 home runs for the season. Besides hitting home runs, McGwire also occasionally popped out. Suppose the ball was 3.5 feet above the ground when he hit it straight up with an initial velocity of 80 feet per second. The function $d(t) = 80t - 16t^2 + 3.5$ gives the ball's height above the ground in feet as a function of time in seconds. How long did the catcher have to get into position to catch the ball after it was hit? *This problem will be solved in Example 3.*

A quadratic equation is a polynomial equation with a degree of two. Solving quadratic equations by graphing usually does not yield exact answers. Also, some quadratic expressions are not factorable over the integers. Therefore, alternative strategies for solving these equations are needed. One such alternative is solving quadratic equations by **completing the square.**

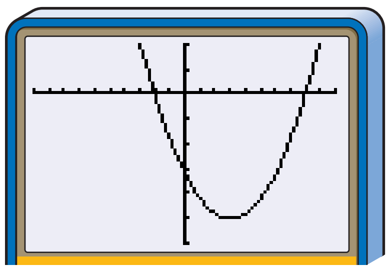
Completing the square is a useful method when the quadratic is not easily factorable. It can be used to solve any quadratic equation. Remember that, for any number b, the square of the binomial $x + b$ has the form $x^2 + 2bx + b^2$. When completing the square, you know the first term and middle term and need to supply the last term. This term equals the square of half the coefficient of the middle term. For example, to complete the square of $x^2 + 8x$, find $\frac{1}{2}(8)$ and square the result. So, the third term is 16, and the expression becomes $x^2 + 8x + 16$. Note that this technique works only if the coefficient of x^2 is 1.

Example 1 **Solve $x^2 - 6x - 16 = 0$.**

This equation can be solved by graphing, factoring, or completing the square.

Method 1

Solve the equation by graphing the related function $f(x) = x^2 - 6x - 16$. The zeros of the function appear to be -2 and 8.



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Method 2

Solve the equation by factoring.

$$x^2 - 6x - 16 = 0$$

$$(x + 2)(x - 8) = 0 \quad \textit{Factor.}$$

$$x + 2 = 0 \quad \text{or} \quad x - 8 = 0$$

$$x = -2 \qquad\qquad x = 8$$

The roots of the equation are -2 and 8.

Method 3

Solve the equation by completing the square.

$$
\begin{aligned}
x^2 - 6x - 16 &= 0 \\
x^2 - 6x &= 16 && \textit{Add 16 to each side.} \\
x^2 - 6x + 9 &= 16 + 9 && \textit{Complete the square by adding } \left(\frac{-6}{2}\right)^2 \textit{ or 9 to each side.} \\
(x - 3)^2 &= 25 && \textit{Factor the perfect square trinomial.} \\
x - 3 &= \pm 5 && \textit{Take the square root of each side.}
\end{aligned}
$$

$$
\begin{aligned}
x - 3 &= 5 \quad \text{or} \quad & x - 3 &= -5 \\
x &= 8 & x &= -2
\end{aligned}
$$

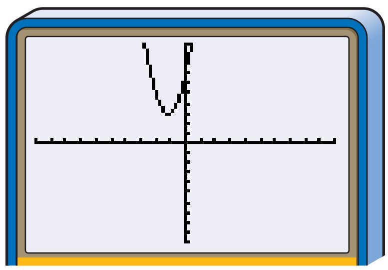
The roots of the equation are 8 and -2.

Although factoring may be an easier method to solve this particular equation, completing the square can always be used to solve any quadratic equation.

When solving a quadratic equation by completing the square, the leading coefficient must be 1. When the leading coefficient of a quadratic equation is not 1, you must first divide each side of the equation by that coefficient before completing the square.

Example 2 **Solve $3n^2 + 7n + 7 = 0$ by completing the square.**

Notice that the graph of the related function, $y = 3x^2 + 7x + 7$, does not cross the x-axis. Therefore, the roots of the equation are imaginary numbers. Completing the square can be used to find the roots of any equation, including one with no real roots.



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$$
\begin{aligned}
3n^2 + 7n + 7 &= 0 \\
n^2 + \frac{7}{3}n + \frac{7}{3} &= 0 && \textit{Divide each side by 3.} \\
n^2 + \frac{7}{3}n &= -\frac{7}{3} && \textit{Subtract } \frac{7}{3} \textit{ from each side.} \\
n^2 + \frac{7}{3}n + \frac{49}{36} &= -\frac{7}{3} + \frac{49}{36} && \textit{Complete the square by adding } \left(\frac{7}{6}\right)^2 \textit{ or } \frac{49}{36} \textit{ to each side.} \\
\left(n + \frac{7}{6}\right)^2 &= -\frac{35}{36} && \textit{Factor the perfect square trinomial.} \\
n + \frac{7}{6} &= \pm \boldsymbol{i}\frac{\sqrt{35}}{6} && \textit{Take the square root of each side.} \\
n &= -\frac{7}{6} \pm \boldsymbol{i}\frac{\sqrt{35}}{6} && \textit{Subtract } \frac{7}{6} \textit{ from each side.}
\end{aligned}
$$

The roots of the equation are $-\frac{7}{6} \pm \boldsymbol{i}\frac{\sqrt{35}}{6}$ or $\frac{-7 \pm \boldsymbol{i}\sqrt{35}}{6}$.

Completing the square can be used to develop a general formula for solving any quadratic equation of the form $ax^2 + bx + c = 0$. This formula is called the **Quadratic Formula.**

Quadratic Formula

The roots of a quadratic equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$ are given by the following formula.

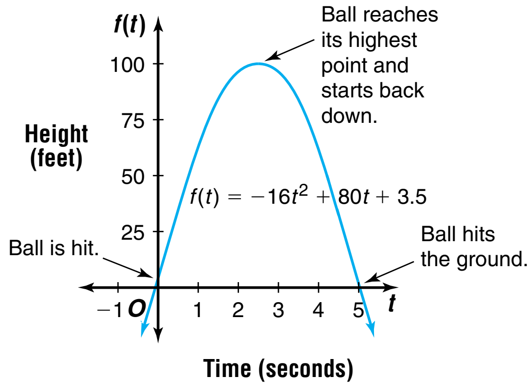
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula can be used to solve any quadratic equation. It is usually easier than completing the square.

Example 3 Real World Application

BASEBALL **Refer to the application at the beginning of the lesson. How long did the catcher have to get into position to catch the ball after if was hit?**

The catcher must get into position to catch the ball before $80t - 16t^2 + 3.5 = 0$. This equation can be written as $-16t^2 + 80t + 3.5 = 0$. Use the Quadratic Formula to solve this equation.



$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(3.5)}}{2(-16)} \qquad a = -16,\ b = 80, \text{ and } c = 3.5$$

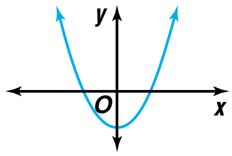
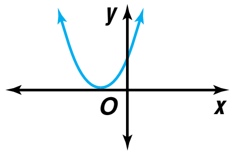
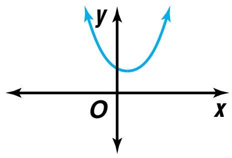
$$t = \frac{-80 \pm \sqrt{6624}}{-32}$$

$$t = \frac{-80 + \sqrt{6624}}{-32} \quad \text{or} \quad t = \frac{-80 - \sqrt{6624}}{-32}$$

$$t \approx -0.04 \qquad\qquad t \approx 5.04$$

The roots of the equation are about -0.04 and 5.04. Since the catcher has a positive amount of time to catch the ball, he will have about 5 seconds to get into position to catch the ball.

In the quadratic formula, the radicand $b^2 - 4ac$ is called the **discriminant** of the equation. The discriminant tells the nature of the roots of a quadratic equation or the zeros of the related quadratic function.

Discriminant	Nature of Roots/Zeros	Graph
$b^2 - 4ac > 0$	two distinct real roots/zeros	
$b^2 - 4ac = 0$	exactly one real root/zero (The one real root is actually a double root.)	
$b^2 - 4ac < 0$	no real roots/zero (two distinct imaginary roots/zeros)	

Example 4 **Find the discriminant of $x^2 - 4x + 15 = 0$ and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.**

The value of the discriminant, $b^2 - 4ac$, is $(-4)^2 - 4(1)(15)$ or -44. Since the value of the discriminant is less than zero, there are no real roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

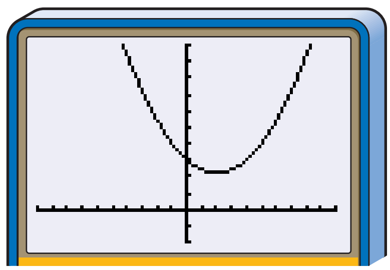
$$x = \frac{-(-4) \pm \sqrt{-44}}{2(1)}$$

$$x = \frac{4 \pm 2\boldsymbol{i}\sqrt{11}}{2}$$

$$x = 2 \pm \boldsymbol{i}\sqrt{11}$$

The roots are $2 + \boldsymbol{i}\sqrt{11}$ and $2 - \boldsymbol{i}\sqrt{11}$.

The graph of $y = x^2 - 4x + 15$ verifies that there are no real roots.



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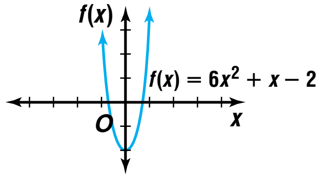
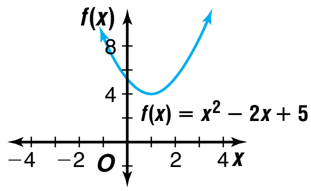
The roots of the equation in Example 4 are the complex numbers $2 + \boldsymbol{i}\sqrt{11}$ and $2 - \boldsymbol{i}\sqrt{11}$. A pair of complex numbers in the form $a + b\boldsymbol{i}$ and $a - b\boldsymbol{i}$ are called **conjugates.** Imaginary roots of polynomial equations with real coefficients always occur in conjugate pairs. Some other examples of complex conjugates are listed below.

$\boldsymbol{i}$ and $-\boldsymbol{i}$ $\qquad$ $-1 + \boldsymbol{i}$ and $-1 - \boldsymbol{i}$ $\qquad$ $-\boldsymbol{i}\sqrt{2}$ and $\boldsymbol{i}\sqrt{2}$

Complex Conjugates Theorem

Suppose a and b are real numbers with $b \neq 0$. If $a + b\boldsymbol{i}$ is a root of a polynomial equation with real coefficients, then $a - b\boldsymbol{i}$ is also a root of the equation. $a + b\boldsymbol{i}$ and $a - b\boldsymbol{i}$ are conjugate pairs.

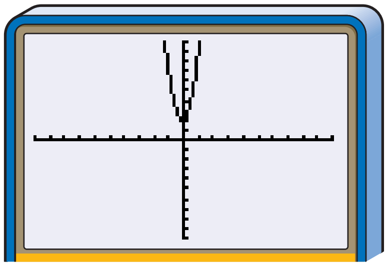
There are four methods used to solve quadratic equations. Two methods work for any quadratic equation. One method approximates any real roots, and one method only works for equations that can be factored over the integers.

Solution Method	Situation	Examples
Graphing	Usually, only approximate solutions are shown. If roots are imaginary (discriminant is less than zero), the graph has no *x*-intercepts, and the solutions must be found by another method. Graphing is a good method to verify solutions.	$6x^2 + x - 2 = 0$  $x = -\frac{2}{3}$ or $x = \frac{1}{2}$ $x^2 - 2x + 5 = 0$ discriminant: $(-2)^2 - 4(1)(5) = -16$  The equation has no real roots.
Factoring	When *a*, *b*, and *c* are integers and the discriminant is a perfect square or zero, this method is useful. It cannot be used if the discriminant is less than zero.	$g^2 + 2g - 8 = 0$ discriminant: $2^2 - 4(1)(-8) = 36$ $g^2 + 2g - 8 = 0$ $(g + 4)(g - 2) = 0$ $g + 4 = 0$ or $g - 2 = 0$ $g = -4$ $\quad$ $g = 2$
Completing the Square	This method works for any quadratic equation. There is more room for an arithmetic error than when using the Quadratic Formula.	$r^2 + 4r - 6 = 0$ $r^2 + 4r = 6$ $r^2 + 4r + 4 = 6 + 4$ $(r + 2)^2 = 10$ $r + 2 = \pm\sqrt{10}$ $r = -2 \pm \sqrt{10}$
Quadratic Formula	This method works for any quadratic equation.	$2s^2 + 5s + 4 = 0$ $s = \frac{-5 \pm \sqrt{5^2 - 4(2)(4)}}{2(2)}$ $s = \frac{-5 \pm \sqrt{-7}}{4}$ $s = \frac{-5 \pm i\sqrt{7}}{4}$

Example 5 **Solve $6x^2 + x + 2 = 0$.**

Method 1: Graphing

Graph $y = 6x^2 + x + 2$.



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The graph does not touch the x-axis, so there are no real roots for this equation. You cannot determine the roots from the graph.

Method 2: Factoring

Find the discriminant.

$b^2 - 4ac = 1^2 - 4(6)(2)$ or -47

The discrimimant is less than zero, so factoring cannot be used to solve the equation.

Method 3: Completing the Square

$$6x^2 + x + 2 = 0$$
$$x^2 + \frac{1}{6}x + \frac{1}{3} = 0$$
$$x^2 + \frac{1}{6}x = -\frac{1}{3}$$
$$x^2 + \frac{1}{6}x + \frac{1}{144} = -\frac{1}{3} + \frac{1}{144}$$
$$\left(x + \frac{1}{12}\right)^2 = -\frac{47}{144}$$
$$x + \frac{1}{12} = \pm \boldsymbol{i}\frac{\sqrt{47}}{12}$$
$$x = -\frac{1}{12} \pm \boldsymbol{i}\frac{\sqrt{47}}{12}$$
$$x = \frac{-1 \pm \boldsymbol{i}\sqrt{47}}{12}$$

Completing the square works, but this method requires a lot of steps.

Method 4: Quadratic Formula

For this equation, $a = 6$, $b = 1$, $c = 2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(6)(2)}}{2(6)}$$
$$x = \frac{-1 \pm \sqrt{-47}}{12}$$
$$x = \frac{-1 \pm \boldsymbol{i}\sqrt{47}}{12}$$

The Quadratic Formula works and requires fewer steps than completing the square.

The roots of the equation are $\frac{-1 \pm \boldsymbol{i}\sqrt{47}}{12}$.

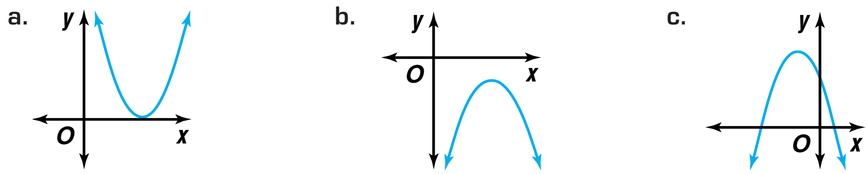
CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a short paragraph explaining how to solve $t^2 - 6t - 4 = 0$ by completing the square.
2. **Discuss** which method of solving $5p^2 - 13p + 7 = 0$ would be most appropriate. Explain. Then solve.

3. **Describe** the discriminant of the equation represented by each graph.



4. *Math Journal* **Solve** $x^2 + 4x - 5 = 0$ using each of the four methods discussed in this lesson. Which method do you prefer? Explain.

Guided Practice

Solve each equation by completing the square.

5. $x^2 + 8x - 20 = 0$
6. $2a^2 + 11a - 21 = 0$

Find the discriminant of each equation and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.

7. $m^2 + 12m + 36 = 0$
8. $t^2 - 6t + 13 = 0$

Solve each equation.

9. $p^2 - 6p + 5 = 0$
10. $r^2 - 4r + 10 = 0$

11. **Electricity** On a cold day, a 12-volt car battery has a resistance of 0.02 ohms. The power available to start the motor is modeled by the equation $P = 12I - 0.02I^2$, where I is the current in amperes. What current is needed to produce 1600 watts of power to start the motor?

EXERCISES

Practice

Solve each equation by completing the square.

12. $z^2 - 2z - 24 = 0$
13. $p^2 - 3p - 88 = 0$
14. $x^2 - 10x + 21 = 0$
15. $d^2 - \frac{3}{4}d + \frac{1}{8} = 0$
16. $3g^2 - 12g = -4$
17. $t^2 - 3t - 7 = 0$

18. What value of c makes $x^2 - x + c$ a perfect square?

19. Describe the nature of the roots of the equation $4n^2 + 6n + 25$. Explain.

Find the discriminant of each equation and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.

20. $6m^2 + 7m - 3 = 0$
21. $s^2 - 5s + 9 = 0$
22. $36d^2 - 84d + 49 = 0$
23. $4x^2 - 2x + 9 = 0$
24. $3p^2 + 4p = 8$
25. $2k^2 + 5k = 9$

26. What is the conjugate of $-7 - \boldsymbol{i}\sqrt{5}$?

27. Name the conjugate of $5 - 2\boldsymbol{i}$.

Solve each equation.

28. $3s^2 - 5s + 9 = 0$

29. $x^2 - 3x - 28 = 0$

30. $4w^2 + 19w - 5 = 0$

31. $4r^2 - r = 5$

32. $p^2 + 2p + 8 = 0$

33. $x^2 - 2x\sqrt{6} - 2 = 0$

Applications and Problem Solving

Real World Application

34. Health Normal systolic blood pressure is a function of age. For a woman, the normal systolic pressure P in millimeters of mercury (mm Hg) is modeled by $P = 0.01A^2 + 0.05A + 107$, where A is age in years.

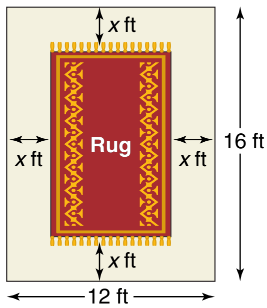
a. Use this model to determine the normal systolic pressure of a 25-year-old woman.

b. Use this model to determine the age of a woman whose normal systolic pressure is 125 mm Hg.

c. Sketch the graph of the function. Describe what happens to the normal systolic pressure as a woman gets older.

35. Critical Thinking Consider the equation $x^2 + 8x + c = 0$. What can you say about the value of c if the equation has two imaginary roots?

36. Interior Design Abey Numkena is an interior designer. She has been asked to locate an oriental rug for a new corporate office. As a rule, the rug should cover $\frac{1}{2}$ of the total floor area with a uniform width surrounding the rug.



a. If the dimensions of the room are 12 feet by 16 feet, write an equation to model the situation.

b. Graph the related function.

c. What are the dimensions of the rug?

37. Entertainment In an action movie, a stuntwoman jumps off a building that is 50 feet tall with an upward initial velocity of 5 feet per second. The distance $d(t)$ traveled by a free falling object can be modeled by the formula $d(t) = v_0t - \frac{1}{2}gt^2$, where v_0 is the initial velocity and g represents the acceleration due to gravity. The acceleration due to gravity is 32 feet per second squared.

a. Draw a graph that relates the woman's distance traveled with the time since the jump.

b. Name the x-intercepts of the graph.

c. What is the meaning of the x-intercepts of the graph?

d. Write an equation that could be used to determine when the stuntwoman will reach the safety pad on the ground. (*Hint:* The ground is -50 feet from the starting point.)

e. How long will it take the stuntwoman to reach the safety pad on the ground?

38. Critical Thinking Derive the quadratic formula by completing the square if $ax^2 + bx + c = 0$, $a \neq 0$.

Extra Practice See p. A32.

CONTENTS

Mixed Review

39. State the number of complex roots of the equation $18a^2 + 3a - 1 = 0$. Then find the roots and graph the related function. *(Lesson 4-1)*

40. Graph $y < |x| - 2$. *(Lesson 3-5)*

41. Find the inverse of $f(x) = (x - 9)^2$. *(Lesson 3-4)*

42. Solve the system of equations, $3x + 4y = 375$ and $5x + 2y = 345$. *(Lesson 2-1)*

43. Sales The Computer Factory is selling a 300 MHz computer system for $595 and a 350 MHz computer system for $619. At this rate, what would be the cost of a 400 MHz computer system? *(Lesson 1-4)*

44. Find the slope of the line whose equation is $3y + 8x = 12$. *(Lesson 1-3)*

45. SAT/ACT Practice The trinomial $x^2 + x - 20$ is exactly divisible by which binomial?

A $x - 4$ **B** $x + 4$ **C** $x + 6$ **D** $x - 10$ **E** $x - 5$

CAREER CHOICES

Environmental Engineering

Would you like a career where you will constantly be learning and have the opportunity to work both outdoors and indoors? Environmental engineering has become an important profession in the past twenty-five years.

As an environmental engineer, you might design, build, or maintain systems for controlling wastes produced by cities or industry. These wastes can include solid waste, waste water, hazardous waste, or air pollutants. You could work for a private company, a consulting firm, or the Environmental Protection Agency. Opportunities for advancement in this field include becoming a supervisor or consultant. You might even have the opportunity to identify a new specialty area in the field of environmental engineering!

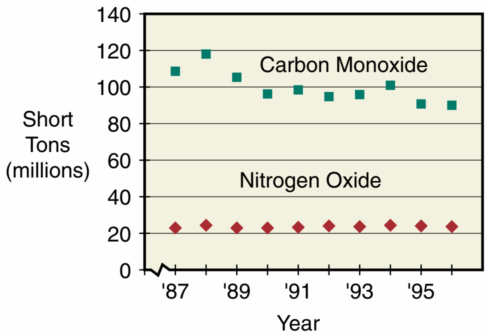
Career Overview

Degree Required:
Bachelor's degree in environmental engineering

Related Courses:
biology, chemistry, mathematics

Outlook:
number of jobs expected to increase though the year 2006

Emissions of Two Air Pollutants in the U.S. 1987-1996



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interNET CONNECTION For more information about environmental engineering, visit: **www.amc.glencoe.com**

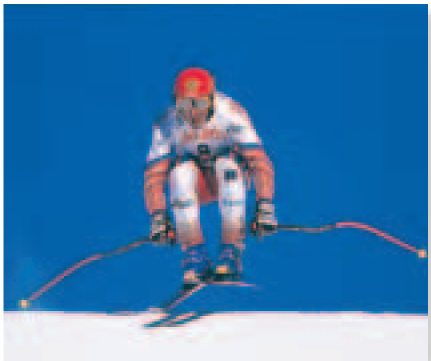
4-3 The Remainder and Factor Theorems

OBJECTIVE

- Find the factors of polynomials using the Remainder and Factor Theorems.

Real World Application

SKIING On December 13, 1998, Olympic champion Hermann (The Herminator) Maier won the super-G at Val d'Isere, France. His average speed was 73 meters per second. The average recreational skier skis at a speed of about 5 meters per second. Suppose you were skiing at a speed of 5 meters per second and heading downhill, accelerating at a rate of 0.8 meter per second squared. How far will you travel in 30 seconds? *This problem will be solved in Example 1.*



Hermann Maier

Consider the polynomial function $f(a) = 2a^2 + 3a - 8$. Since 2 is a factor of 8, it may be possible that $a - 2$ is a factor of $2a^2 + 3a - 8$. Suppose you use long division to divide the polynomial by $a - 2$.

$$\begin{array}{rrl}
 & 2a + 7 & \leftarrow \text{quotient} \\
\text{divisor} \rightarrow a - 2 & \overline{)2a^2 + 3a - 8} & \leftarrow \text{dividend} \\
 & \underline{2a^2 - 4a} & \\
 & 7a - 8 & \\
 & \underline{7a - 14} & \\
 & 6 & \leftarrow \text{remainder}
\end{array}$$

From arithmetic, you may remember that the dividend equals the product of the divisor and the quotient plus the remainder. For example, $44 \div 7 = 6$ R2, so $44 = 7(6) + 2$. This relationship can be applied to polynomials.

You may want to verify that $(a - 2)(2a + 7) + 6 = 2a^2 + 3a - 8$.

$$f(a) = (a - 2)(2a + 7) + 6 \quad \longleftrightarrow \quad f(a) = 2a^2 + 3a - 8$$

Let $a = 2$.

$$f(2) = (2 - 2)[2(2) + 7] + 6 \qquad f(2) = 2(2^2) + 3(2) - 8$$
$$= 0 + 6 \text{ or } 6 \qquad\qquad = 8 + 6 - 8 \text{ or } 6$$

Notice that the value of $f(2)$ is the same as the remainder when the polynomial is divided by $a - 2$. This example illustrates the **Remainder Theorem.**

Remainder Theorem

If a polynomial $P(x)$ is divided by $x - r$, the remainder is a constant $P(r)$, and

$$P(x) = (x - r) \cdot Q(x) + P(r),$$

where $Q(x)$ is a polynomial with degree one less than the degree of $P(x)$.

The Remainder Theorem provides another way to find the value of the polynomial function $P(x)$ for a given value of r. The value will be the remainder when $P(x)$ is divided by $x - r$.

Example 1 Real World Application

SKIING **Refer to the application at the beginning of the lesson. The formula for distance traveled is $d(t) = v_0t + \frac{1}{2}at^2$, where $d(t)$ is the distance traveled, v_0 is the initial velocity, t is the time, and a is the acceleration. Find the distance traveled after 30 seconds.**

The distance formula becomes $d(t) = 5t + \frac{1}{2}(0.8)t^2$ or $d(t) = 0.4t^2 + 5t$. You can use one of two methods to find the distance after 30 seconds.

Method 1

Divide $0.4t^2 + 5t$ by $t - 30$.

$$
\begin{array}{r|l}
 & \;0.4t + 17 \\ \cline{2-2}
t - 30 & 0.4t^2 + 5t \\
 & \underline{0.4t^2 - 12t} \\
 & 17t + 0 \\
 & \underline{17t - 510} \\
 & 510 \rightarrow D(30) = 510
\end{array}
$$

Method 2

Evaluate $d(t)$ for $t = 30$.

$$
\begin{aligned}
d(t) &= 0.4t^2 + 5t \\
d(30) &= 0.4(30^2) + 5(30) \\
&= 0.4(900) + 5(30) \\
&= 510
\end{aligned}
$$

By either method, the result is the same. You will travel 510 meters in 30 seconds.

Long division can be very time consuming. **Synthetic division** is a shortcut for dividing a polynomial by a binomial of the form $x - r$. The steps for dividing $x^3 + 4x^2 - 3x - 5$ by $x + 3$ using synthetic division are shown below.

Step 1 Arrange the terms of the polynomial in descending powers of x. Insert zeros for any missing powers of x. Then, write the coefficients as shown.

$$
\begin{array}{cccc}
x^3 & +\,4x^2 & -\,3x & -\,5 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 4 & -3 & -5
\end{array}
$$

For $x + 3$, the value of r is -3.

Step 2 Write the constant r of the divisor $x - r$. In this case, write -3.

$$
\begin{array}{r|rrrr}
-3 & 1 & 4 & -3 & -5
\end{array}
$$

Step 3 Bring down the first coefficient.

$$
\begin{array}{r|rrrr}
-3 & 1 & 4 & -3 & -5 \\
 & & & & \\ \hline
 & 1 & & &
\end{array}
$$

Step 4 Multiply the first coefficient by r. Then write the product under the next coefficient. Add.

$$
\begin{array}{r|rrrr}
-3 & 1 & 4 & -3 & -5 \\
 & & -3 & & \\ \hline
\times & 1 & 1 & &
\end{array}
$$

Step 5 Multiply the sum by r. Then write the product under the next coefficient. Add.

$$
\begin{array}{r|rrrr}
-3 & 1 & 4 & -3 & -5 \\
 & & -3 & -3 & \\ \hline
\times & 1 & 1 & -6 &
\end{array}
$$

Notice that a vertical bar separates the quotient from the remainder.

Step 6 Repeat Step 5 for all coefficients in the dividend.

$$
\begin{array}{r|rrr|r}
-3 & 1 & 4 & -3 & -5 \\
 & & -3 & -3 & 18 \\ \hline
\times & 1 & 1 & -6 & 13
\end{array}
$$

Step 7 The final sum represents the remainder, which in this case is 13. The other numbers are the coefficients of the quotient polynomial, which has a degree one less than the dividend. Write the quotient $x^2 + x - 6$ with remainder 13. *Check the results using long division.*

Example 2 **Divide $x^3 - x^2 + 2$ by $x + 1$ using synthetic division.**

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

Notice there is no x term. A zero is placed in this position as a placeholder.

The quotient is $x^2 - 2x + 2$ with a remainder of 0.

In Example 2, the remainder is 0. Therefore, $x + 1$ is a factor of $x^3 - x^2 + 2$. If $f(x) = x^3 - x^2 + 2$, then $f(-1) = (-1)^3 - (-1)^2 + 2$ or 0. This illustrates the **Factor Theorem,** which is a special case of the Remainder Theorem.

Factor Theorem The binomial $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.

Example 3 **Use the Remainder Theorem to find the remainder when $2x^3 - 3x^2 + x$ is divided by $x - 1$. State whether the binomial is a factor of the polynomial. Explain.**

Find $f(1)$ to see if $x - 1$ is a factor.

$f(x) = 2x^3 - 3x^2 + x$

$f(1) = 2(1^3) - 3(1^2) + 1$ *Replace x with 1.*

$= 2(1) - 3(1) + 1$ or 0

Since $f(1) = 0$, the remainder is 0. So the binomial $(x - 1)$ is a factor of the polynomial by the Factor Theorem.

When a polynomial is divided by one of its binomial factors $x - r$, the quotient is called a **depressed polynomial.** A depressed polynomial has a degree less than the original polynomial. In Example 3, $x - 1$ is a factor of $2x^3 - 3x^2 + x$. Use synthetic division to find the depressed polynomial.

$$\begin{array}{r|rrrr} 1 & 2 & -3 & 1 & 0 \\ & & 2 & -1 & 0 \\ \hline & 2 & -1 & 0 & 0 \end{array}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$2x^2 - 1x + 0$$

Thus, $(2x^3 - 3x^2 + x) \div (x - 1) = 2x^2 - x$.

The depressed polynomial is $2x^2 - x$.

A depressed polynomial may also be the product of two polynomial factors, which would give you other zeros of the polynomial function. In this case, $2x^2 - x$ equals $x(2x - 1)$. So, the zeros of the polynomial function $f(x) = 2x^3 - 3x^2 + x$ are 0, $\frac{1}{2}$, and 1.

Note that the values of r where no remainder occurs are also factors of the constant term of the polynomial.

You can also find factors of a polynomial such as $x^3 + 2x^2 - 16x - 32$ by using a shortened form of synthetic division to test several values of r. In the table, the first column contains various values of r. The next three columns show the coefficients of the depressed polynomial. The fifth column shows the remainder. Any value of r that results in a remainder of zero indicates that $x - r$ is a factor of the polynomial. The factors of the original polynomial are $x + 4$, $x + 2$, and $x - 4$.

r	1	2	−16	−32
−4	1	−2	−8	0
−3	1	−1	−13	7
−2	1	0	−16	0
−1	1	1	−17	−15
0	1	2	−16	−32
1	1	3	−13	−45
2	1	4	−8	−48
3	1	5	−1	−35
4	1	6	8	0

Look at the pattern of values in the last column. Notice that when r = 1, 2, and 3, the values of f(x) decrease and then increase. This indicates that there is an x-coordinate of a relative minimum between 1 and 3.

Example 4 **Determine the binomial factors of $x^3 - 7x + 6$.**

Graphing Calculator Tip

The **TABLE** feature can help locate integral zeros. Enter the polynomial function as **Y1** in the **Y=** menu and press **TABLE**. Search the **Y1** column to find 0 and then look at the corresponding x-value.

Method 1 Use synthetic division.

r	1	0	−7	6	
−4	1	−4	9	−30	
−3	1	−3	2	0	*x + 3 is a factor.*
−2	1	−2	−3	12	
−1	1	−1	−6	12	
0	1	0	−7	6	
1	1	1	−6	0	*x − 1 is a factor.*
2	1	2	−3	0	*x − 2 is a factor.*

Method 2 Test some values using the Factor Theorem.

$f(x) = x^3 - 7x + 6$
$f(-1) = (-1)^3 - 7(-1) + 6$ or 12
$f(1) = 1^3 - 7(1) + 6$ or 0

Because $f(1) = 0$, $x - 1$ is a factor. Find the depressed polynomial.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

The depressed polynomial is $x^2 + x - 6$. Factor the depressed polynomial.

$x^2 + x - 6 = (x + 3)(x - 2)$

The factors of $x^3 + x - 6$ are $x + 3$, $x - 1$, and $x - 2$. *Verify the results.*

The Remainder Theorem can be used to determine missing coefficients.

Example 5 **Find the value of k so that the remainder of $(x^3 + 3x^2 - kx - 24) \div (x + 3)$ is 0.**

If the remainder is to be 0, $x + 3$ must be a factor of $x^3 + 3x^2 - kx - 24$. So, $f(-3)$ must equal 0.

$$\begin{aligned} f(x) &= x^3 + 3x^2 - kx - 24 \\ f(-3) &= (-3)^3 + 3(-3)^2 - k(-3) - 24 \\ 0 &= -27 + 27 + 3k - 24 \quad \textit{Replace } f(-3) \textit{ with } 0. \\ 0 &= 3k - 24 \\ 8 &= k \end{aligned}$$

The value of k is 8. Check using synthetic division.

$$\begin{array}{r|rrrr} -3 & 1 & 3 & -8 & -24 \\ & & -3 & 0 & 24 \\ \hline & 1 & 0 & -8 & 0 \end{array}$$ ✓

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how the Remainder Theorem and the Factor Theorem are related.

2. **Write** the division problem illustrated by the synthetic division. What is the quotient? What is the remainder?

$$\begin{array}{r|rrrr} 5 & 1 & -4 & -7 & 8 \\ & & 5 & 5 & -10 \\ \hline & 1 & 1 & -2 & -2 \end{array}$$

3. **Compare** the degree of a polynomial and its depressed polynomial.

4. **You Decide** Brittany tells Isabel that if $x + 3$ is a factor of the polynomial function $f(x)$, then $f(3) = 0$. Isabel argues that if $x + 3$ is a factor of $f(x)$, then $f(-3) = 0$. Who is correct? Explain.

Guided Practice

interNET CONNECTION

Graphing Calculator Program
For a graphing calculator program that computes the value of a function visit **www.amc.glencoe.com**

Divide using synthetic division.

5. $(x^2 - x + 4) \div (x - 2)$

6. $(x^3 + x^2 - 17x + 15) \div (x + 5)$

Use the Remainder Theorem to find the remainder for each division. State whether the binomial is a factor of the polynomial.

7. $(x^2 + 2x - 15) \div (x - 3)$

8. $(x^4 + x^2 + 2) \div (x - 3)$

Determine the binomial factors of each polynomial.

9. $x^3 - 5x^2 - x + 5$

10. $x^3 - 6x^2 + 11x - 6$

11. Find the value of k so that the remainder of $(x^3 - 7x + k) \div (x + 1)$ is 2.

12. Let $f(x) = x^7 + x^9 + x^{12} - 2x^2$.
   a. State the degree of $f(x)$.
   b. State the number of complex zeros that $f(x)$ has.
   c. State the degree of the depressed polynomial that would result from dividing $f(x)$ by $x - a$.
   d. Find one factor of $f(x)$.

13. **Geometry** A cylinder has a height 4 inches greater than the radius of its base. Find the radius and the height to the nearest inch if the volume of the cylinder is 5π cubic inches.

EXERCISES

Practice

Divide using synthetic division.

14. $(x^2 + 20x + 91) \div (x + 7)$

15. $(x^3 - 9x^2 + 27x - 28) \div (x - 3)$

16. $(x^4 + x^3 - 1) \div (x - 2)$

17. $(x^4 - 8x^2 + 16) \div (x + 2)$

18. $(3x^4 - 2x^3 + 5x^2 - 4x - 2) \div (x + 1)$

19. $(2x^3 - 2x - 3) \div (x - 1)$

Use the Remainder Theorem to find the remainder for each division. State whether the binomial is a factor of the polynomial.

20. $(x^2 - 2) \div (x - 1)$

21. $(x^5 + 32) \div (x + 2)$

22. $(x^4 - 6x^2 + 8) \div (x - \sqrt{2})$

23. $(x^3 - x + 6) \div (x - 2)$

24. $(4x^3 + 4x^2 + 2x + 3) \div (x - 1)$

25. $(2x^3 - 3x^2 + x) \div (x - 1)$

www.amc.glencoe.com/self_check_quiz
CONTENTS

26. Which binomial is a factor of the polynomial $x^3 + 3x^2 - 2x - 8$?
   a. $x - 1$ b. $x + 1$ c. $x - 2$ d. $x + 2$

27. Verify that $x - \sqrt{6}$ is a factor of $x^4 - 36$.

28. Use synthetic division to find all the factors of $x^3 + 7x^2 - x - 7$ if one of the factors is $x + 1$.

Determine the binomial factors of each polynomial.

29. $x^3 + x^2 - 4x - 4$
30. $x^3 - x^2 - 49x + 49$
31. $x^3 - 5x^2 + 2x + 8$
32. $x^3 - 2x^2 - 4x + 8$
33. $x^3 + 4x^2 - x - 4$
34. $x^3 + 3x^2 + 3x + 1$

35. How many times is 2 a root of $x^6 - 9x^4 + 24x^2 - 16 = 0$?

36. Determine how many times -1 is a root of $x^3 + 2x^2 - x - 2 = 0$. Then find the other roots.

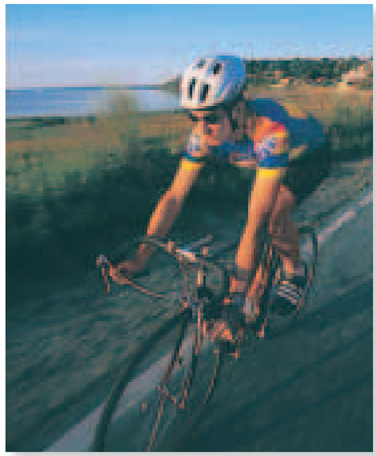
Find the value of *k* so that each remainder is zero.

37. $(2x^3 - x^2 + x + k) \div (x - 1)$
38. $(x^3 - kx^2 + 2x - 4) \div (x - 2)$
39. $(x^3 + 18x^2 + kx + 4) \div (x + 2)$
40. $(x^3 + 4x^2 - kx + 1) \div (x + 1)$

Applications and Problem Solving

Real World Application

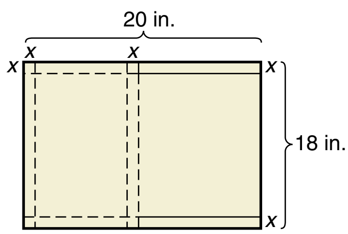
41. **Bicycling** Matthew is cycling at a speed of 4 meters per second. When he starts down a hill, the bike accelerates at a rate of 0.4 meter per second squared. The vertical distance from the top of the hill to the bottom of the hill is 25 meters. Use the equation $d(t) = v_0 t + \frac{1}{2}at^2$ to find how long it will take Matthew to ride down the hill.



42. **Critical Thinking** Determine a and b so that when $x^4 + x^3 - 7x^2 + ax + b$ is divided by $(x - 1)(x + 2)$, the remainder is 0.

43. **Sculpting** Esteban is preparing to start an ice sculpture. He has a block of ice that is 3 feet by 4 feet by 5 feet. Before he starts, he wants to reduce the volume of the ice by shaving off the same amount from the length, the width, and the height.
   a. Write a polynomial function to model the situation.
   b. Graph the function.
   c. He wants to reduce the volume of the ice to $\frac{3}{5}$ of the original volume. Write an equation to model the situation.
   d. How much should he take from each dimension?

44. **Manufacturing** An 18-inch by 20-inch sheet of cardboard is cut and folded to make a box for the Great Pecan Company.
   a. Write an polynomial function to model the volume of the box.
   b. Graph the function.
   c. The company wants the box to have a volume of 224 cubic inches. Write an equation to model this situation.
   d. Find a positive integer for x.



45. **Critical Thinking** Find a, b, and c for $P(x) = ax^2 + bx + c$ if $P(3 + 4\boldsymbol{i}) = 0$ and $P(3 - 4\boldsymbol{i}) = 0$.

Mixed Review

46. Solve $r^2 + 5r - 8 = 0$ by completing the square. *(Lesson 4-2)*

47. Determine whether each number is a root of $x^4 - 4x^3 - x^2 + 4x = 0$. *(Lesson 4-1)*
    a. 2 **b.** 0 **c.** -2 **d.** 4

48. Find the critical points of the graph of $f(x) = x^5 - 32$. Determine whether each represents a *maximum,* a *minimum,* or a *point of inflection.* *(Lesson 3-6)*

49. Describe the transformation(s) that have taken place between the parent graph of $y = x^2$ and the graph of $y = 0.5(x + 1)^2$. *(Lesson 3-2)*

50. **Business** Pristine Pipes Inc. produces plastic pipe for use in newly-built homes. Two of the basic types of pipe have different diameters, wall thickness, and strengths. The strength of a pipe is increased by mixing a special additive into the plastic before it is molded. The table below shows the resources needed to produce 100 feet of each type of pipe and the amount of the resource available each week.

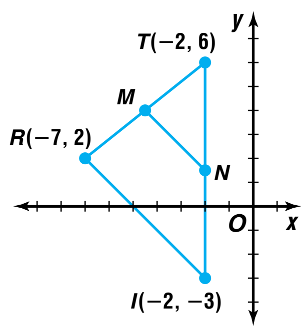
Resource	Pipe A	Pipe B	Resource Availability
Extrusion Dept.	4 hours	6 hours	48 hours
Packaging Dept.	2 hours	2 hours	18 hours
Strengthening Additive	2 pounds	1 pound	16 pounds

If the profit on 100 feet of type A pipe is \$34 and of type B pipe is \$40, how much of each should be produced to maximize the profit? *(Lesson 2-7)*

51. Solve the system of equations. *(Lesson 2-2)*

$$4x + 2y + 3z = 6$$
$$2x + 7y = 3z$$
$$-3x - 9y + 13 = -2z$$

52. **Geometry** Show that the line segment connecting the midpoints of sides $\overline{TR}$ and $\overline{TI}$ is parallel to $\overline{RI}$. *(Lesson 1-5)*



53. **SAT/ACT Practice** If $a > b$ and $c < 0$, which of the following are true?
I. $ac < bc$
II. $a + c > b + c$
III. $a - c < b - c$

A I only **B** II only **C** III only
D I and II only **E** I, II, and III

Extra Practice See p. A32.

4-4

The Rational Root Theorem

OBJECTIVES

- Identify all possible rational roots of a polynomial equation by using the Rational Root Theorem.
- Determine the number of positive and negative real roots a polynomial function has.

Real World Application

CONSTRUCTION The longest and largest canal tunnel in the world was the Rove Tunnel on the Canal de Marseille au Rhone in the south of France. In 1963, the tunnel collapsed and excavation engineers were trying to duplicate the original tunnel. The height of the tunnel was 1 foot more than half its width. The length was 32 feet more than 324 times its width. The volume of the tunnel was 62,231,040 cubic feet. If the tunnel was a rectangular prism, find its original dimensions. *This problem will be solved in Example 4.*

The formula for the volume of a rectangular prism is $V = \ell wh$, where ℓ is the length, w is the width, and h is the height. From the information above, the height of the tunnel is $\frac{1}{2}w + 1$, and its length is $324w + 32$.

$$
\begin{aligned}
V &= \ell wh \\
62{,}231{,}040 &= (324w + 32)w\left(\frac{1}{2}w + 1\right) && \ell = 324w + 32,\ h = \frac{1}{2}w + 1 \\
62{,}231{,}040 &= 162w^3 + 340w^2 + 32w \\
0 &= 162w^3 + 340w^2 + 32w - 62{,}231{,}040 \\
0 &= 81w^3 + 170w^2 + 16w - 31{,}115{,}520 && \text{Divide each side by 2.}
\end{aligned}
$$

We could use synthetic substitution to test possible zeros, but with such large numbers, this is not practical. In situations like this, the **Rational Root Theorem** can give direction in testing possible zeros.

Rational Root Theorem

Let $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ represent a polynomial equation of degree n with integral coefficients. If a rational number $\frac{p}{q}$, where p and q have no common factors, is a root of the equation, then p is a factor of a_n and q is a factor of a_0.

Example 1 **List the possible rational roots of $6x^3 + 11x^2 - 3x - 2 = 0$. Then determine the rational roots.**

According to the Rational Root Theorem, if $\frac{p}{q}$ is a root of the equation, then p is a factor of 2 and q is a factor of 6.

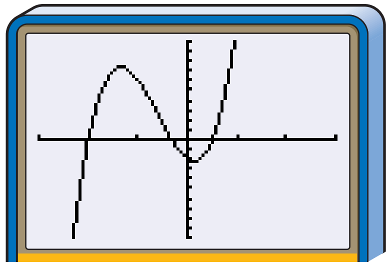
possible values of p: $\pm 1, \pm 2$

possible values of q: $\pm 1, \pm 2, \pm 3, \pm 6$

possible rational roots, $\frac{p}{q}$: $\pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}$

(continued on the next page)

You can use a graphing utility to narrow down the possibilities. You know that all possible rational roots fall in the domain $-2 \le x \le 2$. So set your x-axis viewing window at $[-3, 3]$. Graph the related function $f(x) = 6x^3 + 11x^2 - 3x - 2$. A zero appears to occur at -2. Use synthetic division to check that -2 is a zero.



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$$\begin{array}{r|rrrr} -2 & 6 & 11 & -3 & -2 \\ & & -12 & 2 & 2 \\ \hline & 6 & -1 & -1 & 0 \end{array}$$

Thus, $6x^3 + 11x^2 - 3x - 2 = (x + 2)(6x^2 - x - 1)$. Factoring $6x^2 - x - 1$ yields $(3x + 1)(2x - 1)$. The roots of $6x^3 + 11x^2 - 3x - 2 = 0$ are -2, $-\frac{1}{3}$, and $\frac{1}{2}$.

A corollary to the Rational Root Theorem, called the **Integral Root Theorem,** states that if the leading coefficient a_0 has a value of 1, then any rational roots must be factors of a_n, $a_n \neq 0$.

Integral Root Theorem

Let $x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$ represent a polynomial equation that has a leading coefficient of 1, integral coefficients, and $a_n \neq 0$. Any rational roots of this equation must be integral factors of a_n.

Example 2 **Find the roots of $x^3 + 8x^2 + 16x + 5 = 0$.**

There are three complex roots. According to the Integral Root Theorem, the possible rational roots of the equation are factors of 5. The possibilities are ± 5 and ± 1.

r	1	8	16	5
5	1	13	81	410
−5	1	3	1	0

← *There is a root at $x = -5$.*

The depressed polynomial is $x^2 + 3x + 1$. Use the quadratic formula to find the other two roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(1)}}{2(1)} \qquad a = 1,\ b = 3,\ c = 1$$

$$x = \frac{-3 \pm \sqrt{5}}{2}$$

The three roots of the equation are -5, $\frac{-3 - \sqrt{5}}{2}$, and $\frac{-3 + \sqrt{5}}{2}$.

Descartes' Rule of Signs can be used to determine the possible number of positive real zeros a polynomial has. It is named after the French mathematician René Descartes, who first proved the theorem in 1637. *In Descartes' Rule of Signs, when we speak of a zero of the polynomial, we mean a zero of the corresponding polynomial function.*

Descartes' Rule of Signs

Suppose $P(x)$ is a polynomial whose terms are arranged in descending powers of the variable. Then the number of positive real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms or is less than this by an even number. The number of negative real zeros of $P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or less than this number by an even number.

Ignore zero coefficients when using this rule.

Example 3 **Find the number of possible positive real zeros and the number of possible negative real zeros for $f(x) = 2x^5 + 3x^4 - 6x^3 + 6x^2 - 8x + 3$. Then determine the rational zeros.**

To determine the number of possible positive real zeros, count the sign changes for the coefficients.

$$\begin{array}{lccccccccccc} f(x) = & 2x^5 & + & 3x^4 & - & 6x^3 & + & 6x^2 & - & 8x & + & 3 \\ & 2 & & 3 & & -6 & & 6 & & -8 & & 3 \\ & & \text{no} & & \text{yes} & & \text{yes} & & \text{yes} & & \text{yes} & \end{array}$$

There are four changes. So, there are four, two, or zero positive real zeros.

To determine the number of possible negative real zeros, find $f(-x)$ and count the number of sign changes.

$$\begin{array}{lccccccccccc} f(-x) = & 2(-x)^5 & + & 3(-x)^4 & - & 6(-x)^3 & + & 6(-x)^2 & - & 8(-x) & + & 3 \\ f(-x) = & -2x^5 & + & 3x^4 & + & 6x^3 & + & 6x^2 & + & 8 & + & 3 \\ & -2 & & 3 & & 6 & & 6 & & 8 & & 3 \\ & & \text{yes} & & \text{no} & & \text{no} & & \text{no} & & \text{no} & \end{array}$$

There is one change. So, there is one negative real zero.

Determine the possible zeros.

possible values of p: $\pm 1, \pm 3$

possible values of q: $\pm 1, \pm 2$

possible rational zeros, $\frac{p}{q}$: $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$

Test the possible zeros using the synthetic division and the Remainder Theorem.

r	2	3	−6	6	−8	3
1	2	5	−1	5	−3	0
−1	2	1	−7	13	−21	24
3	2	9	21	69	199	600
−3	2	−3	3	−3	1	0

1 is a zero.

−3 is a zero.

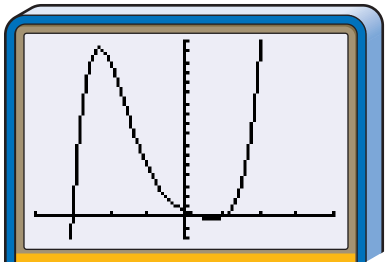
(continued on the next page)

Since there is only one negative real zero and -3 is a zero, you do not need to test any other negative possibilities.

$\frac{1}{2}$	2	4	-4	4	-6	0
$\frac{3}{2}$	2	6	3	$10\frac{1}{2}$	$7\frac{3}{4}$	$14\frac{5}{8}$

$\frac{1}{2}$ is a zero.

All possible rational roots have been considered. There are two positive zeros and one negative zero. The rational zeros for $f(x) = 2x^5 + 3x^4 - 6x^3 + 6x^2 - 8x + 3$ are -3, $\frac{1}{2}$, and 1. Use a graphing utility to check these zeros. Note that there appears to be three x-intercepts. You can use the zero function on the **CALC** menu to verify that the zeros you found are correct.



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You can use a graphing calculator to study Descartes' Rule of Signs.

GRAPHING CALCULATOR EXPLORATION

Remember that you can determine the location of the zeros of a function by analyzing its graph.

TRY THESE **Graph each function to determine how many zeros appear to exist. Use the zero function in the CALC menu to approximate each zero.**

1. $f(x) = x^4 + 4x^3 + 3x^2 - 4x - 4$

2. $f(x) = x^3 - 3x - 2$

WHAT DO YOU THINK?

3. Use Descartes' Rule of Signs to determine the possible positive and negative real zeros of each function.

4. How do your results from Exercise 3 compare with your results using the **TABLE** feature? Explain.

5. What do you think the term "double zero" means?

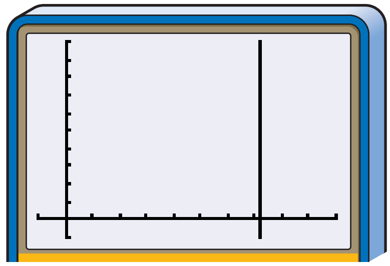
Because the zeros of a polynomial function are the roots of a polynomial equation, Descartes' Rule of Signs can be used to determine the types of roots of the equation.

Example 4 Real World Application

CONSTRUCTION **Refer to the application at the beginning of the lesson. Find the original dimensions of the Rove Tunnel.**

To determine the dimensions of the tunnel, we must solve the equation $0 = 81w^3 + 170w^2 + 16w - 31{,}115{,}520$. According to Descartes' Rule of Signs, there is one positive real root and two or zero negative real roots. Since dimensions are always positive, we are only concerned with the one positive real root.

Use a graphing utility to graph the related function $V(w) = 81w^3 + 170w^2 + 16w - 31{,}115{,}520$. Since the graph has 10 unit increments, the zero is between 70 and 80.



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To help determine the possible rational roots, find the prime factorization of 31,115,520.

$31{,}115{,}520 = 2^8 \times 3^2 \times 5 \times 37 \times 73$

Possible rational roots between 70 and 80 are 72 $(2^3 \times 3^2)$, 73, and 74 (2×37). Use the Factor Theorem until the one zero is found.

$$\begin{aligned}
V(w) &= 81w^3 + 170w^2 + 16w - 31{,}115{,}520\\
V(72) &= 81(72^3) + 170(72^2) + 16(72) - 31{,}115{,}520\\
V(72) &= 30{,}233{,}088 + 881{,}280 + 1152 - 31{,}115{,}520\\
V(72) &= 0
\end{aligned}$$

The original width was 72 feet, the original height was $\frac{1}{2}(72) + 1$ or 37 feet, and the original length was $324(72) + 32$ or 23,360 feet, which is over 4 miles.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Identify** the possible rational roots for the equation $x^4 - 3x^2 + 6 = 0$.
2. **Explain** why the Integral Root Theorem is a corollary to the Rational Root Theorem.
3. **Write** a polynomial function $f(x)$ whose coefficients have three sign changes. Find the number of sign changes that $f(-x)$ has. Describe the nature of the zeros.
4. *Math Journal* **Describe** several methods you could use to determine the rational zeros of a polynomial function. Which would you choose to use first? Explain.

Guided Practice

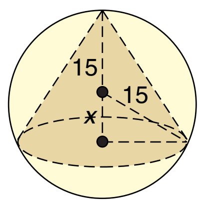
List the possible rational roots of each equation. Then determine the rational roots.

5. $x^3 - 4x^2 + x + 2 = 0$
6. $2x^3 + 3x^2 - 8x + 3 = 0$

Find the number of possible positive real zeros and the number of possible negative real zeros for each function. Then determine the rational zeros.

7. $f(x) = 8x^3 - 6x^2 - 23x + 6$
8. $f(x) = x^3 + 7x^2 + 7x - 15$

9. **Geometry** A cone is inscribed in a sphere with a radius of 15 centimeters. If the volume of the cone is 1152π cubic centimeters, find the length represented by x.



EXERCISES

Practice

List the possible rational roots of each equation. Then determine the rational roots.

10. $x^3 + 2x^2 - 5x - 6 = 0$

11. $x^3 - 2x^2 + x + 18 = 0$

12. $x^4 - 5x^3 + 9x^2 - 7x + 2 = 0$

13. $x^3 - 5x^2 - 4x + 20 = 0$

14. $2x^4 - x^3 - 6x + 3 = 0$

15. $6x^4 + 35x^3 - x^2 - 7x - 1 = 0$

16. State the number of complex roots, the number of positive real roots, and the number of negative real roots of $x^4 - 2x^3 + 7x^2 + 4x - 15 = 0$.

Find the number of possible positive real zeros and the number of possible negative real zeros for each function. Then determine the rational zeros.

17. $f(x) = x^3 - 7x - 6$

18. $f(x) = x^3 - 2x^2 - 8x$

19. $f(x) = x^3 + 3x^2 - 10x - 24$

20. $f(x) = 10x^3 - 17x^2 - 7x + 2$

21. $f(x) = x^4 + 2x^3 - 9x^2 - 2x + 8$

22. $f(x) = x^4 - 5x^2 + 4$

23. Suppose $f(x) = (x - 2)(x + 2)(x + 1)^2$.

a. Determine the zeros of the function.

b. Write $f(x)$ as a polynomial function.

c. Use Descartes' Rule of Signs to find the number of possible positive real roots and the number of possible negative real roots.

d. Compare your answers to part **a** and part **c**. Explain.

Applications and Problem Solving

Real World Application

24. Manufacturing The specifications for a new cardboard container require that the width for the container be 4 inches less than the length and the height be 1 inch less than twice the length.

a. Write a polynomial function that models the volume of the container in terms of its length.

b. Write an equation if the volume must be 2208 cubic inches.

c. Find the dimensions of the new container.

25. Critical Thinking Write a polynomial equation for each restriction.

a. fourth degree with no positive real roots

b. third degree with no negative real roots

c. third degree with exactly one positive root and exactly one negative real root

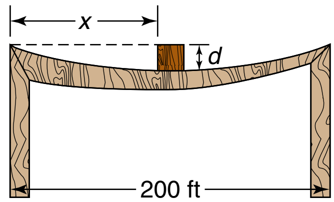
26. Architecture A hotel in Las Vegas, Nevada, is the largest pyramid in the United States. Prior to the construction of the building, the architects designed a scale model.

a. If the height of the scale model was 9 inches less than its length and its base is a square, write a polynomial function that describes the volume of the model in terms of its length.

b. If the volume of the model is 6300 cubic inches, write an equation describing the situation.

c. What were the dimensions of the scale model?

27. **Construction** A steel beam is supported by two pilings 200 feet apart. If a weight is placed x feet from the piling on the left, a vertical deflection d equals $0.0000008x^2(200 - x)$. How far is the weight if the vertical deflection is 0.8 feet?



28. **Critical Thinking** Compare and contrast the graphs and zeros of $f(x) = x^3 + 3x^2 - 6x - 8$ and $g(x) = -x^3 - 3x^2 + 6x + 8$.

Mixed Review

29. Divide $x^2 - x - 56$ by $x + 7$ using synthetic division. *(Lesson 4-3)*

30. Find the discriminant of $4x^2 + 6x + 25 = 0$ and describe the nature of the roots. *(Lesson 4-2)*

31. Write the polynomial equation of least degree whose roots are 1, -1, 2, and -2. *(Lesson 4-1)*

32. **Business** The prediction equation for a set of data relating the year in which a car was rented as the independent variable to the weekly car rental fee as the dependent variable is $y = 4.3x - 8424.3$. Predict the average cost of renting a car in 2008. *(Lesson 1-6)*

33. **SAT/ACT Practice** If $\frac{2x - 3}{x} = \frac{3 - x}{2}$, which of the following could be a value for x?

A -3 **B** -1 **C** 37 **D** 5 **E** 15

MID-CHAPTER QUIZ

1. Write the polynomial equation of least degree with roots 1, -1, $2\boldsymbol{i}$, and $-2\boldsymbol{i}$. (Lesson 4-1)

2. State the number of complex roots of $x^3 - 11x^2 + 30x = 0$. Then find the roots. (Lesson 4-1)

3. Solve $x^2 + 5x = 150$ by completing the square. (Lesson 4-2)

4. Find the discriminant of $6b^2 - 39b + 45 = 0$ and describe the nature of the roots of the equation. Then solve the equation by using the quadratic formula. (Lesson 4-2)

5. Divide $x^3 + 3x^2 - 2x - 8$ by $x + 2$ using synthetic division. (Lesson 4-3)

6. Use the Remainder Theorem to find the remainder for $(x^3 - 4x^2 + 2x - 6) \div (x - 4)$. State whether the binomial is a factor of the polynomial. (Lesson 4-3)

7. Determine the binomial factors of $x^3 - 2x^2 - 5x + 6$. (Lesson 4-3)

8. List the possible rational roots of $x^3 + 6x^2 + 10x + 3 = 0$. Then determine the rational roots. (Lesson 4-4)

9. Find the number of possible positive zeros and the number of possible negative zeros for $F(x) = x^4 + 4x^3 + 3x^2 - 4x - 4$. Then determine the rational zeros. (Lesson 4-4)

10. **Manufacturing** The Universal Paper Product Company makes cone-shaped drinking cups. The height of each cup is 6 centimeters more than the radius. If the volume of each cup is 27π cubic centimeters, find the dimensions of the cup. (Lesson 4-4)

Extra Practice See p. A32.

CONTENTS

4-5

Locating Zeros of a Polynomial Function

OBJECTIVE

- Approximate the real zeros of a polynomial function.

Real World Application

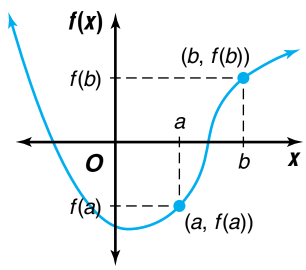
ECONOMY Layoffs at large corporations can cause the unemployment rate to increase, while low interest rates can bolster employment. From October 1997 to November 1998, the Texas economy was strong. The Texas jobless rate during that period can be modeled by the function $f(x) = -0.0003x^4 + 0.0066x^3 - 0.0257x^2 - 0.1345x + 5.35$, where x represents the number of months since October 1997 and $f(x)$ represents the unemployment rate as a percent. Use this model to predict when the unemployment will be 2.5%. *This problem will be solved in Example 4.*

The function $f(x) = -0.0003x^4 + 0.0066x^3 - 0.0257x^2 - 0.1345x + 5.35$ has four complex zeros. According to the Descartes' Rule of Signs, there are three or one positive real zeros and one negative zero. If you used a spreadsheet to evaluate the possible rational zeros, you will discover that none of the possible values is a zero of the function. This means that the zeros are not rational numbers. Another method, called the **Location Principle,** can be used to help determine the zeros of a function.

The Location Principle	Suppose $y = f(x)$ represents a polynomial function with real coefficients. If a and b are two numbers with $f(a)$ negative and $f(b)$ positive, the function has at least one real zero between a and b.

If $f(a) > 0$ and $f(b) < 0$, then the function also has at least one real zero between a and b.

This principle is illustrated by the graph. The graph of $y = f(x)$ is a continuous curve. At $x = a$, $f(a)$ is negative. At $x = b$, $f(b)$ is positive. Therefore, between the x-values of a and b, the graph must cross the x-axis. Thus, a zero exists somewhere between a and b.



Example 1 **Determine between which consecutive integers the real zeros of $f(x) = x^3 - 4x^2 - 2x + 8$ are located.**

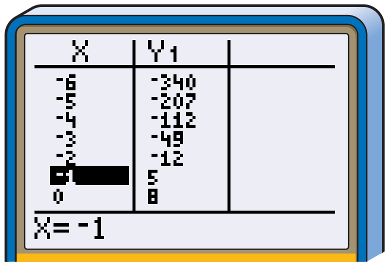
There are three complex zeros for this function. According to Descartes' Rule of Signs, there are two or zero positive real roots and one negative real root. You can use substitution, synthetic division, or the **TABLE** feature on a graphing calculator to evaluate the function for consecutive integral values of x.

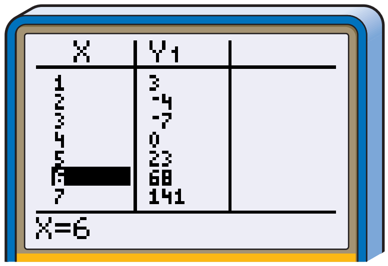
Method 1: Synthetic Division

r	1	−4	−2	8	
−3	1	−7	19	−49	
−2	1	−6	10	−12	*change*
−1	1	−5	3	5	*in signs*
0	1	−4	−2	8	
1	1	−3	−5	3	*change*
2	1	−2	−6	−4	*in signs*
3	1	−1	−5	−7	
4	1	0	−2	0	*zero*
5	1	1	3	23	

Method 2: Graphing Calculator

Use the **TABLE** feature.





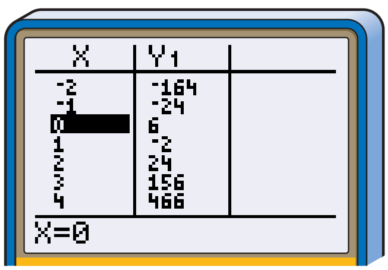
There is a zero at 4. The changes in sign indicate that there are also zeros between −2 and −1 and between 1 and 2. This result is consistent with the Descartes' Rule of Signs.

Once you know two integers between which a zero will fall, you can use substitution or a graphing calculator to approximate the zeros.

Example 2 **Approximate the real zeros of $f(x) = 12x^3 - 19x^2 - x + 6$ to the nearest tenth.**

There are three complex zeros for this function. According to Descartes' Rule of Signs, there are two or zero positive real roots and one negative real root.

Use the **TABLE** feature of a graphing calculator. There are zeros between −1 and 0, between 0 and 1, and between 1 and 2. To find the zeros to the nearest tenth, use the **TBLSET** feature changing **ΔTbl** to 0.1.

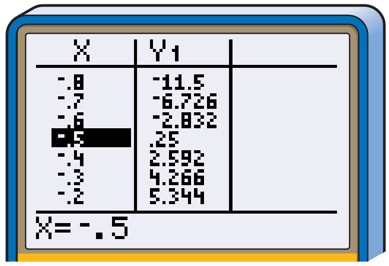


(continued on the next page)

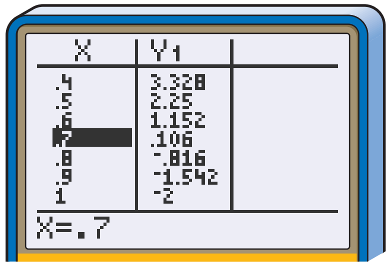
Graphing Calculator Tip

You can check these zeros by graphing the function and selecting **zero** under the **CALC** menu.

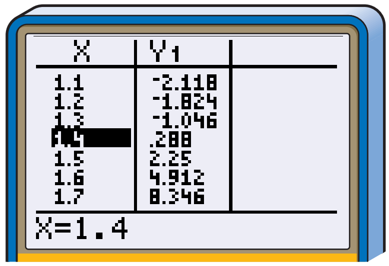
Since 0.25 is closer to zero than -2.832, the zero is about -0.5.



Since 0.106 is closer to zero than -0.816, the zero is about 0.7.



Since 0.288 is closer to zero than -1.046, the zero is about 1.4.



The zeros are about -0.5, 0.7, and 1.4.
If you need a closer approximation, change ΔTbl to 0.01.

The **Upper Bound Theorem** will help you confirm whether you have determined all of the real zeros. An **upper bound** is an integer greater than or equal to the greatest real zero.

Upper Bound Theorem

Suppose c is a positive real number and $P(x)$ is divided by $x - c$. If the resulting quotient and remainder have no change in sign, then $P(x)$ has no real zero greater than c. Thus, c is an upper bound of the zeros of $P(x)$.

Zero coefficients are ignored when counting sign changes.

The synthetic division in Example 1 indicates that 5 is an upper bound of the zeros of $f(x) = x^3 - 4x^2 - 2x + 8$ because there are no change of signs in the quotient and remainder.

A **lower bound** is an integer less than or equal to the least real zero. A lower bound of the zeros of $P(x)$ can be found by determining an upper bound for the zeros of $P(-x)$.

Lower Bound Theorem

If c is an upper bound of the zeros of $P(-x)$, then $-c$ is a lower bound of the zeros of $P(x)$.

Examples

3 **Use the Upper Bound Theorem to find an integral upper bound and the Lower Bound Theorem to find an integral lower bound of the zeros of $f(x) = x^3 + 3x^2 - 5x - 10$.**

The Rational Root Theorem tells us that ± 1, ± 2, ± 5, and ± 10 might be roots of the polynomial equation $x^3 + 3x^2 - 5x - 10 = 0$. These possible zeros of the function are good starting places for finding an upper bound.

$f(x) = x^3 + 3x^2 - 5x - 10$				
r	**1**	**3**	**−5**	**−10**
1	1	4	−1	−11
2	1	5	5	0

$f(-x) = -x^3 + 3x^2 + 5x - 10$				
r	**−1**	**3**	**5**	**−10**
1	−1	2	7	−3
2	−1	1	7	4
3	−1	0	5	5
4	−1	−1	1	−6
5	−1	−2	−5	−35

An upper bound is 2. Since 5 is an upper bound of $f(-x)$, -5 is a lower bound of $f(x)$. This means that all real zeros of $f(x)$ can be found in the interval $-5 \le x \le 2$.

4 **ECONOMICS** **Refer to the application at the beginning of the lesson. Use the model to determine when the unemployment will be 2.5%.**

Real World Application

You need to know when $f(x)$ has a value of 2.5.

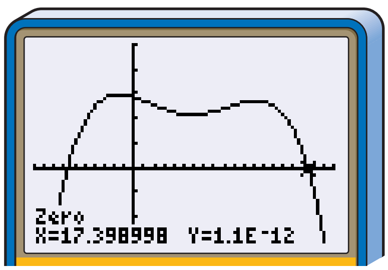
$$2.5 = -0.0003x^4 + 0.0066x^3 - 0.0257x^2 - 0.1345x + 5.35$$
$$0 = -0.0003x^4 + 0.0066x^3 - 0.0257x^2 - 0.1345x + 2.85$$

Now search for the zero of the related function,
$g(x) = -0.0003x^4 + 0.0066x^3 - 0.0257x^2 - 0.1345x + 2.85$

r	**−0.0003**	**0.0066**	**−0.0257**	**−0.1345**	**2.85**
17	−0.0003	0.0015	−0.0002	−0.1379	0.5057
18	−0.0003	0.0012	−0.0041	−0.2083	−0.8994

There is a zero between 17 and 18 months.

Confirm this zero using a graphing calculator. The zero is about 17.4 months. So, about 17 months after October, 1997 or March, 1999, the unemployment rate would be 2.5%. *Equations dealing with unemployment change as the economic conditions change. Therefore, long range predictions may not be accurate.*



[−10, 20] scl:1 by [−3, 5] scl:1

CHECK FOR UNDERSTANDING

Communicating Mathematics

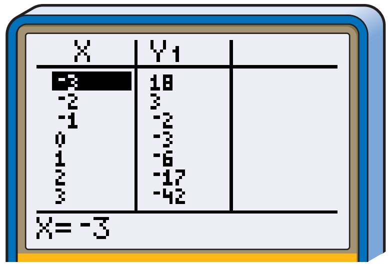
Read and study the lesson to answer each question.

1. **Write a convincing argument** that the Location Principle works. Include a labeled graph.

2. **Explain** how to use synthetic division to determine which consecutive integers the real zeros of a polynomial function are located.

3. **Describe** how you find an upper bound and a lower bound for the zeros of a polynomial function.

4. **You Decide** After looking at the table on a graphing calculator, Tiffany tells Nikki that there is a zero between -1 and 0. Nikki argues that the zero is between -2 and -1. Who is correct? Explain.



Guided Practice

Determine between which consecutive integers the real zeros of each function are located.

5. $f(x) = x^2 - 4x - 2$

6. $f(x) = x^3 - 3x^2 - 2x + 4$

Approximate the real zeros of each function to the nearest tenth.

7. $f(x) = 2x^3 - 4x^2 - 3$

8. $f(x) = x^2 + 3x + 2$

Use the Upper Bound Theorem to find an integral upper bound and the Lower Bound Theorem to find an integral lower bound of the zeros of each function.

9. $f(x) = x^4 - 8x + 2$

10. $f(x) = x^4 + x^2 - 3$

11. **Manufacturing** The It's A Snap Puzzle Company is designing new boxes for their 2000 piece 3-D puzzles. The old box measured 25 centimeters by 30 centimeters by 5 centimeters. For the new box, the designer wants to increase each dimension by a uniform amount.
    a. Write a polynomial function that models the volume of the new box.
    b. The volume of the new box must be 1.5 times the volume of the old box to hold the increase in puzzle pieces. Write an equation that models this situation.
    c. Find the dimensions of the new box.

EXERCISES

Practice

Determine between which consecutive integers the real zeros of each function are located.

12. $f(x) = x^3 - 2$

13. $f(x) = 2x^2 - 5x + 1$

14. $f(x) = x^4 - 2x^3 + x - 2$

15. $f(x) = x^4 - 8x^2 + 10$

16. $f(x) = x^3 - 3x + 1$

17. $f(x) = 2x^4 + x^2 - 3x + 3$

18. Is there a zero of $f(x) = 6x^3 + 24x^2 - 54x - 3$ between -6 and -5? Explain.

Approximate the real zeros of each function to the nearest tenth.

19. $f(x) = 3x^4 + x^2 - 1$

20. $f(x) = x^2 + 3x + 1$

21. $f(x) = x^3 - 4x + 6$

22. $f(x) = x^4 - 5x^3 + 6x^2 - x - 2$

23. $f(x) = 2x^4 - x^3 + x - 2$

24. $f(x) = x^5 - 7x^4 - 3x^3 + 2x^2 - 4x + 9$

25. Approximate the real zero of $f(x) = x^3 - 2x^2 + 5$ to the nearest hundredth.

Use the Upper Bound Theorem to find an integral upper bound and the Lower Bound Theorem to find an integral lower bound of the zeros of each function.

26. $f(x) = 3x^3 - 2x^2 + 5x - 1$

27. $f(x) = x^2 - x - 1$

28. $f(x) = x^4 - 6x^3 + 2x^2 + 6x - 13$

29. $f(x) = x^3 + 5x^2 - 3x - 20$

30. $f(x) = x^4 - 3x^3 - 2x^2 + 3x - 5$

31. $f(x) = x^5 + 5x^4 - 3x^3 + 20x^2 - 15$

32. Analyze the zeros of $f(x) = x^4 - 3x^3 - 2x^2 + 3x - 5$.

a. Determine the number of complex zeros.

b. List the possible rational zeros.

c. Determine the number of possible positive real zeros and the number of possible negative real zeros.

d. Determine the integral intervals where the zeros are located.

e. Determine an integral upper bound of the zeros and an integral lower bound of the zeros.

f. Determine the zeros to the nearest tenth.

Applications and Problem Solving

Real World Application

33. Population The population of Manhattan Island in New York City between 1890 to 1970 can be modeled by $P(x) = -0.78x^4 + 133x^3 - 7500x^2 + 147{,}500x + 1{,}440{,}000$, where $P(x)$ represents the population and x represents the number of years since 1890.

a. According to the data at the right, how valid is the model?

b. Use the model to predict the population in 1980.

c. According to this model, what happens between 1970 and 1980?

d. Do you think this model is valid for any time? Explain.

Population of Manhattan Island

Year	Population
1890	1,441,216
1910	2,331,542
1930	1,867,312
1950	1,960,101
1970	1,539,233

Source: U.S. Bureau of the Census

34. Critical Thinking Write a third-degree integral polynomial function with one zero at $\sqrt{2}$. State the zeros of the function. Draw a graph to support your answer.

35. Medicine A doctor tells Masa to take 60 milligrams of medication each morning for three days. The amount of medication remaining in his body on the fourth day is modeled by $M(x) = 60x^3 + 60x^2 + 60x$, where x represents the absorption rate per day. Suppose Masa has 37.44 milligrams of medication in his body on the fourth day.

a. Write an equation to model this situation.

b. Write the related function for the equation.

c. Graph this function and estimate the absorption rate.

d. Find the absorption rate of the medication.

36. Critical Thinking Write a polynomial function with an upper bound of 1 and a lower bound of -1.

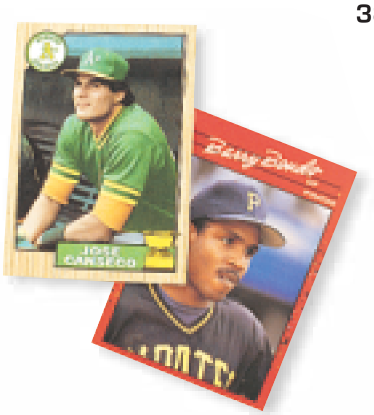
37. **Ecology** In the early 1900s, the deer population of the Kaibab Plateau in Arizona experienced a rapid increase because hunters had reduced the number of natural predators. The food supply was not great enough to support the increased population, eventually causing the population to decline. The deer population for the period 1905 to 1930 can be modeled by $f(x) = -0.125x^5 + 3.125x^4 + 4000$, where x is the number of years from 1905.

a. Graph the function.

b. Use the model to determine the population in 1905.

c. Use the model to determine the population in 1920.

d. According to this model, when did the deer population become zero?



38. **Investments** Instead of investing in the stock market, many people invest in collectibles, like baseball cards. Each year, Anna uses some of the money she receives for her birthday to buy one special baseball card. For the last four birthdays, she purchased cards for \$6, \$18, \$24, and \$18. The current value of these cards is modeled by $T(x) = 6x^4 + 18x^3 + 24x^2 + 18x$, where x represents the average rate of return plus one.

a. If the cards are worth \$81.58, write an equation to model this situation.

b. Find the value of x.

c. What is the average rate of return on Anna's investments?

Mixed Review

39. Find the number of possible positive real zeros and the number of possible negative real zeros for $f(x) = 2x^3 - 5x^2 - 28x + 15$. Then determine the rational zeros. *(Lesson 4-4)*

40. **Physics** The distance $d(t)$ fallen by a free-falling body can be modeled by the formula $d(t) = v_0t - \frac{1}{2}gt^2$, where v_0 is the initial velocity and g represents the acceleration due to gravity. The acceleration due to gravity is 9.8 meters per second squared. If a rock is thrown upward with an initial velocity of +4 meters per second from the edge of the North Rim of the Grand Canyon, which is 1750 meters deep, determine how long it will take the rock to reach the bottom of the Grand Canyon. (*Hint:* The distance to the bottom of the canyon is −1750 meters from the rim.) *(Lesson 4-2)*

41. Graph $y = \dfrac{4x}{x-1}$. *(Lesson 3-7)*

42. Find the value of $\begin{vmatrix} 7 & 9 \\ 3 & 6 \end{vmatrix}$. *(Lesson 2-5)*

43. Find the coordinates of the midpoint of $\overline{FG}$ given its endpoints $F(-3, -2)$ and $G(8, 4)$. *(Lesson 1-5)*

44. Name the slope and y-intercept of the graph of $x - 2y - 4 = 0$. *(Lesson 1-4)*

45. **SAT/ACT Practice** $\triangle ABC$ and $\triangle ABD$ are right triangles that share side $\overline{AB}$. $\triangle ABC$ has area x, and $\triangle ABD$ has area y. If $\overline{AD}$ is longer than $\overline{AC}$ and $\overline{BD}$ is longer than $\overline{BC}$, which of the following cannot be true?

A $y > x$ **B** $y < x$ **C** $y \geq x$ **D** $y \neq x$ **E** $\frac{y}{x} \geq 1$

Extra Practice See p. A33.

4-6

Rational Equations and Partial Fractions

OBJECTIVES
- Solve rational equations and inequalities.
- Decompose a fraction into partial fractions.

Real World Application

SCUBA DIVING If a scuba diver goes to depths greater than 33 feet, the function $T(d) = \frac{1700}{d - 33}$ gives the maximum time a diver can remain down and still surface at a steady rate with no decompression stops. In this function, $T(d)$ represents the dive time in minutes, and d represents the depth in feet. If a diver is planning a 45-minute dive, what is the maximum depth the diver can go without decompression stops on the way back up? *This problem will be solved in Example 1.*

To solve this problem, you need to solve the equation $45 = \frac{1700}{d - 33}$. This type of equation is called a **rational equation.** A rational equation has one or more rational expressions. One way to solve a rational equation is to multiply each side of the equation by the least common denominator (LCD).

Example 1 Real World Application

SCUBA DIVING **What is the maximum depth the diver can go without decompression stops on the way back up?**

$$T(d) = \frac{1700}{d - 33}$$

$$45 = \frac{1700}{d - 33} \quad \text{Replace } T(d) \text{ with 45, the dive time in minutes.}$$

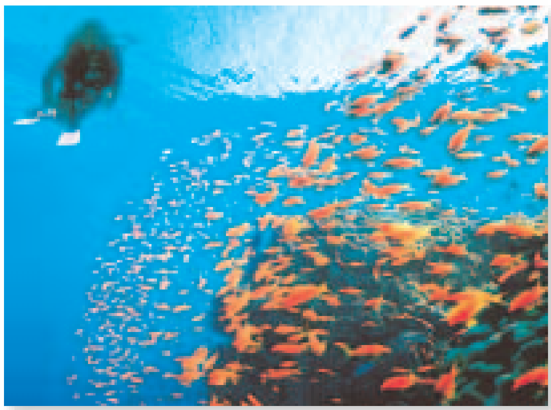
$$45(d - 33) = \frac{1700}{d - 33}(d - 33) \quad \text{Multiply each side by the LCD, } d - 33.$$

$$45d - 1485 = 1700$$

$$45d = 3185$$

$$d \approx 70.78$$

The diver can go to a depth of about 70 feet and surface without decompression stops.



Any possible solution that results in a zero in the denominator must be excluded from your list of solutions. So, in Example 1, the solution could not be 33. Always check your solutions by substituting them into the original equation.

Example 2 Solve $a + \frac{a^2 - 5}{a^2 - 1} = \frac{a^2 + a + 2}{a + 1}$.

$$a + \frac{a^2 - 5}{a^2 - 1} = \frac{a^2 + a + 2}{a + 1}$$

$$\left(a + \frac{a^2 - 5}{a^2 - 1}\right)(a^2 - 1) = \left(\frac{a^2 + a + 2}{\cancel{a + 1}_{1}}\right)(\cancel{a^2 - 1}^{\,a-1})$$

Multiply each side by the LCD, $(a - 1)(a + 1)$ or $a^2 - 1$.

$$a(a^2 - 1) + (a^2 - 5) = (a^2 + a + 2)(a - 1)$$
$$a^3 - a + a^2 - 5 = a^3 + a - 2$$
$$a^2 - 2a - 3 = 0$$
$$(a - 3)(a + 1) = 0 \quad \textit{Factor.}$$

$a - 3 = 0 \qquad a + 1 = 0$
$a = 3 \qquad a = -1$

When you check your solutions, you find that a cannot equal -1 because a zero denominator results. Since -1 cannot be a solution, the only solution is 3.

In order to add or subtract fractions with unlike denominators, you must first find a common denominator. Suppose you have a rational expression and you want to know what fractions were added or subtracted to obtain that expression. Finding these fractions is called decomposing the fraction into **partial fractions.** Decomposing fractions is a skill used in calculus and other advanced mathematics courses.

Example 3 Decompose $\frac{8y + 7}{y^2 + y - 2}$ into partial fractions.

First factor the denominator. $\quad y^2 + y - 2 = (y - 1)(y + 2)$

Express the factored form as the sum of two fractions using A and B as numerators and the factors as denominators.

$$\frac{8y + 7}{y^2 + y - 2} = \frac{A}{y - 1} + \frac{B}{y + 2}$$

Eliminate the denominators by multiplying each side by the LCD, $(y - 1)(y - 2)$.

$$8y + 7 = A(y + 2) + B(y - 1)$$

Eliminate B by letting $y = 1$ so that $y - 1$ becomes 0.

$$8y + 7 = A(y + 2) + B(y - 1)$$
$$8(1) + 7 = A(1 + 2) + B(1 - 1)$$
$$15 = 3A$$
$$5 = A$$

Eliminate A by letting $y = -2$ so that $y + 2$ becomes 0.

$$8y + 7 = A(y + 2) + B(y - 1)$$
$$8(-2) + 7 = A(-2 + 2) + B(-2 - 1)$$
$$9 = -3B$$
$$3 = B$$

Now substitute the values for A and B to determine the partial fractions.

$$\frac{A}{y - 1} + \frac{B}{y + 2} = \frac{5}{y - 1} + \frac{3}{y + 2}$$

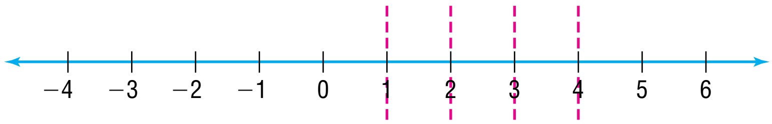
So, $\frac{8y + 7}{y^2 + y - 2} = \frac{5}{y - 1} + \frac{3}{y + 2}$.

Check to see if the sum of the two partial fractions equals the original fraction.

The process used to solve rational equations can be used to solve **rational inequalities.**

Example 4 **Solve $\frac{(x-2)(x-1)}{(x-3)(x-4)^2} < 0$.**

Let $f(x) = \frac{(x-2)(x-1)}{(x-3)(x-4)^2}$. On a number line, mark the zeros of $f(x)$ and the excluded values for $f(x)$ with vertical dashed lines. The zeros of $f(x)$ are the same values that make $(x-2)(x-1) = 0$. These zeros are 2 and 1. Excluded values for $f(x)$ are the values that make $(x-3)(x-4)^2 = 0$. These excluded values are 3 and 4.



The vertical dashed lines separate the number line into intervals. Test a convenient value within each interval in the original rational inequality to see if the test value is a solution. If the value in the interval is a solution, all values are solutions. In this problem, it is not necessary to find the exact value of the expression.

For $x < 1$, test $x = 0$: $\frac{(0-2)(0-1)}{(0-3)(0-4)^2} \rightarrow \frac{(-)(-)}{(-)(-)(-)} \rightarrow -$

So in the interval $x < 1$, $f(x) < 0$. Thus, $x < 1$ is a solution.

For $1 < x < 2$, test $x = 1.5$: $\frac{(1.5-2)(1.5-1)}{(1.5-3)(1.5-4)^2} \rightarrow \frac{(-)(+)}{(-)(-)(-)} \rightarrow +$

So in the interval $1 < x < 2$, $f(x) > 0$. Thus, $1 < x < 2$ is not a solution.

For $2 < x < 3$, test $x = 2.5$: $\frac{(2.5-2)(2.5-1)}{(2.5-3)(2.5-4)^2} \rightarrow \frac{(+)(+)}{(-)(-)(-)} \rightarrow -$

So in the interval $2 < x < 3$, $f(x) < 0$. Thus, $2 < x < 3$ is a solution.

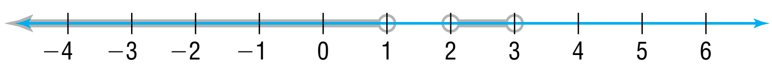
For $3 < x < 4$, test $x = 3.5$: $\frac{(3.5-2)(3.5-1)}{(3.5-3)(3.5-4)^2} \rightarrow \frac{(+)(+)}{(+)(-)(-)} \rightarrow +$

So in the interval $3 < x < 4$, $f(x) > 0$. Thus, $3 < x < 4$ is not a solution.

For $4 < x$, test $x = 5$: $\frac{(5-2)(5-1)}{(5-3)(5-4)^2} \rightarrow \frac{(+)(+)}{(+)(+)(+)} \rightarrow +$

So in the interval $x > 4$, $f(x) > 0$. Thus, $4 < x$ is not a solution.

The solution is $x < 1$ or $2 < x < 3$. This solution can be graphed on a number line.



Example 5 Solve $\frac{2}{3a} + \frac{5}{6a} > \frac{3}{4}$.

The inequality can be written as $\frac{2}{3a} + \frac{5}{6a} - \frac{3}{4} > 0$. The related function is $f(a) = \frac{2}{3a} + \frac{5}{6a} - \frac{3}{4}$. Find the zeros of this function.

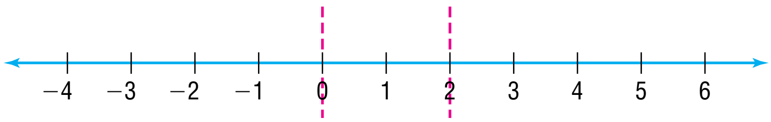
$$\frac{2}{3a} + \frac{5}{6a} - \frac{3}{4} = 0$$

$$\frac{2}{3a}(12a) + \frac{5}{6a}(12a) - \frac{3}{4}(12a) = 0(12a) \quad \textit{The LCD is 12a.}$$

$$8 + 10 - 9a = 0$$

$$2 = a$$

The zero is 2. The excluded value is 0. On a number line, mark these values with vertical dashed lines. The vertical dashed lines separate the number line into intervals.



Now test a sample value in each interval to determine if the values in the interval satisfy the inequality.

For $a < 0$, test $x = -1$:

$$\frac{2}{3(-1)} + \frac{5}{6(-1)} \stackrel{?}{>} \frac{3}{4}$$

$$-\frac{2}{3} - \frac{5}{6} \stackrel{?}{>} \frac{3}{4}$$

$$-\frac{3}{2} \not> \frac{3}{4} \quad \textit{a < 0 is not a solution.}$$

For $0 < a < 2$, test $x = 1$:

$$\frac{2}{3(1)} + \frac{5}{6(1)} \stackrel{?}{>} \frac{3}{4}$$

$$\frac{2}{3} + \frac{5}{6} \stackrel{?}{>} \frac{3}{4}$$

$$\frac{3}{2} > \frac{3}{4} \quad \textit{0 < a < 2 is a solution.}$$

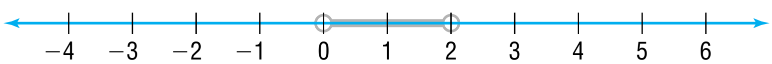
For $2 < a$, test $x = 3$:

$$\frac{2}{3(3)} + \frac{5}{6(3)} \stackrel{?}{>} \frac{3}{4}$$

$$\frac{2}{9} + \frac{5}{18} \stackrel{?}{>} \frac{3}{4}$$

$$\frac{1}{2} \not> \frac{3}{4} \quad \textit{2 < x is not a solution.}$$

The solution is $0 < a < 2$. This solution can be graphed on a number line.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the process used to solve $\frac{b+1}{3(b-2)} = \frac{5b}{6} + \frac{1}{b-2}$.
2. **Write** a sentence explaining why all solutions of a rational equation must be checked.
3. **Explain** what is meant by decomposing a fraction into partial fractions.
4. **Explain** why $x + \frac{2}{x-2} = 2 + \frac{2}{x-2}$ has no solution.

Guided Practice

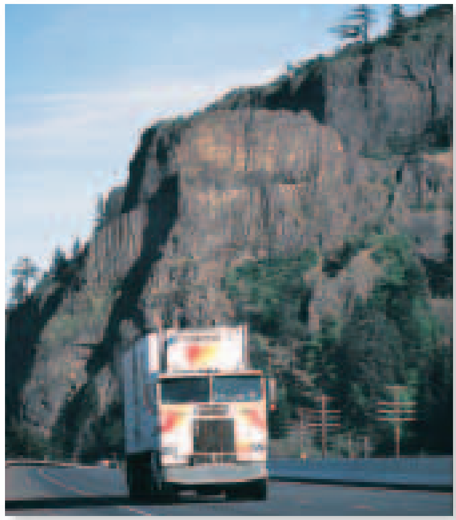
Solve each equation.

5. $a - \frac{5}{a} = 4$
6. $\frac{9}{b+5} = \frac{3}{b-3}$
7. $\frac{t+4}{t} + \frac{3}{t-4} = \frac{-16}{t^2-4t}$
8. Decompose $\frac{3p-1}{p^2-1}$ into partial fractions.

Solve each inequality.

9. $5 + \frac{1}{x} > \frac{16}{x}$
10. $1 + \frac{5}{a-1} \leq \frac{7}{6}$

11. **Interstate Commerce** When truckers are on long-haul drives, their driving logs must reflect their average speed. Average speed is the total distance driven divided by the total time spent driving. A trucker drove 3 hours on a freeway at 60 miles per hour and then drove 20 miles in the city. The trucker's average speed was 57.14 miles per hour.
    a. Write an equation that models the situation.
    b. How long was the trucker driving in the city to the nearest hundredth of an hour?



EXERCISES

Practice

Solve each equation.

12. $\frac{12}{t} + t - 8 = 0$
13. $\frac{1}{m} = \frac{m-34}{2m^2}$
14. $\frac{2}{y+2} + \frac{3}{y} = \frac{-y}{y+2}$
15. $\frac{10}{n^2-1} + \frac{2n-5}{n-1} = \frac{2n+5}{n+1}$
16. $\frac{1}{b+2} + \frac{1}{b+2} = \frac{3}{b+1}$
17. $\frac{7a}{3a+3} - \frac{5}{4a-4} = \frac{3a}{2a+2}$
18. $1 = \frac{1}{1-a} + \frac{a}{a-1}$
19. $\frac{2q}{2q+3} - \frac{2q}{2q-3} = 1$
20. $\frac{1}{3m} + \frac{6m-9}{3m} = \frac{3m-3}{4m}$
21. $\frac{-4}{x-1} = \frac{7}{2-x} + \frac{3}{x+1}$

22. Consider the equation $1 + \frac{n+6}{n+1} = \frac{4}{n-2}$.

a. What is the LCD of the rational expressions?

b. What values must be excluded from the list of possible solutions.

c. What is the solution of the equation?

Decompose each expression into partial fractions.

23. $\frac{x-6}{x^2-2x}$ **24.** $\frac{5m-4}{m^2-4}$ **25.** $\frac{-4y}{3y^2-4y+1}$

26. Find two rational expressions that have a sum of $\frac{9-9x}{x^2-9}$.

27. Consider the inequality $\frac{a-2}{a} < \frac{a-4}{a-6}$.

a. What is the LCD of the rational expressions?

b. Find the zero(s) of the related function.

c. Find the excluded value(s) of the related function.

d. Solve the inequality.

Solve each inequality.

28. $\frac{2}{w} + 3 > \frac{29}{w}$ **29.** $\frac{(x-3)(x-4)}{(x-5)(x-6)^2} \leq 0$

30. $\frac{x^2-16}{x^2-4x-5} \geq 0$ **31.** $\frac{1}{4a} + \frac{5}{8a} > \frac{1}{2}$

32. $\frac{1}{2b+1} + \frac{1}{b+1} > \frac{8}{15}$ **33.** $\frac{7}{y+1} > 7$

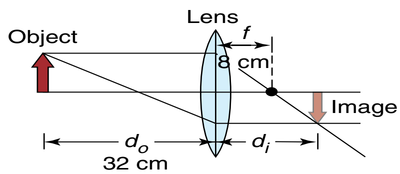
34. Four times the multiplicative inverse of a number is added to the number. The result is $10\frac{2}{5}$. What is the number?

35. The ratio of $x + 2$ to $x - 5$ is greater than 30%. Solve for x.

Applications and Problem Solving

Real World Application

36. Optics The lens equation is $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_o}$, where f is the focal length, d_i is the distance from the lens to the image, and d_o is the distance from the lens to the object. Suppose the object is 32 centimeters from the lens and the focal length is 8 centimeters.



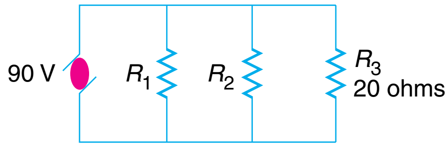
a. Write a rational equation to model the situation.

b. Find the distance from the lens to the image.

37. Critical Thinking Write a rational equation that cannot have 3 or -2 as a solution.

38. Trucking Two trucks can carry loads of coal in a ratio of 5 to 2. The smaller truck has a capacity 3 tons less than that of the larger truck. What is the capacity of the larger truck?

39. Electricity The diagram of an electric circuit shows three parallel resistors. If R represents the equivalent resistance of the three resistors, then $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. In this circuit, R_1 represents twice the resistance of R_2, and R_3 equals 20 ohms. Suppose the equivalent resistance equals 10 ohms.



a. Write a rational equation to model the situation.

b. Find R_1 and R_2.

40. Education Todd has answered 11 of his last 20 daily quiz questions correctly. His baseball coach told him that he must bring his average up to at least 70% if he wants to play in the season opener. Todd vows to study diligently and answer all of the daily quiz questions correctly in the future. How many consecutive daily quiz questions must he answer correctly to bring his average up to 70%?

41. Aviation An aircraft flies 1062 miles with the wind at its tail. In the same amount of time, a similar aircraft flies against the wind 738 miles. If the air speed of each plane is 200 miles per hour, what is the speed of the wind? (*Hint:* Time equals the distance divided by the speed.)

42. Critical Thinking Solve for a if $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$.

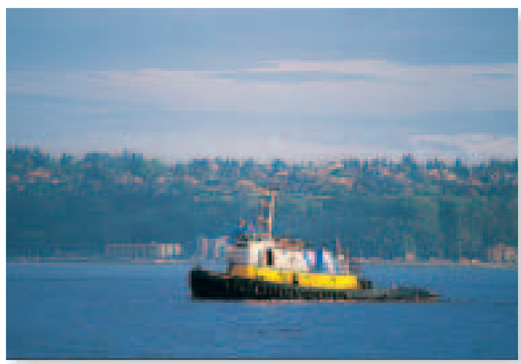
43. Statistics A number x is said to be the *harmonic mean* of y and z if $\frac{1}{x}$ is the average of $\frac{1}{y}$ and $\frac{1}{z}$.

a. Write an equation whose solution is the harmonic mean of 30 and 45.

b. Find the harmonic mean of 30 and 45.

44. Economics Darnell drives about 15,000 miles each year. He is planning to buy a new car. The car he wants to buy averages 20 miles on one gallon of gasoline. He has decided he would buy another car if he could save at least $200 per year in gasoline expenses. Assume gasoline costs $1.20 per gallon. What is the minimum number of miles per gallon that would fulfill Darnell's criteria?

45. Navigation The speed of the current in Puget Sound is 5 miles per hour. A barge travels with the current 26 miles and returns in $10\frac{2}{3}$ hours. What is its speed in still water?



Puget Sound

46. Critical Thinking If $\frac{3x}{5y} = 11$, find the value of $\frac{3x - 5y}{5y}$.

Mixed Review

47. Determine between which consecutive integers the real zeros of $f(x) = x^3 + 2x^2 - 3x - 5$ are located. *(Lesson 4-5)*

48. Use the Remainder Theorem to find the remainder for $(x^3 - 30x) \div (x + 5)$. State whether the binomial is a factor of the polynomial. *(Lesson 4-3)*

49. State the number of complex roots of $12x^2 + 8x - 15 = 0$. Then find the roots. *(Lesson 4-1)*

50. Name all the values of x that are not in the domain of the function $f(x) = \frac{x}{|3x| - 12}$. *(Lesson 3-7)*

51. Determine if (6, 3) is a solution for $y \leq \frac{2x + 3}{x}$. *(Lesson 3-3)*

52. Determine if the graph of $y^2 = 121x^2$ is symmetric with respect to each line. *(Lesson 3-1)*

a. the x-axis

b. the y-axis

c. the line $y = x$

d. the line $y = -x$

53. Education The semester test in your English class consists of short answer and essay questions. Each short answer question is worth 5 points, and each essay question is worth 15 points. You may choose up to 20 questions of any type to answer. It takes 2 minutes to answer each short answer question and 12 minutes to answer each essay question. *(Lesson 2-7)*

a. You have one hour to complete the test. Assuming that you answer all of the questions that you attempt correctly, how many of each type should you answer to earn the highest score?

b. You have two hours to complete the test. Assuming that you answer all of the questions that you attempt correctly, how many of each type should you answer to earn the highest score?

54. Find matrix X in the equation $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -3 & -5 \end{bmatrix} = X$. *(Lesson 2-3)*

55. Write the standard form of the equation of the line that is parallel to the graph of $y = 2x - 10$ and passes through the point at $(-3, 1)$. *(Lesson 1-5)*

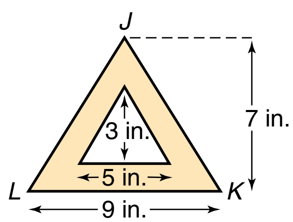
56. Manufacturing It costs ABC Corporation $3000 to produce 20 color televisions and $5000 to produce 60 of the same color televisions. *(Lesson 1-4)*

a. Find the cost function.

b. Determine the fixed cost and the variable cost per unit.

c. Sketch the graph of the cost function.

57. SAT Practice **Grid-In** Find the area of the shaded region in square inches.



Extra Practice See p. A33.

CONTENTS

4-7

Radical Equations and Inequalities

OBJECTIVE
- Solve radical equations and inequalities.

Real World Application

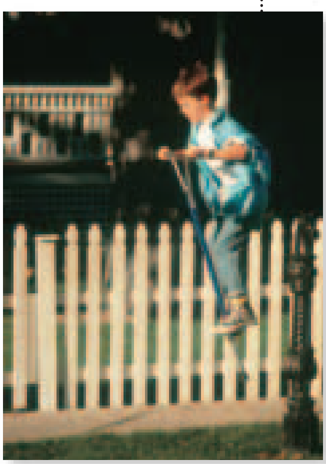
RECREATION A pogo stick stores energy in a spring. When a jumper compresses the spring from its natural length x_0 to a length x, the maximum height h reached by the bottom of the pogo stick is related to x_0 and x by the equation $x_0 = x + \sqrt{\frac{2mgh}{k}}$, where m is the combined mass of the jumper and the pogo stick, g is the acceleration due to gravity (9.80 meters per second squared), and k is a constant that depends on the spring. If the combined mass of the jumper and the stick is 50 kilograms, the spring compressed from its natural length of 1 meter to the length of 0.9 meter. If $k = 1.2 \times 10^4$, find the maximum height reached by the bottom of the pogo stick. *This problem will be solved in Example 1.*

Equations in which radical expressions include variables, such as the equation above, are known as **radical equations.** To solve radical equations, the first step is to isolate the radical on one side of the equation. Then raise each side of the equation to the proper power to eliminate the radical expression.

The process of raising each side of an equation to a power sometimes produces **extraneous solutions.** These are solutions that do not satisfy the original equation. Therefore, it is important to check all possible solutions in the original equation to determine if any of them should be eliminated from the solution set.

Examples **1** **RECREATION** **Find the maximum height reached by the bottom of the pogo stick.**

Real World Application



$$x_0 = x + \sqrt{\frac{2mgh}{k}}$$

$$1 = 0.9 + \sqrt{\frac{2 \times 50 \times 9.80 \times h}{1.2 \times 10^4}} \qquad x_0 = 1,\ x = 0.9,\ m = 50,\ g = 9.80,\ k = 1.2 \times 10^4$$

$$0.1 = \sqrt{\frac{980h}{1.2 \times 10^4}} \qquad \text{Isolate the radical.}$$

$$0.01 = \frac{980h}{1.2 \times 10^4} \qquad \text{Square each side.}$$

$$0.1224489796 = h \qquad \text{Solve for } h.$$

A possible solution is about 0.12. Check this solution.

(continued on the next page)

Check: $x_0 = x + \sqrt{\dfrac{2mgh}{k}}$

$$1 \stackrel{?}{=} 0.9 + \sqrt{\frac{2 \times 50 \times 9.80 \times 0.12}{1.2 \times 10^4}}$$

$$1 \approx 0.9989949494 \checkmark$$

The solution checks. The maximum height is about 0.12 meter.

2 **Solve $x = \sqrt{x + 7} + 5$.**

$$x = \sqrt{x+7} + 5$$

$$x - 5 = \sqrt{x+7} \quad \textit{Isolate the radical.}$$

$$x^2 - 10x + 25 = x + 7 \quad \textit{Square each side.}$$

$$x^2 - 11x + 18 = 0$$

$$(x - 9)(x - 2) = 0 \quad \textit{Factor.}$$

$$x - 9 = 0 \qquad x - 2 = 0$$

$$x = 9 \qquad x = 2$$

Check both solutions to make sure they are not extraneous.

Check $x = 9$:

$$x = \sqrt{x+7} + 5$$

$$9 \stackrel{?}{=} \sqrt{9+7} + 5$$

$$9 \stackrel{?}{=} \sqrt{16} + 5$$

$$9 \stackrel{?}{=} 4 + 5$$

$$9 = 9 \checkmark$$

Check $x = 2$:

$$x = \sqrt{x+7} + 5$$

$$2 \stackrel{?}{=} \sqrt{2+7} + 5$$

$$2 \stackrel{?}{=} \sqrt{9} + 5$$

$$2 \stackrel{?}{=} 3 + 5$$

$$2 \neq 8$$

One solution checks and the other solution does not check. The solution is 9.

The same method of solution works for nth root equations.

Example 3 **Solve $4 = \sqrt[3]{x + 2} + 8$.**

$$4 = \sqrt[3]{x+2} + 8$$

$$-4 = \sqrt[3]{x+2} \quad \textit{Isolate the cube root.}$$

$$-64 = x + 2 \quad \textit{Cube each side. } \left(\sqrt[3]{x+2}\right)^3 = x + 2.$$

$$-66 = x$$

Check the solution.

Check:

$$4 = \sqrt[3]{x+2} + 8$$

$$4 \stackrel{?}{=} \sqrt[3]{-66+2} + 8$$

$$4 \stackrel{?}{=} \sqrt[3]{-64} + 8$$

$$4 \stackrel{?}{=} -4 + 8$$

$$4 = 4 \checkmark$$

The solution checks. The solution is -66.

If there is more than one radical in an equation, you may need to repeat the process for solving radical equations more than once until all radicals have been eliminated.

Example 4 **Solve $\sqrt{x + 10} = 5 - \sqrt{3 - x}$.**

Graphing Calculator Tip

To solve a radical equation with a graphing calculator, graph each side of the equation as a separate function. The solution is any value(s) of x for points common to both graphs.

Leave the radical signs on opposite sides of the equation. If you square the sum or difference of two radical expressions, the value under the radical sign in the resulting product will be more complicated.

$$\sqrt{x + 10} = 5 - \sqrt{3 - x}$$

$$x + 10 = 25 - 10\sqrt{3 - x} + 3 - x \quad \textit{Square each side.}$$

$$2x - 18 = -10\sqrt{3 - x} \quad \textit{Isolate the radical.}$$

$$4x^2 - 72x + 324 = 100(3 - x) \quad \textit{Square each side.}$$

$$4x^2 - 72x + 324 = 300 - 100x$$

$$4x^2 + 28x + 24 = 0$$

$$x^2 + 7x + 6 = 0 \quad \textit{Divide each side by 4.}$$

$$(x + 6)(x + 1) = 0$$

$x + 6 = 0$ $\quad$ $x + 1 = 0$

$x = -6$ $\quad$ $x = -1$

Check both solutions to make sure they are not extraneous.

Check $x = -6$:

$$\sqrt{x + 10} = 5 - \sqrt{3 - x}$$

$$\sqrt{-6 + 10} \stackrel{?}{=} 5 - \sqrt{3 - (-6)}$$

$$\sqrt{4} \stackrel{?}{=} 5 - \sqrt{9}$$

$$2 \stackrel{?}{=} 5 - 3$$

$$2 = 2 \checkmark$$

Check $x = -1$:

$$\sqrt{x + 10} = 5 - \sqrt{3 - x}$$

$$\sqrt{-1 + 10} \stackrel{?}{=} 5 - \sqrt{3 - (-1)}$$

$$\sqrt{9} \stackrel{?}{=} 5 - \sqrt{4}$$

$$3 \stackrel{?}{=} 5 - 2$$

$$3 = 3 \checkmark$$

Both solutions check. The solutions are -6 and -1.

Use the same procedures to solve **radical inequalities.**

Example 5 **Solve $\sqrt{4x + 5} \leq 10$.**

$$\sqrt{4x + 5} \leq 10$$

$$4x + 5 \leq 100 \quad \textit{Square each side.}$$

$$4x \leq 95$$

$$x \leq 23.75$$

In order for $\sqrt{4x + 5}$ to be a real number, $4x + 5$ must be greater than or equal to zero.

$$4x + 5 \geq 0$$

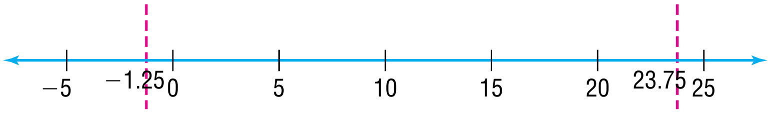
$$4x \geq -5$$

$$x \geq -1.25$$

So the solution is $-1.25 \leq x \leq 23.75$. Check this solution by testing values in the intervals defined by the solution.

(continued on the next page)

On a number line, mark -1.25 and 23.75 with vertical dashed lines. The dashed lines separate the number line into intervals. Check the solution by testing values for x from each interval into the original inequality.



For $x \leq -1.25$, test $x = -2$: $\sqrt{4(-2) + 5} \stackrel{?}{\leq} 10$

$\sqrt{-3} \stackrel{?}{\leq} 10$ *This statement is meaningless. $x \leq -1.25$ is not a solution.*

For $-1.25 \leq x \leq 23.75$, test $x = 0$: $\sqrt{4(0) + 5} \stackrel{?}{\leq} 10$

$\sqrt{5} \stackrel{?}{\leq} 10$

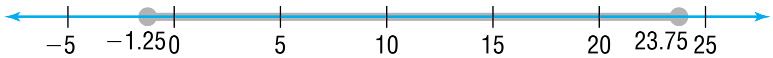
$2.236 \leq 10$ *$-1.25 \leq x \leq 23.75$ is a solution.*

For $23.75 \leq x$, test $x = 25$: $\sqrt{4(25) + 5} \stackrel{?}{\leq} 10$

$\sqrt{105} \stackrel{?}{\leq} 10$

$10.247 \not\leq 10$ *$23.75 \leq x$ is not a solution.*

The solution checks and is graphed on the number line below.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why the first step in solving $5 + \sqrt{x + 1} = x$ should be to isolate the radical.
2. **Write** several sentences explaining why it is necessary to check for extraneous solutions in radical equations.
3. **Explain** the difference between solving an equation with one radical and solving an equation with more than one radical.

Guided Practice

Solve each equation.

4. $\sqrt{1 - 4t} = 2$
5. $\sqrt[3]{x + 4} + 12 = 3$
6. $5 + \sqrt{x - 4} = 2$
7. $\sqrt{6x - 4} = \sqrt{2x + 10}$
8. $\sqrt{a + 4} + \sqrt{a - 3} = 7$

Solve each inequality.

9. $\sqrt{5x + 4} \leq 8$
10. $3 + \sqrt{4a - 5} \leq 10$

CONTENTS

11. **Amusement Parks** The velocity of a roller coaster as it moves down a hill is $v = \sqrt{v_0^2 + 64h}$, where v_0 is the initial velocity and h is the vertical drop in feet. The designer of a coaster wants the coaster to have a velocity of 90 feet per second when it reaches the bottom of the hill.
   a. If the initial velocity of the coaster at the top of the hill is 10 feet per second, write an equation that models the situation.
   b. How high should the designer make the hill?

EXERCISES

Practice

Solve each equation.

12. $\sqrt{x+8} = 5$
13. $\sqrt[3]{y-7} = 4$
14. $\sqrt{8n-5} - 1 = 2$
15. $\sqrt{x+16} = \sqrt{x} + 4$
16. $4\sqrt{3m^2 - 15} = 4$
17. $\sqrt{9u-4} = \sqrt{7u-20}$
18. $\sqrt[3]{6u-5} + 2 = -3$
19. $\sqrt{4m^2 - 3m + 2} - 2m - 5 = 0$
20. $\sqrt{k+9} - \sqrt{k} = \sqrt{3}$
21. $\sqrt{a+21} - 1 = \sqrt{a+12}$
22. $\sqrt{3x+4} - \sqrt{2x-7} = 3$
23. $2\sqrt[3]{7b-1} - 4 = 0$
24. $\sqrt[4]{3t} - 2 = 0$
25. $\sqrt{x+2} - 7 = \sqrt{x+9}$
26. $\sqrt{2x+1} + \sqrt{2x+6} = 5$
27. $\sqrt{3x+10} = \sqrt{x+11} - 1$

28. Consider the equation $\sqrt{3t-14} + t = 6$.
   a. Name any extraneous solutions of the equation.
   b. What is the solution of the equation?

Solve each inequality.

29. $\sqrt{2x-7} \geq 5$
30. $\sqrt{b+4} \leq 6$
31. $\sqrt{a-5} \leq 4$
32. $\sqrt{2x-5} \leq 6$
33. $\sqrt[4]{5y-9} \leq 2$
34. $\sqrt{m+2} \leq \sqrt{3m+4}$

35. What values of c make $\sqrt{2c-5}$ greater than 7?

Applications and Problem Solving

Real World Application

36. **Physics** The time t in seconds that it takes an object at rest to fall a distance of s meters is given by the formula $t = \sqrt{\frac{2s}{g}}$. In this formula, g is the acceleration due to gravity in meters per second squared. On the moon, a rock falls 7.2 meters in 3 seconds.
   a. Write an equation that models the situation.
   b. What is the acceleration due to gravity on the moon?

37. **Critical Thinking** Solve $\sqrt{x-5} = \sqrt[3]{x-3}$.

38. **Driving** After an accident, police can determine how fast a car was traveling before the driver put on his breaks by using the equation $s = \sqrt{30fd}$. In this equation, s represents the speed in miles per hour, f represents the coefficient of friction, and d represents the length of the skid in feet. The coefficient of friction varies with road conditions. Suppose the coefficient of friction is 0.6.

a. Find the speed of a car that skids 25 feet.

b. If you were driving 35 miles per hour, how many feet would it take you to stop?

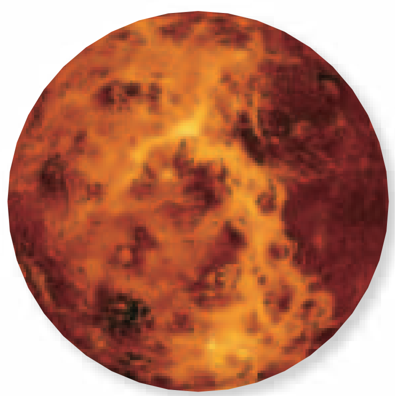
c. If the speed is doubled, will the skid be twice as long? Explain.

39. **Physics** The period of a pendulum (the time required for one back and forth swing) can be determined by the formula $T = 2\pi\sqrt{\frac{\ell}{g}}$. In this formula, T represents the period, ℓ represents the length of the pendulum, and g represents acceleration due to gravity.

a. Determine the period of a 1-meter pendulum on Earth if the acceleration due to gravity at Earth's surface is 9.8 meters per second squared.

b. Suppose the acceleration due to gravity on the surface of Venus is 8.9 meters per second squared. Calculate the period of the pendulum on Venus.

c. How must the length of the pendulum be changed to double the period?



Venus

40. **Astronomy** Johann Kepler (1571–1630) determined the relationship of the time of revolution of two planets and their average distance from the sun. This relationship can be expressed as $\frac{T_a}{T_b} = \sqrt{\left(\frac{r_a}{r_b}\right)^3}$. In this equation, T_a represents the time it takes planet a to orbit the sun, and r_a represents the average distance between planet a and the sun. Likewise, T_b represents the time it takes planet b to orbit the sun, and r_b represents the average distance between planet b and the sun. The average distance between Venus and the sun is 67,200,000 miles, and it takes Venus about 225 days to orbit the sun. If it takes Mars 687 days to orbit the sun, what is its average distance from the sun?

41. **Critical Thinking** For what values of a and b will the equation $\sqrt{2x-9} - a = b$ have no real solution?

42. **Engineering** Engineers are often required to determine the stress on building materials. Tensile stress can be found by using the formula $T = \frac{t+c}{2} + \sqrt{\left(\frac{t-c}{2}\right)^2 + p^2}$. In this formula, T represents the tensile stress, t represents the tension, c represents the compression, and p represents the pounds of pressure per square inch. If the tensile stress is 108 pounds per square inch, the pressure is 50 pounds per square inch, and the compression is -200 pounds per square inch, what is the tension?

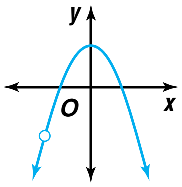
Mixed Review

43. Solve $\frac{a+2}{2a+1} = \frac{a}{3} + \frac{3}{4a+2}$. *(Lesson 4-6)*

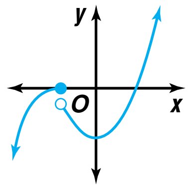
44. List the possible rational roots of $x^4 + 5x^3 + 5x^2 - 5x - 6 = 0$. Then determine the rational roots. *(Lesson 4-4)*

45. Determine whether each graph has *infinite discontinuity, jump discontinuity,* or *point discontinuity,* or is *continuous.* *(Lesson 3-7)*

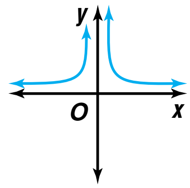
a.



b.



c.



46. Music The frequency of a sound wave is called its pitch. The pitch p of a musical tone and its wavelength w are related by the equation $p = \frac{v}{w}$, where v is the velocity of sound through air. Suppose a sound wave has a velocity of 1056 feet per second. *(Lesson 3-6)*

a. Graph the equation $p = \frac{v}{w}$.

b. What lines are close to the maximum values for the pitch and the wavelength?

c. What happens to the pitch of the tone as the wavelength decreases?

d. If the wavelength is doubled, what happens to the pitch of the tone?

47. Find the product of the matrices $\begin{bmatrix} 4 & -1 & 6 \\ 4 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 3 \\ 2 & -2 \\ 5 & 1 \end{bmatrix}$. *(Lesson 2-3)*

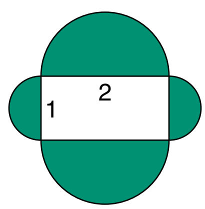
48. Solve the system of equations. *(Lesson 2-2)*

$a + b + c = 6$
$2a - 3b + 4c = 3$
$4a - 8b + 4c = 12$

49. Education The regression equation for a set of data is $y = -3.54x + 7107.7$, where x represents the year and y represents the average number of students assigned to each advisor in a certain business school. Use the equation to predict the number of students assigned to each adviser in the year 2005. *(Lesson 1-6)*

50. Write the slope-intercept form of the equation that is perpendicular to $7y + 4x - 3 = 0$ and passes through the point with coordinates (2, 5). *(Lesson 1-5)*

51. SAT/ACT Practice In the figure at the right, four semicircles are drawn on the four sides of a rectangle. What is the total area of the shaded regions?



A 5π **B** $\frac{5\pi}{2}$ **C** $\frac{5\pi}{4}$

D $\frac{5\pi}{8}$ **E** $\frac{5\pi}{16}$

Extra Practice See p. A33.

CONTENTS

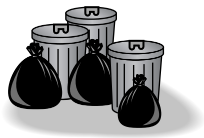
4-8 Modeling Real-World Data with Polynomial Functions

OBJECTIVES

- Write polynomial functions to model real-world data.
- Use polynomial functions to interpret real-world data.

Real World Application

WASTE MANAGEMENT The average daily amount of waste generated by each person in the United States is given below. This includes all wastes such as industrial wastes, demolition wastes, and sewage.



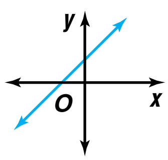
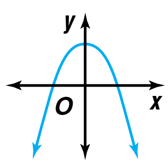
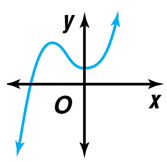
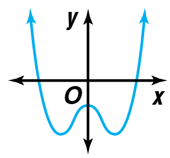
What a Waste!

Year	1980	1985	1990	1991	1992	1993	1994	1995	1996
Pounds of Waste per Person per Day	3.7	3.8	4.5	4.4	4.5	4.5	4.5	4.4	4.3

Source: Franklin Associates, Ltd.

What polynomial function could be used to model these data? *This problem will be solved in Example 3.*

In order to model real world data using polynomial functions, you must be able to identify the general shape of the graph of each type of polynomial function.

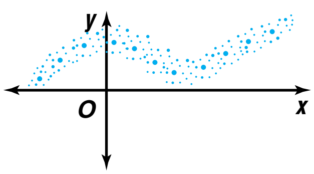
Function	**Linear** $y = ax + b$	**Quadratic** $y = ax^2 + bx + c$	**Cubic** $y = ax^3 + bx^2 + cx + d$	**Quartic** $y = ax^4 + bx^3 + cx^2 + dx + e$
Typical Graph				
Direction Changes	0	1	2	3

Example 1

Determine the type of polynomial function that could be used to represent the data in each scatter plot.

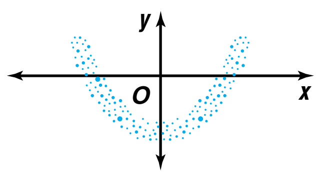
Look Back
You can refer to Lesson 1-6 to review scatter plots.

a.



The scatter plot seems to change direction two times, so a cubic function would best fit the plot.

b.



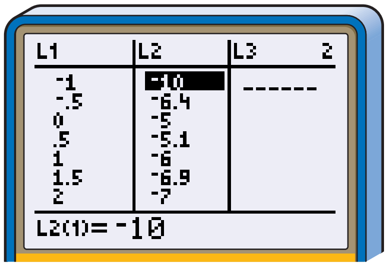
The scatter plot seems to change direction one time, so a quadratic function would best fit the plot.

You can use a graphing calculator to determine a polynomial function that models a set of data.

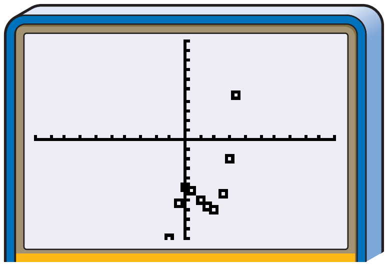
Example 2 **Use a graphing calculator to write a polynomial function to model the set of data.**

x	−1	−0.5	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x)$	−10	−6.4	−5	−5.1	−6	−6.9	−7	−5.6	−2	4.6	15

Clear the statistical memory and input the data.



Using the standard viewing window, graph the data. The scatter plot seems to change direction two times, so a cubic function would best fit the scatter plot.



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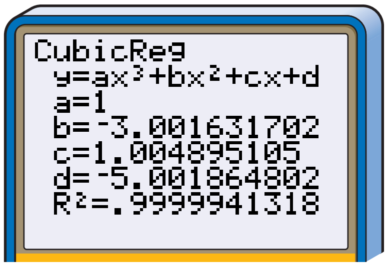
Graphing Calculator Tip

You can change the appearance of the data points in the **STAT PLOT** menu. The data can appear as squares, dots, or + signs.

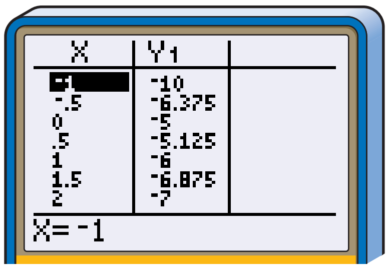
Press STAT and highlight **CALC**. Since the scatter plot seems to be a cubic function, choose **6:CubicReg**.

Press 2nd **[L1]** , 2nd **[L2]** ENTER.



Rounding the coefficients to the nearest whole number, $f(x) = x^3 - 3x^2 + x - 5$ models the data. Since the value of the coefficient of determination r^2 is very close to 1, the function is an excellent fit. However it may not be the best model for the situation.

To check the polynomial function, enter the function in the **Y=** list. Then use the **TABLE** feature for x values from −1 with an interval of 0.5. Compare the values of y with those given in the data. The values are similar, and the polynomial function seems to be a good fit.



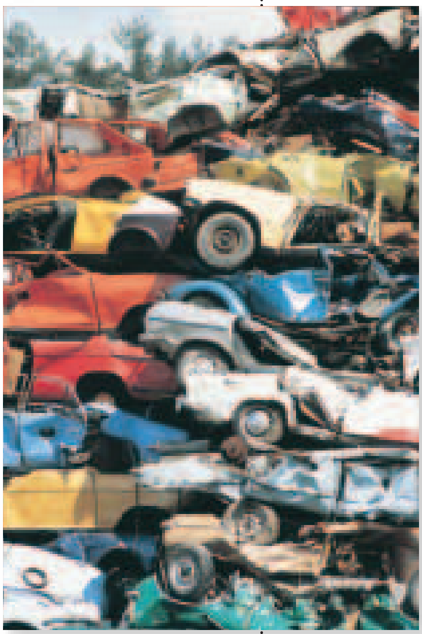
Example 3 Real World Application

WASTE MANAGEMENT **Refer to the application at the beginning of the lesson.**

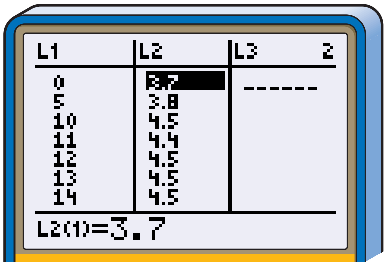
a. What polynomial function could be used to model these data?

b. Use the model to predict the amount of waste produced per day in 2010.

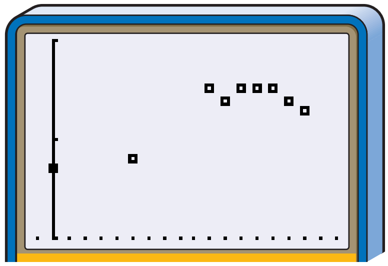
c. Use the model to predict when the amount of waste will drop to 3 pounds per day.



a. Let **L1** be the number of years since 1980 and **L2** be the pounds of solid waste per person per day. Enter this data into the calculator.



Adjust the window to an appropriate setting and graph the statistical data. The data seems to change direction one time, so a quadratic function will fit the scatter plot.



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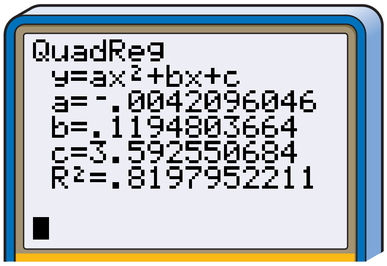
Press STAT, highlight **CALC**, and choose **5:QuadReg**.

Press 2nd [L1] , 2nd [L2] ENTER.



Rounding the coefficients to the nearest thousandth, $f(x) = -0.004x^2 + 0.119x + 3.593$ models the data. Since the value of the coefficient of determination r^2 is close to 1, the model is a good fit.

b. Since 2010 is 30 years after 1980, find $f(30)$.

$$f(x) = -0.004x^2 + 0.119x + 3.593$$

$$f(30) = -0.004(30^2) + 0.119(30) + 3.593$$

$$= 3.563$$

According to the model, each person will produce about 3.6 pounds of waste per day in 2010.

c. To find when the amount of waste will be reduced to 3 pounds per person per day, solve the equation $3 = -0.004x^2 + 0.119x + 3.593$.

$$3 = -0.004x^2 + 0.119x + 3.593$$

$$0 = -0.004x^2 + 0.119x + 0.593 \qquad \textit{Add } -3 \textit{ to each side.}$$

Use the Quadratic Formula to solve for x.

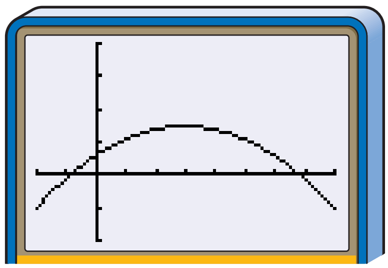
$$x = \frac{-0.119 \pm \sqrt{0.119^2 - 4(-0.004)(0.593)}}{2(-0.004)} \qquad a = -0.004,\ b = 0.119,\ c = 0.593$$

$$x = \frac{-0.119 \pm \sqrt{0.023649}}{-0.008}$$

$$x \approx -4 \text{ or } 34$$

According to the model, 3 pounds per day per person occurs 4 years before 1980 (in 1976) or 34 years after 1980 (in 2014). Since you want to know when the amount of waste will drop to 3 pounds per day, the answer is in 2014.

To check the answer, graph the related function $f(x) = -0.004x^2 + 0.119x + 0.593$. Use the **CALC** menu to determine the zero at the right. The answer of 34 years checks.



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The waste should reduce to 3 pounds per person per day about 34 years after 1980 or in 2014.

Check for Understanding

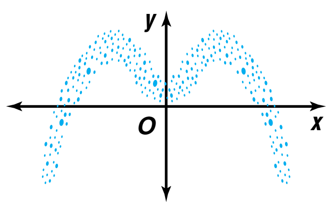
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Draw** an example of a scatter plot that can be represented by each type of function.
   a. quartic b. quadratic c. cubic
2. **Explain** why it is important to recognize the shape of the graph of each type of polynomial function.
3. **List** reasons why the amount of waste per person may vary from the model in Example 3.

Guided Practice

4. Determine the type of polynomial function that could be used to represent the data in the scatter plot.



Graphing Calculator

Use a graphing calculator to write a polynomial function to model each set of data.

5.

x	−3.5	−3	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2
***f*(*x*)**	103	32	−1	−11	−9	−2	3	5	4	4	12	37

6.

x	−3	−2.5	−1.5	−0.5	0	1	2	2.5	3.5
***f*(*x*)**	19	11	−1	−7	−8	−5	4	11	29

7. Population The percent of the United States population living in metropolitan areas has increased since 1950.

Year	1950	1960	1970	1980	1990	1996
Population living in metropolitan areas	56.1%	63%	68.6%	74.8%	74.8%	79.9%

Source: *American Demographics*

a. Write a model that relates the percent as a function of the number of years since 1950.

b. Use the model to predict the percent of the population that will be living in metropolitan areas in 2010.

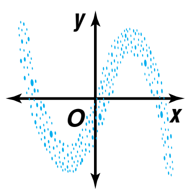
c. Use the model to predict what year will have 85% of the population living in metropolitan areas.

EXERCISES

Practice

Determine the type of polynomial function that could be used to represent the data in scatter plot.

8.



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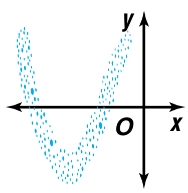
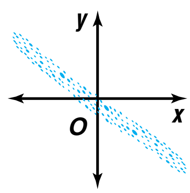


10.



11. What type of polynomial function would be the best model for the set of data?

x	1	2	3	4	5	7	8
***f*(*x*)**	15	7	2	−1	3	10	15

Graphing Calculator

Use a graphing calculator to write a polynomial function to model each set of data.

12.

x	−3	−2	−1	0	1	2	3
***f*(*x*)**	8.75	7.5	6.25	5	3.75	2.5	1.25

13.

x	−2	−1	0	1	2	3
***f*(*x*)**	29	2	−9	−4	17	54

14.

x	−1	−0.5	0	0.5	1	1.5	2	2.5	3	3.5
***f*(*x*)**	13	3	1	2	3	3	1	−1	1	10

15.

x	5	7	8	10	11	12	15	16
$f(x)$	2	5	6	4	−1	−3	5	9

16.

x	30	35	40	45	50	55	60	65	70	75
$f(x)$	52	41	32	44	61	88	72	59	66	93

17.

x	−17	−6	−1	2	8	12	15
$f(x)$	51	29	−6	41	57	37	19

18. Consider the set of data.

x	−2.5	−2	−1.5	−1	−0.5	0	0.5	1	1.5
$f(x)$	23	11	7	6	6	5	3	2	4

a. What quadratic polynomial function models the data?

b. What cubic polynomial function models the data?

c. Which model do you think is more appropriate? Explain.

Applications and Problem Solving

interNET CONNECTION

Data Update
To compare current census data with predictions using your models, visit our website at **www.amc.glencoe.com**

19. **Marketing** The United States Census Bureau has projected the median age of the U.S. population to the year 2080. A fast-food chain wants to target its marketing towards customers that are about the median age.

Year	1900	1930	1960	1990	2020	2050	2080
Median age	22.9	26.5	29.5	33.0	40.2	42.7	43.9

a. Write a model that relates the median age as a function of the number of years since 1900.

b. Use the model to predict what age the fast-food chain should target in the year 2005.

c. Use the model to predict what age the fast-food chain should target in the year 2025.

20. **Critical Thinking** Write a set of data that could be best represented by a cubic polynomial function.

21. **Consumer Credit** The amount of consumer credit as a percent of disposable personal income is given below.

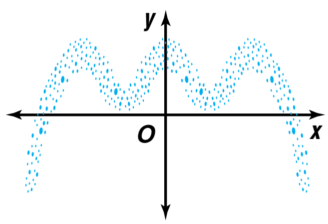
Year	1988	1989	1990	1991	1992	1995	1996	1997
Consumer credit	23%	24%	23%	22%	19%	21%	24%	22%

Source: *The World Almanac and Book of Facts*

a. Write a model that relates the percent of consumer credit as a function of the number of years since 1988.

b. Use the model to estimate the percent of consumer credit in 1994.

22. **Critical Thinking** What type of polynomial function should be used to model the scatter plot? Explain.



23. **Baseball** The attendance at major league baseball games for various years is listed below.

Year	1985	1987	1990	1992	1993	1994	1995	1996
Attendance (in millions)	48	53	56	57	71	50	51	62

Source: *Statistical Abstract of the United States*

a. Write a model that relates the attendance in millions as a function of the number of years since 1985.

b. Use the model to predict when the attendance will reach 71 million again.

c. In 1998, Mark McGwire and Sammy Sosa raced to break the homerun record. That year the attendance reached 70.6 million. Did the attendance for 1998 follow your model? Do you think the race to break the homerun record affected attendance? Why or why not?

24. **Communication** Cellular phones are becoming more popular. The numbers of cellular subscribers are listed in below.

Year	1983	1984	1986	1989	1990	1992	1994	1996	1997
Subscribers (in millions)	0.25	3.0	6.5	3.8	5.3	11.0	24.1	44.0	55.3

Source: *Statistical Abstract of the United States*

a. Write a model that relates the millions of subscribers as a function of the number of years since 1983.

b. According to the model, how many subscribers were there in 1995?

c. Use the model to predict when there will be 150 million subscribers.

Mixed Review

25. Solve $5 - \sqrt{b + 2} = 0$. *(Lesson 4-7)*

26. Solve $\frac{6}{p + 3} + \frac{p}{p - 3} = 1$. *(Lesson 4-6)*

27. Approximate the real zeros of $f(x) = 2x^4 - x^3 + x - 2$ to the nearest tenth. *(Lesson 4-5)*

28. **Agriculture** If Wesley Jackson harvests his apple crop now, the yield will average 120 pounds per tree. Also, he will be able to sell the apples for \$0.48 per pound. However, he knows that if he waits, his yield will increase by about 10 pounds per week, while the selling price will decrease by \$0.03 per pound per week. *(Lesson 3-6)*

a. How many weeks should Mr. Jackson wait in order to maximize the profit?

b. What is the maximum profit?

29. **SAT/ACT Practice** A college graduate goes to work for $\$x$ per week. After several months, the company falls on hard times and gives all the employees a 10% pay cut. A few months later, business picks up and the company gives all employees a 10% raise. What is the college graduate's new salary?

A $\$0.90x$ **B** $\$0.99x$ **C** $\$x$

D $\$1.01x$ **E** $\$1.11x$

Extra Practice See p. A33.

GRAPHING CALCULATOR EXPLORATION

4-8B Fitting a Polynomial Function to a Set of Points

An Extension of Lesson 4-8

OBJECTIVE

- Use matrices to find a polynomial function whose graph passes through a given set of points in the coordinate plane.

The **STAT CALC** menu of a graphing calculator has built-in programs that allow you to find a polynomial function whose graph passes through as many as five arbitrarily selected points in the coordinate plane if no two points are on the same vertical line. For example, for a set of four points, you can enter the x-coordinates in list **L1** and the y-coordinates in list **L2**. Then, select **6: CubicReg** from the **STAT CALC** menu to display the coefficients for a cubic function whose graph passes through the four points.

For any finite set of points where no two points are on the same vertical line, you can perform calculations to find a polynomial function whose graph passes exactly through those points. For a set of n points (where $n \geq 1$), the polynomial function will have a degree of $n - 1$.

Example

Find a polynomial function whose graph passes through points $A(-2, 69)$, $B(-1.5, 19.75)$, $C(-1, 7)$, $D(2, 25)$, $E(2.5, 42)$, and $F(3, 49)$.

The equation of a polynomial function through the six points is of the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. The coordinates of each point must satisfy the equation for the function. If you substitute the x- and y-coordinates of the points into the equation, you obtain a system of six linear equations in six variables.

A: $(-2)^5a + (-2)^4b + (-2)^3c + (-2)^2d + (-2)e + 1f = 69$

B: $(-1.5)^5a + (-1.5)^4b + (-1.5)^3c + (-1.5)^2d + (-1.5)e + 1f = 19.75$

C: $(-1)^5a + (-1)^4b + (-1)^3c + (-1)^2d + (-1)e + 1f = 7$

D: $(2)^5a + (2)^4b + (2)^3c + (2)^2d + (2)e + 1f = 25$

E: $(2.5)^5a + (2.5)^4b + (2.5)^3c + (2.5)^2d + (2.5)e + 1f = 42$

F: $(3)^5a + (3)^4b + (3)^3c + (3)^2d + (3)e + 1f = 49$

Perform the following steps on a graphing calculator.

Step 1 Press STAT, select **Edit**, and enter three lists. In list **L1**, enter the x-coordinates of the six points in the order they appear as coefficients of e. In list **L2**, enter the y-coordinates in the order they appear after the equal signs. In list **L3**, enter six 1s.

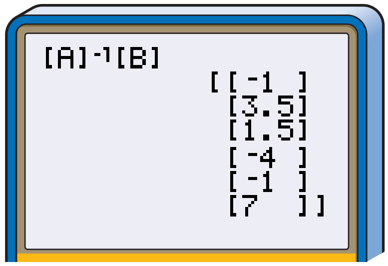
Step 2 Notice that the columns of coefficients of a, b, c, and d are powers of the column of numbers that you entered for **L1**. To solve the linear system, you need to enter the columns of coefficients as columns of a 6×6 matrix. So, press the MATRX key, and set the dimensions of matrix **[A]** to 6×6. Next use **9: List▶matr** from the **MATRX MATH** menu to put the columns of coefficients into matrix **[A]**. On the home screen, enter the following expression.

List▶matr (L1^5, L1^4, L1^3, L1^2, L1, L3, [A])

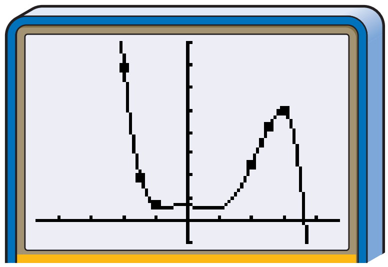
(continued on the next page)

Step 3 Put the column of y-coordinates (the numbers after the equals signs) into matrix **[B]**. Enter **List▶matr (L2, [B])** and press ENTER.

Step 4 Enter **[A]$^{-1}$[B]** and press ENTER. The calculator will display the solution of the linear system as a column matrix. The elements of this matrix are the coefficients for the polynomial being sought. The top element is the coefficient of x^5, the second element is the coefficient of x^4, and so on. The function is $f(x) = -x^5 + 3.5x^4 + 1.5x^3 - 4x^2 - x + 7$.



Step 5 Enter the polynomial function on the **Y=** list and display its graph. If you use **STAT PLOT** and display the data for the coordinates in **L1** and **L2** as a scatter plot, you will have visual confirmation that the graph of the polynomial function passes through the given points.



[5, 5] scl:1 by [−10, 80] scl:10

For problems of this kind, the coefficients in the solution matrix for the linear system are often decimals. You can enter the equation for the function on the **Y=** list efficiently and accurately by first changing the column matrix for the solution into a list. On the home screen, enter **Matr▶list ([A]$^{-1}$[B], 1, L4)**. Then go to the **Y=** list and enter the equation **Y1 = L4(1)X^5 + L4(2)X^4 + L4(3)X^3 + L4(4)X^2 + L4(5)X + L4(6)**.

TRY THESE

1. Use the steps given above to find a cubic polynomial function whose graph passes through points with coordinates $(-2, -23)$, $(-1, 21)$, $(0, 15)$, and $(1, 1)$.
2. Use **6: CubicReg** on the **STAT CALC** menu to fit a cubic model for the points in Exercise 1. Is the resulting cubic function the same as the one you found in Exercise 1?
3. Find a polynomial function whose graph passes through points with coordinates $(-3, 116)$, $(-2, 384)$, $(-0.4, -5.888)$, $(0, -4)$, $(1, 36)$, $(2, 256)$, and $(2.6, 34.1056)$.

WHAT DO YOU THINK?

4. How many polynomial functions have graphs that pass through a given set of points, no two of which are on the same vertical line? Explain your thinking.
5. In Step 2, why was it necessary to define the list **L3**? Would it have been possible to use **L1^0** instead of **L3** in the **List▶matr** expression used to define matrix **[A]**? Explain your thinking.

VOCABULARY

completing the square (p. 213)
complex number (p. 206)
conjugate (p. 216)
degree (p. 206)
depressed polynomial (p. 224)
Descartes' Rule of Signs (p. 231)
discriminant (p. 215)
extraneous solution (p. 251)
Factor Theorem (p. 224)
Fundamental Theorem of Algebra (p. 207)
imaginary number (p. 206)
Integral Root Theorem (p. 230)
leading coefficient (p. 206)
Location Principle (p. 236)
lower bound (p. 238)
Lower Bound Theorem (p. 238)
partial fractions (p. 244)
polynomial equation (p. 206)
polynomial function (p. 206)
polynomial in one variable (p. 205)
pure imaginary number (p. 206)
Quadratic Formula (p. 215)
radical equation (p. 251)
radical inequality (p. 253)
rational equation (p. 243)
rational inequality (p. 245)
Rational Root Theorem (p. 229)
Remainder Theorem (p. 222)
root (p. 206)
synthetic division (p. 223)
upper bound (p. 238)
Upper Bound Theorem (p. 238)
zero (p. 206)

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term from the list to complete each sentence.

1. The ___?___ can be used to solve any quadratic equation.
2. The ___?___ states that if the leading coefficient of a polynomial equation a_0 has a value of 1, then any rational root must be factors of a_n.
3. For a rational equation, any possible solution that results with a ___?___ in the denominator must be excluded from the list of solutions.
4. The ___?___ states that the binomial $x - r$ is a factor of the polynomial $P(x)$ if and only if $P(r) = 0$.
5. Descartes' Rule of Signs can be used to determine the possible number of positive real zeros a ___?___ has.
6. A(n) ___?___ of the zeros of $P(x)$ can be found by determining an upper bound for the zeros of $P(-x)$.
7. ___?___ solutions do not satisfy the original equation.
8. Since the x-axis only represents real numbers, ___?___ of a polynomial function cannot be determined by using a graph.
9. The Fundamental Theorem of Algebra states that every polynomial equation with degree greater than zero has at least one root in the set of ___?___
10. A ___?___ is a special polynomial equation with a degree of two.

complex numbers
complex roots
discriminant
extraneous
Factor Theorem
Integral Root Theorem
lower bound
polynomial function
quadratic equation
Quadratic Formula
radical equation
synthetic division
zero

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 4-1 Determine roots of polynomial equations.

- Determine whether 2 is a root of $x^4 - 3x^3 - x^2 - x = 0$. Explain.

  $f(2) = 2^4 - 3(2^3) - 2^2 - 2$

  $f(2) = 16 - 24 - 4 - 2$ or -14

  Since $f(2) \neq 0$, 2 is not a root of $x^4 - 3x^3 - x^2 - x = 0$.

REVIEW EXERCISES

Determine whether each number is a root of $a^3 - 3a^2 - 3a - 4 = 0$. Explain.

11. 0 **12.** 4 **13.** -2

14. Is -3 a root of $t^4 - 2t^2 - 3t + 1 = 0$?

15. State the number of complex roots of the equation $x^3 + 2x^2 - 3x = 0$. Then find the roots and graph the related function.

Lesson 4-2 Solve quadratic equations.

- Find the discriminant of $3x^2 - 2x - 5 = 0$ and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.

  The value of the discriminant, $b^2 - 4ac$, is $(-2)^2 - 4(3)(-5)$ or 64. Since the value of the discriminant is greater than zero, there are two distinct real roots.

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  $$x = \frac{-(-2) \pm \sqrt{64}}{2(3)}$$

  $$x = \frac{2 \pm 8}{6}$$

  $$x = -1 \text{ or } \frac{5}{3}$$

Find the discriminant of each equation and describe the nature of the roots of the equation. Then solve the equation by using the Quadratic Formula.

16. $2x^2 - 7x - 4 = 0$

17. $3m^2 - 10m + 5 = 0$

18. $x^2 - x + 6 = 0$

19. $-2y^2 + 3y + 8 = 0$

20. $a^2 + 4a + 4 = 0$

21. $5r^2 - r + 10 = 0$

Lesson 4-3 Find the factors of polynomials using the Remainder and Factor Theorems.

- Use the Remainder Theorem to find the remainder when $(x^3 + 2x^2 - 5x - 9)$ is divided by $(x + 3)$. State whether the binomial is a factor of the polynomial.

  $f(x) = x^3 + 2x^2 - 5x - 9$

  $f(-3) = (-3)^3 + 2(-3)^2 - 5(-3) - 9$

  $= -27 + 18 + 15 - 9$ or -3

  Since $f(-3) = -3$, the remainder is -3. So the binomial $x + 3$ is not a factor of the polynomial by the Remainder Theorem.

Use the Remainder Theorem to find the remainder for each division. State whether the binomial is a factor of the polynomial.

22. $(x^3 - x^2 - 10x - 8) \div (x + 2)$

23. $(2x^3 - 5x^2 + 7x + 1) \div (x - 5)$

24. $(4x^3 - 7x + 1) \div \left(x + \frac{1}{2}\right)$

25. $(x^4 - 10x^2 + 9) \div (x - 3)$

OBJECTIVES AND EXAMPLES

Lesson 4-4 Identify all possible rational roots of a polynomial equation by using the Rational Root Theorem.

- List the possible rational roots of $4x^3 - x^2 - x - 5 = 0$. Then determine the rational roots.

  If $\frac{p}{q}$ is a root of the equation, then p is a factor of 5 and q is a factor of 4.
  possible values of p: $\pm 1, \pm 5$
  possible values of q: $\pm 1, \pm 2, \pm 4$

  possible rational roots: $\pm 1, \pm 5, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{5}{2}, \pm \frac{5}{4}$

  Graphing and substitution show a zero at $\frac{5}{4}$.

REVIEW EXERCISES

List the possible rational roots of each equation. Then determine the rational roots.

26. $x^3 - 2x^2 - x + 2 = 0$
27. $x^4 - x^2 + x - 1 = 0$
28. $2x^3 - 2x^2 - 2x - 4 = 0$
29. $2x^4 + 3x^3 - 6x^2 - 11x - 3 = 0$
30. $x^5 - 7x^3 + x^2 + 12x - 4 = 0$
31. $3x^3 + 7x^2 - 2x - 8 = 0$
32. $4x^3 + x^2 + 8x + 2 = 0$
33. $x^4 + 4x^2 - 5 = 0$

Lesson 4-4 Determine the number and type of real roots a polynomial function has.

- For $f(x) = 3x^4 - 9x^3 + 4x - 6$, there are three sign changes. So there are three or one positive real zeros.

  For $f(-x) = 3x^4 + 9x^3 - 4x - 6$, there is one sign change. So there is one negative real zero.

Find the number of possible positive real zeros and the number of possible negative real zeros for each function. Then determine the rational zeros.

34. $f(x) = x^3 - x^2 - 34x - 56$
35. $f(x) = 2x^3 - 11x^2 + 12x + 9$
36. $f(x) = x^4 - 13x^2 + 36$

Lesson 4-5 Approximate the real zeros of a polynomial function.

- Determine between which consecutive integers the real zeros of $f(x) = x^3 + 4x^2 + x - 2$ are located.
  Use synthetic division.

r	1	4	1	−2	
−4	1	0	1	−6	*change*
−3	1	1	2	4	*in signs*
−2	1	2	−3	4	
−1	1	3	−2	0	*zero*
0	1	4	1	−2	*change*
1	1	5	6	4	*in signs*

  One zero is −1. Another is located between −4 and −3. The other is between 0 and 1.

Determine between which consecutive integers the real zeros of each function are located.

37. $g(x) = 3x^3 + 1$
38. $f(x) = x^2 - 4x + 2$
39. $g(x) = x^2 - 3x - 3$
40. $f(x) = x^3 - x^2 + 1$
41. $g(x) = 4x^3 + x^2 - 11x + 3$
42. $f(x) = -9x^3 + 25x^2 - 24x + 6$
43. Approximate the real zeros of $f(x) = 2x^3 + 9x^2 - 12x - 40$ to the nearest tenth.

OBJECTIVES AND EXAMPLES

Lesson 4-6 Solve rational equations and inequalities.

Solve $\frac{1}{9} + \frac{1}{2a} = \frac{1}{a^2}$.

$$\frac{1}{9} + \frac{1}{2a} = \frac{1}{a^2}$$
$$\left(\frac{1}{9} + \frac{1}{2a}\right)(18a^2) = \left(\frac{1}{a^2}\right)(18a^2)$$
$$2a^2 + 9a = 18$$
$$2a^2 + 9a - 18 = 0$$
$$(2a - 3)(a + 6) = 0$$
$$a = \frac{3}{2} \text{ or } -6$$

REVIEW EXERCISES

Solve each equation or inequality.

44. $n - \frac{6}{n} + 5 = 0$

45. $\frac{1}{x} = \frac{x+3}{2x^2}$

46. $\frac{5}{6} = \frac{2m}{2m+2} - \frac{1}{3m-3}$

47. $\frac{3}{y} - 2 < \frac{5}{y}$

48. $\frac{2}{x+1} < 1 - \frac{1}{x-1}$

Lesson 4-7 Solve radical equations and inequalities.

Solve $9 + \sqrt{x-1} = 1$.

$$9 + \sqrt{x-1} = 1$$
$$\sqrt{x-1} = -8$$
$$x - 1 = 64$$
$$x = 65$$

Solve each equation or inequality.

49. $5 - \sqrt{x+2} = 0$

50. $\sqrt[3]{4a-1} + 8 = 5$

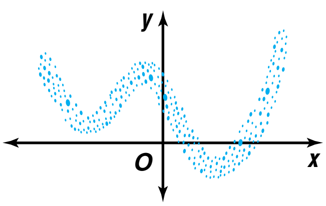
51. $3 + \sqrt{x+8} = \sqrt{x+35}$

52. $\sqrt{x-5} < 7$

53. $4 + \sqrt{2a+7} \geq 6$

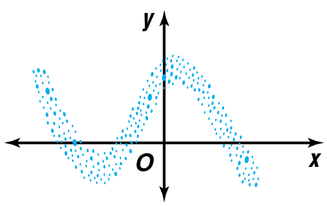
Lesson 4-8 Write polynomial functions to model real-world data.

Determine the type of polynomial function that would best fit the data in the scatter plot.



The scatter plot seems to change direction three times. So a quartic function would best fit the scatter plot.

54. Determine the type of polynomial function that would best fit the scatter plot.



55. Write a polynomial function to model the data.

x	−3	−1	0	2	4	7
***f*(*x*)**	24	6	3	9	31	94

APPLICATIONS AND PROBLEM SOLVING

56. **Entertainment** The scenery for a new children's show has a playhouse with a painted window. A special gloss paint covers the area of the window to make them look like glass. If the gloss only covers 315 square inches and the window must be 6 inches taller than it is wide, how large should the scenery painters make the window? *(Lesson 4-1)*

57. **Gardening** The length of a rectangular flower garden is 6 feet more than its width. A walkway 3 feet wide surrounds the outside of the garden. The total area of the walkway itself is 288 square feet. Find the dimensions of the garden. *(Lesson 4-2)*

58. **Medicine** Doctors can measure cardiac output in potential heart attack patients by monitoring the concentration of dye after a known amount in injected in a vein near the heart. In a normal heart, the concentration of the dye is given by $g(x) = -0.006x^4 + 0.140x^3 - 0.053x^2 + 1.79x$, where x is the time in seconds. *(Lesson 4-4)*

a. Graph $g(x)$

b. Find all the zeros of this function.

59. **Physics** The formula $T = 2\pi\sqrt{\frac{\ell}{g}}$ is used to find the period T of a oscillating pendulum. In this formula, ℓ is the length of the pendulum, and g is acceleration due to gravity. Acceleration due to gravity is 9.8 meters per second squared. If a pendulum has an oscillation period of 1.6 seconds, determine the length of the pendulum. *(Lesson 4-7)*

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. Write a rational equation that has at least two solutions, one which is 2. Solve your equation.

2. a. Write a radical equation that has solutions of 3 and 6, one of which is extraneous.

b. Solve your equation. Identify the extraneous solution and explain why it is extraneous.

3. a. Write a set of data that is best represented by a cubic equation.

b. Write a polynomial function to model the set of data.

c. Approximate the real zeros of the polynomial function to the nearest tenth.

PORTFOLIO

Explain how you can use the leading coefficient and the degree of a polynomial equation to determine the number of possible roots of the equation.

Unit 1 interNET Project

TELECOMMUNICATION

The Pen is Mightier than the Sword!

- Gather all materials obtained from your research for the mini-projects in Chapters 1, 2, and 3. Decide what types of software would help you to prepare a presentation.
- Research websites that offer downloads of software including work processing, graphics, spreadsheet, and presentation software. Determine whether the software is a demonstration version or free shareware. Select at least two different programs for each of the four categories listed above.
- Prepare a presentation of your Unit 1 project using the software that you found. Be sure that you include graphs and maps in the presentation.

Additional Assessment See p. A59 for Chapter 4 practice test.

THE PRINCETON REVIEW

TEST-TAKING TIP

- Draw diagrams.
- Locate points on the grid or number line.
- Eliminate any choices that are clearly incorrect.

Coordinate Geometry Problems

The ACT test usually includes several coordinate geometry problems. You'll need to know and apply these formulas for points (x_1, y_1) and (x_2, y_2):

Midpoint	Distance	Slope
$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$	$\frac{y_2 - y_1}{x_2 - x_1}$

The SAT test includes problems that involve coordinate points. But they aren't easy!

ACT EXAMPLE

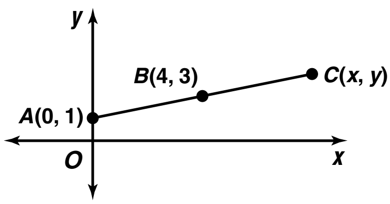
1. Point $B(4,3)$ is the midpoint of line segment AC. If point A has coordinates (0,1), then what are the coordinates of point C?

A $(-4, -1)$ **B** (4, 1) **C** (4, 4)

D (8, 5) **E** (8, 9)

HINT Draw a diagram. You may be able to solve the problem without calculations.

Solution Draw a diagram showing the known quantities and the unknown point C.



Since C lies to the right of B and the x-coordinate of A is not 4, any points with an x-coordinate of 4 or less can be eliminated. So eliminate choices A, B, and C.

Use the Midpoint Formula. Consider the x-coordinates. Write an equation in x.

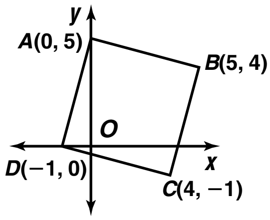
$$\frac{0 + x}{2} = 4$$
$$x = 8$$

Do the same for y.
$$\frac{1 + y}{2} = 3$$
$$y = 5$$

The coordinates of C are (8, 5) The answer is choice **D.**

SAT EXAMPLE

2. What is the area of square $ABCD$ in square units?



A 25 **B** $18\sqrt{2}$ **C** 26

D $25 + \sqrt{2}$ **E** 36

HINT Estimate the answer to eliminate impossible choices and to check your calculations.

Solution First estimate the area. Since the square's side is a little more than 5, the area is a little more than 25. Eliminate choices A and E.

To find the area, find the measure of a side and square it. Choose side AD, because points A and D have simple coordinates. Use the Distance Formula.

$$(\overline{AD})^2 = (-1 - 0)^2 + (0 - 5)^2$$
$$= (-1)^2 + (-5)^2$$
$$= 1 + 25 \text{ or } 26$$

The answer is choice **C.**

Alternate Solution You could also use the Pythagorean Theorem. Draw right triangle DOA, with the right angle at O, the origin. Then DO is 1, OA is 5, and DA is $\sqrt{26}$. So the area is 26 square units.

CONTENTS

SAT AND ACT PRACTICE

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. What is the length of the line segment whose endpoints are represented on the coordinate axis by points at $(-2, 1)$ and $(1, -3)$?

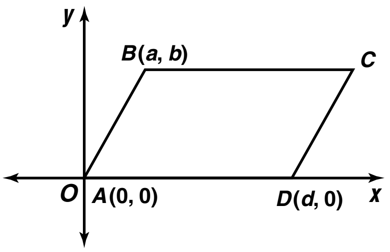
A 3 **B** 4 **C** 5

D 6 **E** 7

2. $\left(\frac{4}{5} \times 3\right)\left(\frac{3}{4} \times 5\right)\left(\frac{5}{3} \times 4\right) =$

A 1 **B** 3 **C** 6 **D** 20 **E** 60

3. In the figure below, *ABCD* is a parallelogram. What are the coordinates of point C?



A (x, y) **B** $(d + a, y)$

C $(d - a, b)$ **D** $(d + x, b)$

E $(d + a, b)$

4. A rectangular garden is surrounded by a 60-foot long fence. One side of the garden is 6 feet longer than the other. Which equation could be used to find s, the shorter side, of the garden?

A $8s + s = 60$

B $4s = 60 + 12$

C $s(s + 6) = 60$

D $2(s - 6) + 2s = 60$

E $2(s + 6) + 2s = 60$

5. What is the slope of a line perpendicular to the line represented by the equation $3x - 6y = 12$?

A -2 **B** $\frac{-1}{2}$ **C** $\frac{1}{3}$

D $\frac{1}{2}$ **E** 2

6. If x is an integer, which of the following could be x^3?

A 2.7×10^{11}

B 2.7×10^{12}

C 2.7×10^{13}

D 2.7×10^{14}

E 2.7×10^{15}

7. If $0x + 5y = 14$ and $4x - y = 2$, then what is the value of $6x + 6y$

A 2 **B** 7 **C** 12 **D** 16 **E** 24

8. What is the midpoint of the line segment whose endpoints are represented on the coordinate grid by points at $(3, 5)$ and $(-4, 3)$?

A $(-2, -5)$ **B** $\left(-\frac{1}{2}, 4\right)$ **C** $(1, 8)$

D $\left(4, \frac{1}{2}\right)$ **E** $(3, 3)$

9. If $ax \neq 0$, which quantity must be non-negative?

A $x^2 - 2ax + a^2$
B $-2ax$
C $2ax$
D $x^2 - a^2$
E $a^2 - x^2$

10. **Grid-In** Points E, F, G, and H lie on a line in that order. If $EG = \frac{5}{3}EF$ and $HF = 5FG$, then what is $\frac{EF}{HG}$?

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