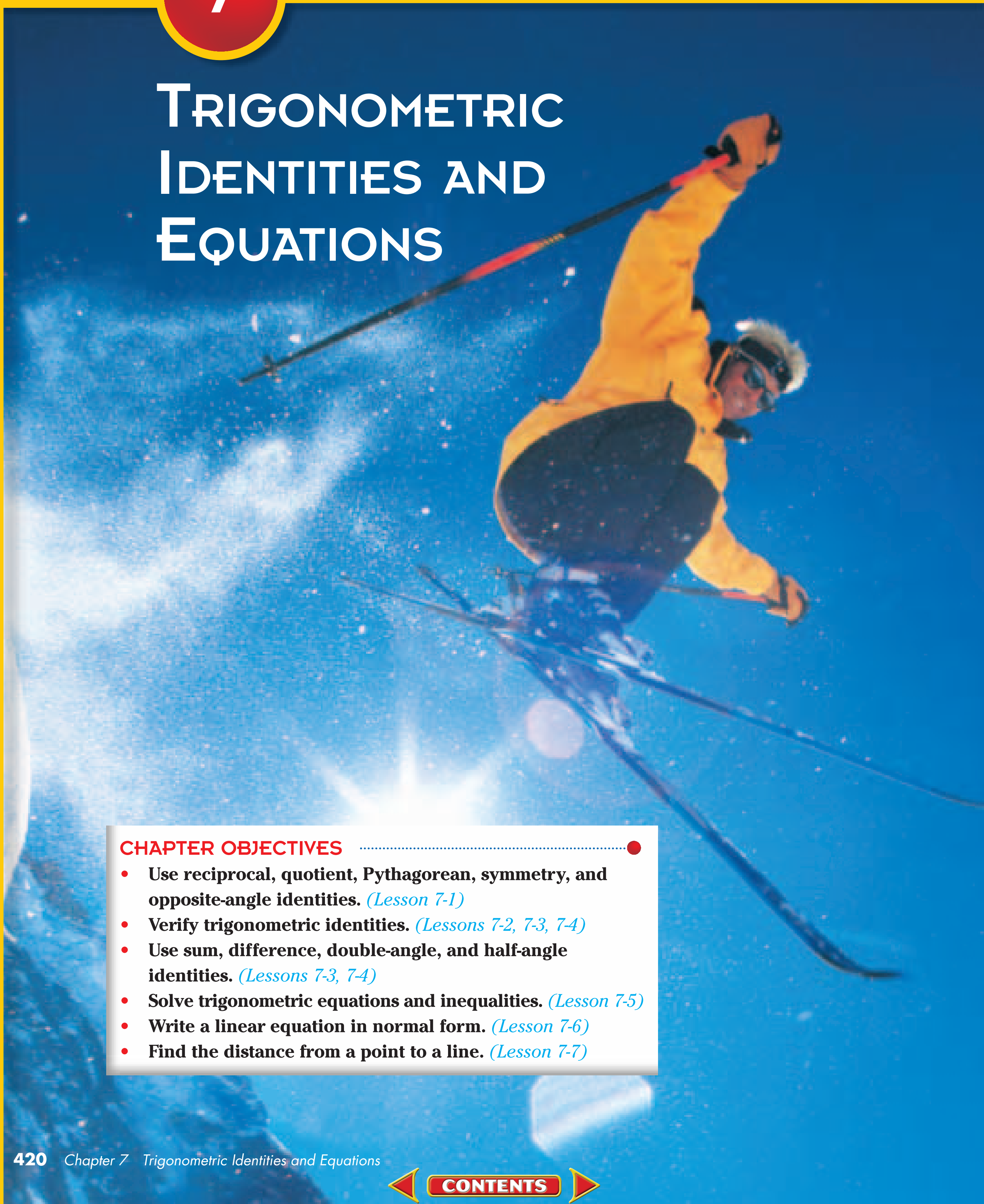
Trigonometric Identities and Equations



CHAPTER OBJECTIVES

- **Use reciprocal, quotient, Pythagorean, symmetry, and opposite-angle identities.** *(Lesson 7-1)*
- **Verify trigonometric identities.** *(Lessons 7-2, 7-3, 7-4)*
- **Use sum, difference, double-angle, and half-angle identities.** *(Lessons 7-3, 7-4)*
- **Solve trigonometric equations and inequalities.** *(Lesson 7-5)*
- **Write a linear equation in normal form.** *(Lesson 7-6)*
- **Find the distance from a point to a line.** *(Lesson 7-7)*

7-1

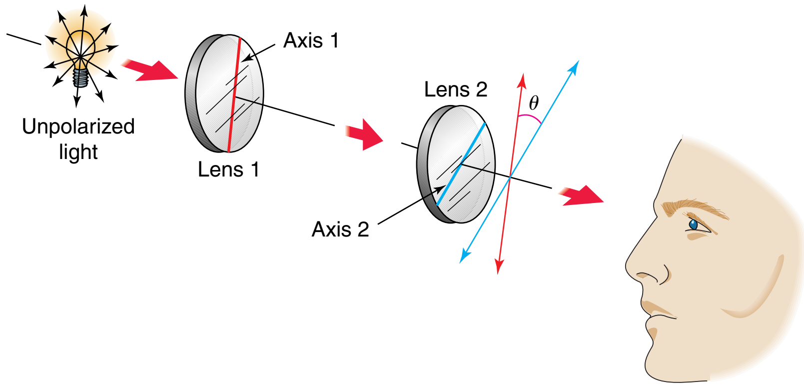
Basic Trigonometric Identities

OBJECTIVE

- Identify and use reciprocal identities, quotient identities, Pythagorean identities, symmetry identities, and opposite-angle identities.

Real World Application

OPTICS Many sunglasses have polarized lenses that reduce the intensity of light. When unpolarized light passes through a polarized lens, the intensity of the light is cut in half. If the light then passes through another polarized lens with its axis at an angle of θ to the first, the intensity of the light is again diminished.



The intensity of the emerging light can be found by using the formula $I = I_0 - \frac{I_0}{\csc^2 \theta}$, where I_0 is the intensity of the light incoming to the second polarized lens, I is the intensity of the emerging light, and θ is the angle between the axes of polarization. Simplify this expression and determine the intensity of light emerging from a polarized lens with its axis at a 30° angle to the original. *This problem will be solved in Example 5.*

In algebra, variables and constants usually represent real numbers. The values of trigonometric functions are also real numbers. Therefore, the language and operations of algebra also apply to trigonometry. Algebraic expressions involve the operations of addition, subtraction, multiplication, division, and exponentiation. These operations are used to form trigonometric expressions. Each expression below is a trigonometric expression.

$$\cos x - x \qquad \sin^2 a + \cos^2 a \qquad \frac{1 - \sec A}{\tan A}$$

A statement of equality between two expressions that is true for *all* values of the variable(s) for which the expressions are defined is called an **identity.** For example, $x^2 - y^2 = (x - y)(x + y)$ is an algebraic identity. An identity involving trigonometric expressions is called a **trigonometric identity.**

If you can show that a specific value of the variable in an equation makes the equation false, then you have produced a *counterexample.* It only takes one counterexample to prove that an equation is not an identity.

Example 1 **Prove that $\sin x \cos x = \tan x$ is *not* a trigonometric identity by producing a counterexample.**

Suppose $x = \frac{\pi}{4}$.

$\sin x \cos x \stackrel{?}{=} \tan x$

$\sin \frac{\pi}{4} \cos \frac{\pi}{4} \stackrel{?}{=} \tan \frac{\pi}{4}$ *Replace x with $\frac{\pi}{4}$.*

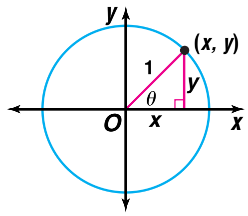
$\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \stackrel{?}{=} 1$

$\frac{1}{2} \neq 1$

Since evaluating each side of the equation for the same value of x produces an inequality, the equation is not an identity.

Although producing a counterexample can show that an equation is not an identity, proving that an equation is an identity generally takes more work. Proving that an equation is an identity requires showing that the equality holds for *all* values of the variable where each expression is defined. Several fundamental trigonometric identities can be verified using geometry.

Recall from Lesson 5-3 that the trigonometric functions can be defined using the unit circle. From the unit circle, $\sin \theta = \frac{y}{1}$, or y and $\csc \theta = \frac{1}{y}$. That is, $\sin \theta = \frac{1}{\csc \theta}$. Identities derived in this manner are called **reciprocal identities.**



Reciprocal Identities

The following trigonometric identities hold for all values of θ where each expression is defined.

$$\sin \theta = \frac{1}{\csc \theta} \qquad \cos \theta = \frac{1}{\sec \theta}$$
$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta}$$
$$\tan \theta = \frac{1}{\cot \theta} \qquad \cot \theta = \frac{1}{\tan \theta}$$

Returning to the unit circle, we can say that $\frac{\sin \theta}{\cos \theta} = \frac{y}{x} = \tan \theta$. This is an example of a **quotient identity.**

Quotient Identities

The following trigonometric identities hold for all values of θ where each expression is defined.

$$\frac{\sin \theta}{\cos \theta} = \tan \theta \qquad \frac{\cos \theta}{\sin \theta} = \cot \theta$$

Since the triangle in the unit circle on the previous page is a right triangle, we may apply the Pythagorean Theorem: $y^2 + x^2 = 1^2$, or $\sin^2 \theta + \cos^2 \theta = 1$. Other identities can be derived from this one.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$ *Divide each side by $\cos^2 \theta$.*

$$\tan^2 \theta + 1 = \sec^2 \theta$$ *Quotient and reciprocal identities*

Likewise, the identity $1 + \cot^2 \theta = \csc^2 \theta$ can be derived by dividing each side of the equation $\sin^2 \theta + \cos^2 \theta = 1$ by $\sin^2 \theta$. These are the **Pythagorean identities.**

Pythagorean Identities

The following trigonometric identities hold for all values of θ where each expression is defined.

$\sin^2 \theta + \cos^2 \theta = 1$ $\qquad$ $\tan^2 \theta + 1 = \sec^2 \theta$ $\qquad$ $1 + \cot^2 \theta = \csc^2 \theta$

You can use the identities to help find the values of trigonometric functions.

Example 2 **Use the given information to find the trigonometric value.**

a. If $\sec \theta = \frac{3}{2}$, find $\cos \theta$.

$$\cos \theta = \frac{1}{\sec \theta}$$ *Choose an identity that involves $\cos \theta$ and $\sec \theta$.*

$$= \frac{1}{\frac{3}{2}} \text{ or } \frac{2}{3}$$ *Substitute $\frac{3}{2}$ for $\sec \theta$ and evaluate.*

b. If $\csc \theta = \frac{4}{3}$, find $\tan \theta$.

Since there are no identities relating $\csc \theta$ and $\tan \theta$, we must use two identities, one relating $\csc \theta$ and $\cot \theta$ and another relating $\cot \theta$ and $\tan \theta$.

$$\csc^2 \theta = 1 + \cot^2 \theta$$ *Pythagorean identity*

$$\left(\frac{4}{3}\right)^2 = 1 + \cot^2 \theta$$ *Substitute $\frac{4}{3}$ for $\csc \theta$.*

$$\frac{16}{9} = 1 + \cot^2 \theta$$

$$\frac{7}{9} = \cot^2 \theta$$

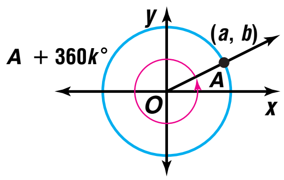
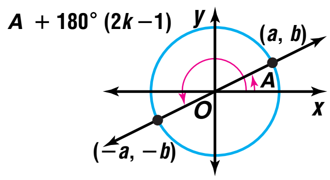
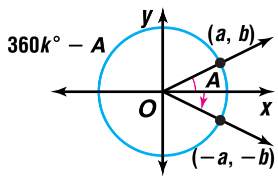
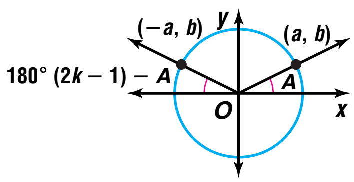
$$\pm\frac{\sqrt{7}}{3} = \cot \theta$$ *Take the square root of each side.*

Now find $\tan \theta$.

$$\tan \theta = \frac{1}{\cot \theta}$$ *Reciprocal identity*

$$= \pm\frac{3\sqrt{7}}{7}, \text{ or about } \pm 1.134$$

To determine the sign of a function value, you need to know the quadrant in which the angle terminates. The signs of function values in different quadrants are related according to the symmetries of the unit circle. Since we can determine the values of tan A, cot A, sec A, and csc A in terms of sin A and/or cos A with the reciprocal and quotient identities, we only need to investigate sin A and cos A.

Case	Relationship between angles *A* and *B*	Diagram	Conclusion
1	The angles differ by a multiple of 360°. $B - A = 360k°$ or $B = A + 360\ k°$		Since A and $A + 360k°$ are coterminal, they share the same value of sine and cosine.
2	The angles differ by an odd multiple of 180°. $B - A = 180°(2k - 1)$ or $B = A + 180°(2k - 1)$		Since A and $A + 180°(2k - 1)$ have terminal sides in diagonally opposite quadrants, the values of both sine and cosine change sign.
3	The sum of the angles is a multiple of 360°. $A + B = 360k°$ or $B = 360k° - A$		Since A and $360k° - A$ lie in vertically adjacent quadrants, the sine values are opposite but the cosine values are the same.
4	The sum of the angles is an odd multiple of 180°. $A + B = 180°\ (2k - 1)$ or $B = 180°(2k - 1) - A$		Since A and $180°(2k - 1) - A$ lie in horizontally adjacent quadrants, the sine values are the same but the cosine values are opposite.

These general rules for sine and cosine are called **symmetry identities.**

Symmetry Identities	The following trigonometric identities hold for any integer k and all values of A.	
Case 1:	$\sin\ (A + 360k°) = \sin A$	$\cos\ (A + 360k°) = \cos A$
Case 2:	$\sin\ (A + 180°(2k - 1)) = -\sin A$	$\cos\ (A + 180°(2k - 1)) = -\cos A$
Case 3:	$\sin\ (360k° - A) = -\sin A$	$\cos\ (360k° - A) = \cos A$
Case 4:	$\sin\ (180°(2k - 1) - A) = \sin A$	$\cos\ (180°(2k - 1) - A) = -\cos A$

To use the symmetry identities with radian measure, replace 180° with π and 360° with 2π.

CONTENTS

Example 3 **Express each value as a trigonometric function of an angle in Quadrant I.**

a. sin 600°

Relate 600° to an angle in Quadrant I.

$600° = 60° + 3(180°)$ *600° and 60° differ by an odd multiple of 180°.*

$\sin 600° = \sin (60° + 3(180°))$ *Case 2, with A = 60° and k = 2*

$= -\sin 60°$

b. $\sin \frac{19\pi}{4}$

The sum of $\frac{19\pi}{4}$ and $\frac{\pi}{4}$, which is $\frac{20\pi}{4}$ or 5π, is an odd multiple of π.

$\frac{19\pi}{4} = 5\pi - \frac{\pi}{4}$

$\sin \frac{19\pi}{4} = \sin \left(5\pi - \frac{\pi}{4}\right)$ *Case 4, with* $A = \frac{\pi}{4}$ *and* $k = 3$

$= \sin \frac{\pi}{4}$

c. cos (−410°)

The sum of −410° and 50° is a multiple of 360°.

$-410° = -360° - 50°$

$\cos (-410°) = \cos (-360° - 50°)$ *Case 3, with A = 50° and k = −1*

$= \cos 50°$

d. $\tan \frac{37\pi}{6}$

$\frac{37\pi}{6}$ and $\frac{\pi}{6}$ differ by a multiple of 2π.

$\frac{37\pi}{6} = 3(2\pi) + \frac{\pi}{6}$ *Case 1, with* $A = \frac{\pi}{6}$ *and* $k = 3$

$\tan \frac{37\pi}{6} = \frac{\sin \frac{37\pi}{6}}{\cos \frac{37\pi}{6}}$ *Rewrite using a quotient identity since the symmetry identities are in terms of sine and cosine.*

$= \frac{\sin \left(3(2\pi) + \frac{\pi}{6}\right)}{\cos \left(3(2\pi) + \frac{\pi}{6}\right)}$

$= \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}}$ or $\tan \frac{\pi}{6}$ *Quotient identity*

Case 3 of the Symmetry Identities can be written as the **opposite-angle identities** when $k = 0$.

Opposite-Angle Identities

The following trigonometric identities hold for all values of A.

$$\sin(-A) = -\sin A$$

$$\cos(-A) = \cos A$$

The basic trigonometric identities can be used to simplify trigonometric expressions. Simplifying a trigonometric expression means that the expression is written using the fewest trigonometric functions possible and as algebraically simplified as possible. This may mean writing the expression as a numerical value.

Examples

4 Simplify $\sin x + \sin x \cot^2 x$.

$$\begin{aligned}
\sin x + \sin x \cot^2 x &= \sin x\,(1 + \cot^2 x) && \textit{Factor.}\\
&= \sin x \csc^2 x && \textit{Pythagorean identity: } 1 + \cot^2 x = \csc^2 x\\
&= \sin x \cdot \frac{1}{\sin^2 x} && \textit{Reciprocal identity}\\
&= \frac{1}{\sin x}\\
&= \csc x && \textit{Reciprocal identity}
\end{aligned}$$

5 OPTICS Refer to the application at the beginning of the lesson.

Real World Application

a. Simplify the formula $I = I_0 - \dfrac{I_0}{\csc^2 \theta}$.

b. Use the simplified formula to determine the intensity of light that passes through a second polarizing lens with axis at 30° to the original.

a. $I = I_0 - \dfrac{I_0}{\csc^2 \theta}$

$I = I_0 - I_0 \sin^2 \theta$ *Reciprocal identity*

$I = I_0(1 - \sin^2 \theta)$ *Factor.*

$I = I_0 \cos^2 \theta$ *$1 - \sin^2 \theta = \cos^2 \theta$*

b. $I = I_0 \cos^2 30°$

$I = I_0\left(\dfrac{\sqrt{3}}{2}\right)^2$

$I = \dfrac{3}{4} I_0$

The light has three-fourths the intensity it had before passing through the second polarizing lens.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Find a counterexample** to show that the equation $1 - \sin x = \cos x$ is not an identity.
2. **Explain** why the Pythagorean and opposite-angle identities are so named.
3. **Write** two reciprocal identities, one quotient identity, and one Pythagorean identity, each of which involves $\cot \theta$.
4. **Prove** that $\tan(-A) = -\tan A$ using the quotient and opposite-angle identities.
5. **You Decide** Claude and Rosalinda are debating whether an equation from their homework assignment is an identity. Claude says that since he has tried ten specific values for the variable and all of them worked, it must be an identity. Rosalinda explained that specific values could only be used as counterexamples to prove that an equation is not an identity. Who is correct? Explain your answer.

Guided Practice

Prove that each equation is *not* a trigonometric identity by producing a counterexample.

6. $\sin \theta + \cos \theta = \tan \theta$
7. $\sec^2 x + \csc^2 x = 1$

Use the given information to determine the exact trigonometric value.

8. $\cos \theta = \frac{2}{3}, 0° < \theta < 90°$; $\sec \theta$
9. $\cot \theta = -\frac{\sqrt{5}}{2}, \frac{\pi}{2} < \theta < \pi$; $\tan \theta$
10. $\sin \theta = -\frac{1}{5}, \pi < \theta < \frac{3\pi}{2}$; $\cos \theta$
11. $\tan \theta = -\frac{4}{7}, 270° < \theta < 360°$; $\sec \theta$

Express each value as a trigonometric function of an angle in Quadrant I.

12. $\cos \frac{7\pi}{3}$
13. $\csc(-330°)$

Simplify each expression.

14. $\frac{\csc \theta}{\cot \theta}$
15. $\cos x \csc x \tan x$
16. $\cos x \cot x + \sin x$

17. **Physics** When there is a current in a wire in a magnetic field, a force acts on the wire. The strength of the magnetic field can be determined using the formula $B = \frac{F \csc \theta}{I\ell}$, where F is the force on the wire, I is the current in the wire, ℓ is the length of the wire, and θ is the angle the wire makes with the magnetic field. Physics texts often write the formula as $F = I\ell B \sin \theta$. Show that the two formulas are equivalent.

EXERCISES

Practice

Prove that each equation is not a trigonometric identity by producing a counterexample.

18. $\sin \theta \cos \theta = \cot \theta$
19. $\frac{\sec \theta}{\tan \theta} = \sin \theta$
20. $\sec^2 x - 1 = \frac{\cos x}{\csc x}$
21. $\sin x + \cos x = 1$
22. $\sin y \tan y = \cos y$
23. $\tan^2 A + \cot^2 A = 1$

24. Find a value of θ for which $\cos\left(\theta + \frac{\pi}{2}\right) \neq \cos\theta + \cos\frac{\pi}{2}$.

Use the given information to determine the exact trigonometric value.

25. $\sin\theta = \frac{2}{5}$, $0° < \theta < 90°$; $\csc\theta$

26. $\tan\theta = \frac{\sqrt{3}}{4}$, $0 < \theta < \frac{\pi}{2}$; $\cot\theta$

27. $\sin\theta = \frac{1}{4}$, $0 < \theta < \frac{\pi}{2}$; $\cos\theta$

28. $\cos\theta = -\frac{2}{3}$, $90° < \theta < 180°$; $\sin\theta$

29. $\csc\theta = \frac{\sqrt{11}}{3}$, $\frac{\pi}{2} < \theta < \pi$; $\cot\theta$

30. $\sec\theta = -\frac{5}{4}$, $90° < \theta < 180°$; $\tan\theta$

31. $\sin\theta = -\frac{1}{3}$, $180° < \theta < 270°$; $\tan\theta$

32. $\tan\theta = \frac{2}{3}$, $\pi < \theta < \frac{3\pi}{2}$; $\cos\theta$

33. $\sec\theta = -\frac{7}{5}$, $180° < \theta < 270°$; $\sin\theta$

34. $\cos\theta = \frac{1}{8}$, $\frac{3\pi}{2} < \theta < 2\pi$; $\tan\theta$

35. $\cot\theta = -\frac{4}{3}$, $270° < \theta < 360°$; $\sin\theta$

36. $\cot\theta = -8$, $\frac{3\pi}{2} < \theta < 2\pi$; $\csc\theta$

37. If A is a second quadrant angle, and $\cos A = -\frac{\sqrt{3}}{4}$, find $\frac{\sec^2 A - \tan^2 A}{2\sin^2 A + 2\cos^2 A}$.

Express each value as a trigonometric function of an angle in Quadrant I.

38. $\sin 390°$

39. $\cos\frac{27\pi}{8}$

40. $\tan\frac{19\pi}{5}$

41. $\csc\frac{10\pi}{3}$

42. $\sec(-1290°)$

43. $\cot(-660°)$

Simplify each expression.

44. $\frac{\sec x}{\tan x}$

45. $\frac{\cot\theta}{\cos\theta}$

46. $\frac{\sin(\theta + \pi)}{\cos(\theta - \pi)}$

47. $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

48. $\sin x \cos x \sec x \cot x$

49. $\cos x \tan x + \sin x \cot x$

50. $(1 + \cos\theta)(\csc\theta - \cot\theta)$

51. $1 + \cot^2\theta - \cos^2\theta - \cos^2\theta\cot^2\theta$

52. $\frac{\sin x}{1 + \cos x} + \frac{\sin x}{1 - \cos x}$

53. $\cos^4\alpha + 2\cos^2\alpha\sin^2\alpha + \sin^4\alpha$

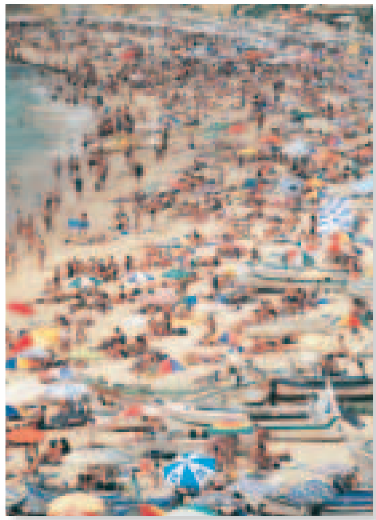
Applications and Problem Solving

Real World Application

54. Optics Refer to the equation derived in Example 5. What angle should the axes of two polarizing lenses make in order to block all light from passing through?

55. Critical Thinking Use the unit circle definitions of sine and cosine to provide a geometric interpretation of the opposite-angle identities.

56. Dermatology It has been shown that skin cancer is related to sun exposure. The rate W at which a person's skin absorbs energy from the sun depends on the energy S, in watts per square meter, provided by the sun, the surface area A exposed to the sun, the ability of the body to absorb energy, and the angle θ between the sun's rays and a line perpendicular to the body. The ability of an object to absorb energy is related to a factor called the *emissivity, e*, of the object. The emissivity can be calculated using the formula $e = \frac{W \sec \theta}{AS}$.



a. Solve this equation for W. Write your answer using only $\sin \theta$ or $\cos \theta$.

b. Find W if $e = 0.80$, $\theta = 40°$, $A = 0.75$ m^2, and $S = 1000$ W/m^2.

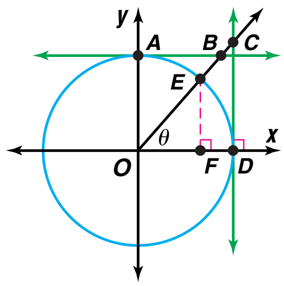
57. Physics A skier of mass m descends a θ-degree hill at a constant speed. When Newton's Laws are applied to the situation, the following system of equations is produced.

$$F_N - mg \cos \theta = 0$$
$$mg \sin \theta - \mu_k F_N = 0$$

where g is the acceleration due to gravity, F_N is the normal force exerted on the skier, and μ_k is the coefficient of friction. Use the system to define μ_k as a function of θ.

58. Geometry Show that the area of a regular polygon of n sides, each of length a, is given by $A = \frac{1}{4} na^2 \cot \left(\frac{180°}{n}\right)$.

59. Critical Thinking The circle at the right is a unit circle with its center at the origin. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are tangent to the circle. State the segments whose measures represent the ratios $\sin \theta$, $\cos \theta$, $\tan \theta$, $\sec \theta$, $\cot \theta$, and $\csc \theta$. Justify your answers.



Mixed Review

60. Find $\text{Cos}^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. *(Lesson 6-8)*

61. Graph $y = \cos\left(x - \frac{\pi}{6}\right)$. *(Lesson 6-5)*

62. Physics A pendulum 20 centimeters long swings 3°30′ on each side of its vertical position. Find the length of the arc formed by the tip of the pendulum as it swings. *(Lesson 6-1)*

63. Angle C of $\triangle ABC$ is a right angle. Solve the triangle if $A = 20°$ and $c = 35$. *(Lesson 5-4)*

64. Find all the rational roots of the equation $2x^3 + x^2 - 8x - 4 = 0$. *(Lesson 4-4)*

65. Solve $2x^2 + 7x - 4 = 0$ by completing the square. *(Lesson 4-2)*

66. Determine whether $f(x) = 3x^3 + 2x - 5$ is continuous or discontinuous at $x = 5$. *(Lesson 3-5)*

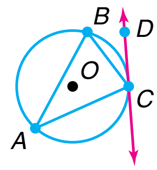
Extra Practice See p. A38.

67. Solve the system of equations algebraically. *(Lesson 2-2)*

$x + y - 2z = 3$
$-4x - y - z = 0$
$-x - 5y + 4z = 11$

68. Write the slope-intercept form of the equation of the line that passes through points at (5, 2) and (−4, 4). *(Lesson 1-4)*

69. **SAT/ACT Practice** Triangle ABC is inscribed in circle O, and $\overleftrightarrow{CD}$ is tangent to circle O at point C. If $m\angle BCD = 40°$, find $m\angle A$.



A 60° **B** 50° **C** 40° **D** 30° **E** 20°

CAREER CHOICES

Cartographer

Do maps fascinate you? Do you like drawing, working with computers, and geography? You may want to consider a career in cartography. As a cartographer, you would make maps, charts, and drawings. Cartography has changed a great deal with modern technology. Computers and satellites have become powerful new tools in making maps. As a cartographer, you may work with manual drafting tools as well as computer software designed for making maps.

The image at the right shows how a cartographer uses a three-dimensional landscape to create a two-dimensional topographic map.

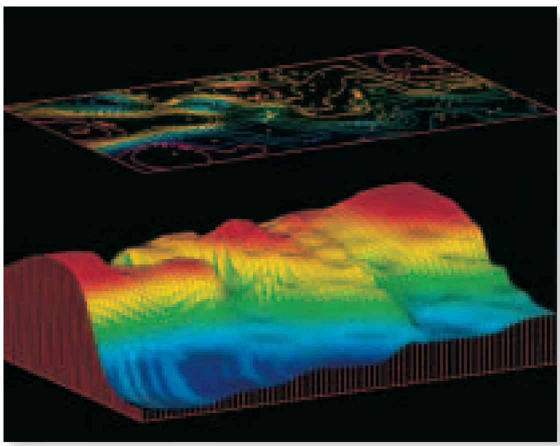
There are several areas of specialization in the field of cartography. Some of these include making maps from political boundaries and natural features, making maps from aerial photographs, and correcting original maps.

CAREER OVERVIEW

Degree Preferred:
bachelor's degree in engineering or a physical science

Related Courses:
mathematics, geography, computer science, mechanical drawing

Outlook:
slower than average through 2006



interNET CONNECTION For more information on careers in cartography, visit: **www.amc.glencoe.com**

7-2

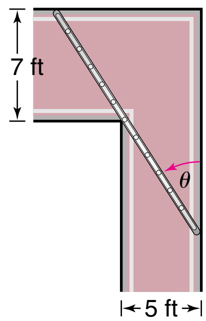
Verifying Trigonometric Identities

OBJECTIVES

- Use the basic trigonometric identities to verify other identities.
- Find numerical values of trigonometric functions.

Real World Application

PROBLEM SOLVING While working on a mathematics assignment, a group of students derived an expression for the length of a ladder that, when held horizontally, would turn from a 5-foot wide corridor into a 7-foot wide corridor. They determined that the maximum length ℓ of a ladder that would fit was given by $\ell(\theta) = \frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta}$, where θ is the angle that the ladder makes with the outer wall of the 5-foot wide corridor. When their teacher worked the problem, she concluded that $\ell(\theta) = 7 \sec \theta + 5 \csc \theta$. Are the two expressions for $\ell(\theta)$ equivalent? *This problem will be solved in Example 2.*



Verifying trigonometric identities algebraically involves transforming one side of the equation into the same form as the other side by using the basic trigonometric identities and the properties of algebra. Either side may be transformed into the other side, or both sides may be transformed separately into forms that are the same.

Suggestions for Verifying Trigonometric Identities

- Transform the more complicated side of the equation into the simpler side.
- Substitute one or more basic trigonometric identities to simplify expressions.
- Factor or multiply to simplify expressions.
- Multiply expressions by an expression equal to 1.
- Express all trigonometric functions in terms of sine and cosine.

You cannot add or subtract quantities from each side of an unverified identity, nor can you perform any other operation on each side, as you often do with equations. An unverified identity is not an equation, so the properties of equality do not apply.

Example 1 **Verify that $\sec^2 x - \tan x \cot x = \tan^2 x$ is an identity.**

Since the left side is more complicated, transform it into the expression on the right.

$$\sec^2 x - \tan x \cot x \stackrel{?}{=} \tan^2 x$$

$$\sec^2 x - \tan x \cdot \frac{1}{\tan x} \stackrel{?}{=} \tan^2 x \qquad \textit{cot } x = \frac{1}{\tan x}$$

$$\sec^2 x - 1 \stackrel{?}{=} \tan^2 x \qquad \textit{Multiply.}$$

$$\tan^2 x + 1 - 1 \stackrel{?}{=} \tan^2 x \qquad \sec^2 x = \tan^2 x + 1$$

$$\tan^2 x = \tan^2 x \qquad \textit{Simplify.}$$

We have transformed the left side into the right side. The identity is verified.

Examples

2 **PROBLEM SOLVING** **Verify that the two expressions for $\ell(\theta)$ in the application at the beginning of the lesson are equivalent. That is, verify that $\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} = 7 \sec \theta + 5 \csc \theta$ is an identity.**

Real World Application

Begin by writing the right side in terms of sine and cosine.

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} \stackrel{?}{=} 7 \sec \theta + 5 \csc \theta$$

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} \stackrel{?}{=} \frac{7}{\cos \theta} + \frac{5}{\sin \theta}$$ *$\sec \theta = \frac{1}{\cos \theta}$, $\csc \theta = \frac{1}{\sin \theta}$*

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} \stackrel{?}{=} \frac{7 \sin \theta}{\sin \theta \cos \theta} + \frac{5 \cos \theta}{\sin \theta \cos \theta}$$ *Find a common denominator.*

$$\frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta} = \frac{7 \sin \theta + 5 \cos \theta}{\sin \theta \cos \theta}$$ *Simplify.*

The students and the teacher derived equivalent expressions for $\ell(\theta)$, the length of the ladder.

3 **Verify that $\frac{\sin A}{\csc A} + \frac{\cos A}{\sec A} = \csc^2 A - \cot^2 A$ is an identity.**

Since the two sides are equally complicated, we will transform each side independently into the same form.

$$\frac{\sin A}{\csc A} + \frac{\cos A}{\sec A} \stackrel{?}{=} \csc^2 A - \cot^2 A$$

$$\frac{\sin A}{\frac{1}{\sin A}} + \frac{\cos A}{\frac{1}{\cos A}} \stackrel{?}{=} (1 + \cot^2 A) - \cot^2 A$$ *Quotient identities; Pythagorean identity*

$$\sin^2 A + \cos^2 A \stackrel{?}{=} 1$$ *Simplify.*

$$1 = 1$$ *$\sin^2 A + \cos^2 A = 1$*

The techniques that you use to verify trigonometric identities can also be used to simplify trigonometric equations. Sometimes you can change an equation into an equivalent equation involving a single trigonometric function.

Example

4 **Find a numerical value of one trigonometric function of x if $\frac{\cot x}{\cos x} = 2$.**

You can simplify the trigonometric expression on the left side by writing it in terms of sine and cosine.

$$\frac{\cot x}{\cos x} = 2$$

$$\frac{\frac{\cos x}{\sin x}}{\cos x} = 2$$ *$\cot x = \frac{\cos x}{\sin x}$*

$$\frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} = 2$$ *Definition of division*

$$\frac{1}{\sin x} = 2 \quad \textit{Simplify.}$$

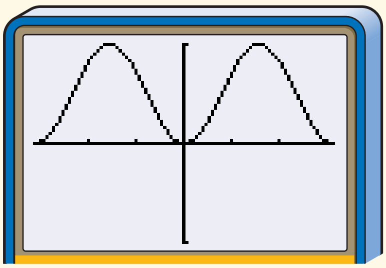
$$\csc x = 2 \quad \frac{1}{\sin x} = \csc x$$

Therefore, if $\frac{\cot x}{\cos x} = 2$, then $\csc x = 2$.

You can use a graphing calculator to investigate whether an equation may be an identity.

GRAPHING CALCULATOR EXPLORATION

- Graph both sides of the equation as two separate functions. For example, to test $\sin^2 x = (1 - \cos x)(1 + \cos x)$, graph $y_1 = \sin^2 x$ and $y_2 = (1 - \cos x)(1 + \cos x)$ on the same screen.
- If the two graphs do not match, then the equation is not an identity.
- If the two sides appear to match in every window you try, then the equation may be an identity.



$[-\pi, \pi]$ scl:1 by $[-1, 1]$ scl:1

TRY THESE **Determine whether each equation could be an identity. Write *yes* or *no*.**

1. $\sin x \csc x - \sin^2 x = \cos^2 x$

2. $\sec x + \csc x = 1$

3. $\sin x - \cos x = \frac{1}{\csc x - \sec x}$

WHAT DO YOU THINK?

4. If the two sides appear to match in every window you try, does that prove that the equation is an identity? Justify your answer.

5. Graph the function $f(x) = \frac{\sec x - \cos x}{\tan x}$. What simpler function could you set equal to $f(x)$ in order to obtain an identity?

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a trigonometric equation that is not an identity. Explain how you know it is not an identity.
2. **Explain** why you cannot square each side of the equation when verifying a trigonometric identity.
3. **Discuss** why both sides of a trigonometric identity are often rewritten in terms of sine and cosine.

4. *Math Journal* **Create** your own trigonometric identity that contains at least three different trigonometric functions. Explain how you created it. Give it to one of your classmates to verify. Compare and contrast your classmate's approach with your approach.

Guided Practice

Verify that each equation is an identity.

5. $\cos x = \dfrac{\cot x}{\csc x}$

6. $\dfrac{1}{\tan x + \sec x} = \dfrac{\cos x}{\sin x + 1}$

7. $\csc \theta - \cot \theta = \dfrac{1}{\csc \theta + \cot \theta}$

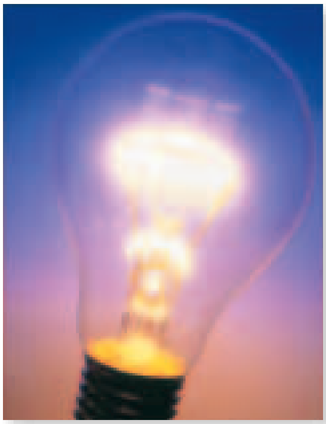
8. $\sin \theta \tan \theta = \sec \theta - \cos \theta$

9. $(\sin A - \cos A)^2 = 1 - 2 \sin^2 A \cot A$

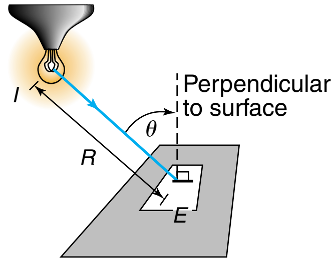
Find a numerical value of one trigonometric function of *x*.

10. $\tan x = \dfrac{1}{4} \sec x$

11. $\cot x + \sin x = -\cos x \cot x$



12. **Optics** The amount of light that a source provides to a surface is called the *illuminance.* The illuminance E in foot candles on a surface that is R feet from a source of light with intensity I candelas is $E = \dfrac{I \cos \theta}{R^2}$, where θ is the measure of the angle between the direction of the light and a line perpendicular to the surface being illuminated. Verify that $E = \dfrac{I \cot \theta}{R^2 \csc \theta}$ is an equivalent formula.



EXERCISES

Practice

Verify that each equation is an identity.

13. $\tan A = \dfrac{\sec A}{\csc A}$

14. $\cos \theta = \sin \theta \cot \theta$

15. $\sec x - \tan x = \dfrac{1 - \sin x}{\cos x}$

16. $\dfrac{1 + \tan x}{\sin x + \cos x} = \sec x$

17. $\sec x \csc x = \tan x + \cot x$

18. $\sin \theta + \cos \theta = \dfrac{2 \sin^2 \theta - 1}{\sin \theta - \cos \theta}$

19. $(\sin A + \cos A)^2 = \dfrac{2 + \sec A \csc A}{\sec A \csc A}$

20. $(\sin \theta - 1)(\tan \theta + \sec \theta) = -\cos \theta$

21. $\dfrac{\cos y}{1 - \sin y} = \dfrac{1 + \sin y}{\cos y}$

22. $\cos \theta \cos (-\theta) - \sin \theta \sin (-\theta) = 1$

23. $\csc x - 1 = \dfrac{\cot^2 x}{\csc x + 1}$

24. $\cos B \cot B = \csc B - \sin B$

25. $\sin \theta \cos \theta \tan \theta + \cos^2 \theta = 1$

26. $(\csc x - \cot x)^2 = \dfrac{1 - \cos x}{1 + \cos x}$

27. $\sin x + \cos x = \dfrac{\cos x}{1 - \tan x} + \dfrac{\sin x}{1 - \cot x}$

28. Show that $\sin \theta + \cos \theta + \tan \theta \sin \theta = \sec \theta + \cos \theta \tan \theta$.

Find a numerical value of one trigonometric function of *x*.

29. $\frac{\csc x}{\cot x} = \sqrt{2}$

30. $\frac{1 + \tan x}{1 + \cot x} = 2$

31. $\frac{1}{\cot x} - \frac{\sec x}{\csc x} = \cos x$

32. $\frac{1 + \cos x}{\sin x} + \frac{\sin x}{1 + \cos x} = 4$

33. $\cos^2 x + 2 \sin x - 2 = 0$

34. $\csc x = \sin x \tan x + \cos x$

35. If $\frac{\tan^3 \theta - 1}{\tan \theta - 1} - \sec^2 \theta - 1 = 0$, find $\cot \theta$.

Graphing Calculator

Use a graphing calculator to determine whether each equation could be an identity.

36. $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = 1$

37. $\cos \theta(\cos \theta - \sec \theta) = -\sin^2 \theta$

38. $2 \sin A + (1 - \sin A)^2 = 2 - \cos^2 A$

39. $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = \sin^2 x + \cos^2 x$

Applications and Problem Solving

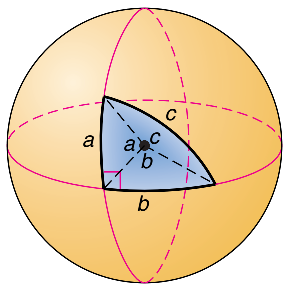
40. Electronics When an alternating current of frequency f and peak current I_0 passes through a resistance R, then the power delivered to the resistance at time t seconds is $P = I_0^2 R \sin^2 2\pi ft$.

a. Write an expression for the power in terms of $\cos^2 2\pi ft$.

b. Write an expression for the power in terms of $\csc^2 2\pi ft$.

41. Critical Thinking Let $x = \frac{1}{2} \tan \theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Write $f(x) = \frac{x}{\sqrt{1 + 4x^2}}$ in terms of a single trigonometric function of θ.

42. Spherical Geometry Spherical geometry is the geometry that takes place on the surface of a sphere. A line segment on the surface of the sphere is measured by the angle it subtends at the center of the sphere. Let a, b, and c be the sides of a right triangle on the surface of the sphere. Let the angles opposite those sides be α, β, and $\gamma = 90°$, respectively. The following equations are true:



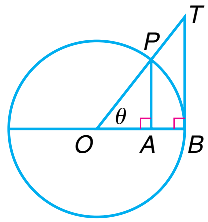
β is the Greek letter beta and γ is the Greek letter gamma.

$$\sin a = \sin \alpha \sin c$$
$$\cos b = \frac{\cos \beta}{\sin \alpha}$$
$$\cos c = \cos a \cos b.$$

Show that $\cos \beta = \tan a \cot c$.

43. Physics When a projectile is fired from the ground, its height y and horizontal displacement x are related by the equation $y = \frac{-gx^2}{2{v_0}^2 \cos^2 \theta} + \frac{x \sin \theta}{\cos \theta}$, where v_0 is the initial velocity of the projectile, θ is the angle at which it was fired, and g is the acceleration due to gravity. Rewrite this equation so that $\tan \theta$ is the only trigonometric function that appears in the equation.

44. Critical Thinking Consider a circle O with radius 1. $\overline{PA}$ and $\overline{TB}$ are each perpendicular to $\overline{OB}$. Determine the area of $ABTP$ as a product of trigonometric functions of θ.



45. Geometry Let a, b, and c be the sides of a triangle. Let α, β, and γ be the respective opposite angles. Show that the area A of the triangle is given by $A = \dfrac{a^2 \sin \beta \sin \gamma}{2 \sin (\beta + \gamma)}$.

Mixed Review

46. Simplify $\dfrac{\tan x + \cos x + \sin x \tan x}{\sec x + \tan x}$. *(Lesson 7-1)*

47. Write an equation of a sine function with amplitude 2, period 180°, and phase shift 45°. *(Lesson 6-5)*

48. Change $\dfrac{15\pi}{16}$ radians to degree measure to the nearest minute. *(Lesson 6-1)*

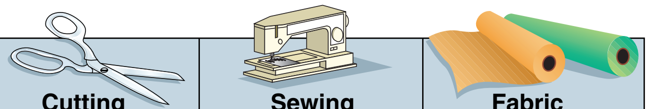
49. Solve $\sqrt[3]{3y - 1} - 2 = 0$. *(Lesson 4-7)*

50. Determine the equations of the vertical and horizontal asymptotes, if any, of $f(x) = \dfrac{3x}{x + 1}$. *(Lesson 3-7)*

51. Manufacturing The Simply Sweats Corporation makes high quality sweatpants and sweatshirts. Each garment passes through the cutting and sewing departments of the factory. The cutting and sewing departments have 100 and 180 worker-hours available each week, respectively. The fabric supplier can provide 195 yards of fabric each week. The hours of work and yards of fabric required for each garment are shown in the table below. If the profit from a sweatshirt is \$5.00 and the profit from a pair of sweatpants is \$4.50, how many of each should the company make for maximum profit? *(Lesson 2-7)*

Simply Sweats Corporation

"Quality Sweatpants and Sweatshirts"



Clothing	Cutting	Sewing	Fabric
Shirt	1 h	2.5 h	1.5 yd
Pants	1.5 h	2 h	3 yd

52. State the domain and range of the relation $\{(16,-4),(16, 4)\}$. Is this relation a function? Explain. *(Lesson 1-1)*

53. SAT/ACT Practice Divide $\dfrac{a - b}{a + b}$ by $\dfrac{b - a}{b + a}$.

A 1 **B** $\dfrac{(a - b)^2}{(a + b)^2}$ **C** $\dfrac{1}{a^2 - b^2}$

D -1 **E** 0

Extra Practice See p. A38.

7-3

Sum and Difference Identities

OBJECTIVE
- Use the sum and difference identities for the sine, cosine, and tangent functions.

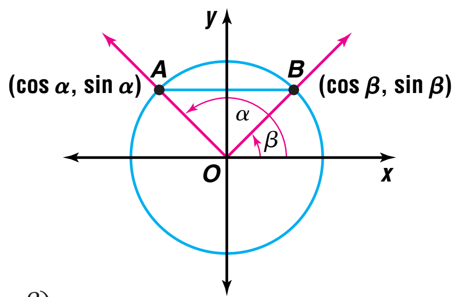
Real World Application

BROADCASTING Have you ever had trouble tuning in your favorite radio station? Does the picture on your TV sometimes appear blurry? Sometimes these problems are caused by *interference.* Interference can result when two waves pass through the same space at the same time. The two kinds of interference are:

- *constructive interference,* which occurs if the amplitude of the sum of the waves is greater than the amplitudes of the two component waves, and
- *destructive interference,* which occurs if the amplitude of the sum is less than the amplitudes of the component waves.

What type of interference results when a signal modeled by the equation $y = 20\sin(3t + 45°)$ is combined with a signal modeled by the equation $y = 20\sin(3t + 225°)$? *This problem will be solved in Example 4.*

Consider two angles α and β in standard position. Let the terminal side of α intersect the unit circle at point $A(\cos\alpha, \sin\alpha)$. Let the terminal side of β intersect the unit circle at $B(\cos\beta, \sin\beta)$. We will calculate $(AB)^2$ in two different ways.



$m\angle BOA = \alpha - \beta$

Look Back
You can refer to Lesson 5-8 to review the Law of Cosines.

First use the Law of Cosines.

$(AB)^2 = (OA)^2 + (OB)^2 - 2(OA)(OB)\cos(\alpha - \beta)$
$(AB)^2 = 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta)$ *$OA = OB = 1$*
$(AB)^2 = 2 - 2\cos(\alpha - \beta)$ *Simplify.*

Now use the distance formula.

$(AB)^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$
$(AB)^2 = \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$
$(AB)^2 = (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$
$(AB)^2 = 1 + 1 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$ *$\cos^2 a + \sin^2 a = 1$*
$(AB)^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$ *Simplify.*

Set the two expressions for $(AB)^2$ equal to each other.

$2 - 2\cos(\alpha - \beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$
$-2\cos(\alpha - \beta) = -2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$ *Subtract 2 from each side.*
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ *Divide each side by −2.*

This equation is known as the **difference identity for cosine.**

The **sum identity for cosine** can be derived by substituting $-\beta$ for β in the difference identity.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha-(-\beta))\\ &= \cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta+\sin\alpha(-\sin\beta) \quad \textit{cos}\,(-\beta)=\textit{cos}\,\beta;\ \textit{sin}\,(-\beta)=-\textit{sin}\,\beta\\ &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\end{aligned}$$

Sum and Difference Identities for the Cosine Function

If α and β represent the measures of two angles, then the following identities hold for all values of α and β.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Notice how the addition and subtraction symbols are related in the sum and difference identities.

You can use the sum and difference identities and the values of the trigonometric functions of common angles to find the values of trigonometric functions of other angles. Note that α and β can be expressed in either degrees or radians.

Example 1

a. Show by producing a counterexample that $\cos(x+y) \neq \cos x + \cos y$.

b. Show that the sum identity for cosine is true for the values used in part a.

a. Let $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$. First find $\cos(x+y)$ for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

$$\begin{aligned}\cos(x+y) &= \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) \quad \textit{Replace } x \textit{ with } \frac{\pi}{3} \textit{ and } y \textit{ with } \frac{\pi}{6}.\\ &= \cos\frac{\pi}{2} \quad \frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}\\ &= 0\end{aligned}$$

Now find $\cos x + \cos y$.

$$\begin{aligned}\cos x+\cos y &= \cos\frac{\pi}{3}+\cos\frac{\pi}{6} \quad \textit{Replace } x \textit{ with } \frac{\pi}{3} \textit{ and } y \textit{ with } \frac{\pi}{6}.\\ &= \frac{1}{2}+\frac{\sqrt{3}}{2} \text{ or } \frac{1+\sqrt{3}}{2}\end{aligned}$$

So, $\cos(x+y) \neq \cos x + \cos y$.

b. Show that $\cos(x+y) = \cos x\cos y - \sin x\sin y$ for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

First find $\cos(x+y)$. From part a, we know that $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = 0$.

Now find $\cos x\cos y - \sin x\sin y$.

$$\begin{aligned}\cos x\cos y-\sin x\sin y &= \cos\frac{\pi}{3}\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\sin\frac{\pi}{6} \quad \textit{Substitute for } x \textit{ and } y.\\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)\\ &= 0\end{aligned}$$

Thus, the sum identity for cosine is true for $x = \frac{\pi}{3}$ and $y = \frac{\pi}{6}$.

Example 2 **Use the sum or difference identity for cosine to find the exact value of $\cos 735°$.**

$735° = 2(360°) + 15°$ *Symmetry identity, Case 1*

$\cos 735° = \cos 15°$

$\cos 15° = \cos (45° - 30°)$ *45° and 30° are two common angles that differ by 15°.*

$= \cos 45° \cos 30° + \sin 45° \sin 30°$ *Difference identity for cosine*

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6} + \sqrt{2}}{4}$

Therefore, $\cos 735° = \frac{\sqrt{6} + \sqrt{2}}{4}$.

We can derive sum and difference identities for the sine function from those for the cosine function. Replace α with $\frac{\pi}{2}$ and β with s in the identities for $\cos (\alpha \pm \beta)$. The following equations result.

These equations are examples of cofunction identities.

$$\cos \left(\frac{\pi}{2} + s\right) = -\sin s \qquad \cos \left(\frac{\pi}{2} - s\right) = \sin s$$

Replace s with $\frac{\pi}{2} + s$ in the equation for $\cos \left(\frac{\pi}{2} + s\right)$ and with $\frac{\pi}{2} - s$ in the equation for $\cos \left(\frac{\pi}{2} - s\right)$ to obtain the following equations.

These equations are other cofunction identities.

$$\cos s = \sin \left(\frac{\pi}{2} + s\right) \qquad \cos s = \sin \left(\frac{\pi}{2} - s\right)$$

Replace s with $(\alpha + \beta)$ in the equation for $\cos \left(\frac{\pi}{2} - s\right)$ to derive an identity for the sine of the sum of two real numbers.

$$\cos \left[\frac{\pi}{2} - (\alpha + \beta)\right] = \sin (\alpha + \beta)$$

$$\cos \left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right] = \sin (\alpha + \beta)$$

$$\cos \left(\frac{\pi}{2} - \alpha\right) \cos \beta + \sin \left(\frac{\pi}{2} - \alpha\right) \sin \beta = \sin \left(\alpha + \beta\right) \quad \textit{Identity for } \cos (\alpha - \beta)$$

$$\sin \alpha \cos \beta + \cos \alpha \sin \beta = \sin (\alpha + \beta) \quad \textit{Substitute.}$$

This equation is known as the **sum identity for sine.**

The **difference identity for sine** can be derived by replacing β with $(-\beta)$ in the sum identity for sine.

$$\sin [\alpha + (-\beta)] = \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta)$$

$$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

Sum and Difference Identities for the Sine Function

If α and β represent the measures of two angles, then the following identities hold for all values of α and β.

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

Examples

3 **Find the value of sin $(x - y)$ if $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, sin $x = \frac{9}{41}$, and sin $y = \frac{7}{25}$.**

In order to use the difference identity for sine, we need to know cos x and cos y. We can use a Pythagorean identity to determine the necessary values.

$\sin^2 \alpha + \cos^2 \alpha = 1 \Rightarrow \cos^2 \alpha = 1 - \sin^2 \alpha$ *Pythagorean identity*

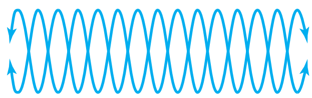
Since we are given that the angles are in Quadrant I, the values of sine and cosine are positive. Therefore, $\cos \alpha = \sqrt{1 - \sin^2 \alpha}$.

$$\cos x = \sqrt{1 - \left(\frac{9}{41}\right)^2}$$
$$= \sqrt{\frac{1600}{1681}} \text{ or } \frac{40}{41}$$
$$\cos y = \sqrt{1 - \left(\frac{7}{25}\right)^2}$$
$$= \sqrt{\frac{576}{625}} \text{ or } \frac{24}{25}$$

Now substitute these values into the difference identity for sine.

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$= \left(\frac{9}{41}\right)\left(\frac{24}{25}\right) - \left(\frac{40}{41}\right)\left(\frac{7}{25}\right)$$
$$= -\frac{64}{1025} \text{ or about } -0.0624$$

4 **BROADCASTING** **Refer to the application at the beginning of the lesson. What type of interference results when signals modeled by the equations $y = 20 \sin(3t + 45°)$ and $y = 20 \sin(3t + 225°)$ are combined?**



Real World Application

Add the two sine functions together and simplify.

$$20 \sin(3t + 45°) + 20 \sin(3t + 225°)$$
$$= 20(\sin 3t \cos 45° + \cos 3t \sin 45°) + 20(\sin 3t \cos 225° + \cos 3t \sin 225°)$$
$$= 20\left[(\sin 3t)\left(\frac{\sqrt{2}}{2}\right) + (\cos 3t)\left(\frac{\sqrt{2}}{2}\right)\right] + 20\left[(\sin 3t)\left(-\frac{\sqrt{2}}{2}\right) + (\cos 3t)\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$= 10\sqrt{2} \sin 3t + 10\sqrt{2} \cos 3t - 10\sqrt{2} \sin 3t - 10\sqrt{2} \cos 3t$$
$$= 0$$

The interference is destructive. The signals cancel each other completely.

You can use the sum and difference identities for the cosine and sine functions to find sum and difference identities for the tangent function.

Sum and Difference Identities for the Tangent Function

If α and β represent the measures of two angles, then the following identities hold for all values of α and β.

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

You will be asked to derive these identities in Exercise 47.

Example 5 **Use the sum or difference identity for tangent to find the exact value of tan 285°.**

$\tan 285° = \tan(240° + 45°)$ — *240° and 45° are common angles whose sum is 285°.*

$= \dfrac{\tan 240° + \tan 45°}{1 - \tan 240° \tan 45°}$ — *Sum identity for tangent*

$= \dfrac{\sqrt{3} + 1}{1 - (\sqrt{3})(1)}$ — *Multiply by $\dfrac{1+\sqrt{3}}{1+\sqrt{3}}$ to simplify.*

$= -2 - \sqrt{3}$

You can use sum and difference identities to verify other identities.

Example 6 **Verify that $\csc\left(\frac{3\pi}{2} + A\right) = -\sec A$ is an identity.**

Transform the left side since it is more complicated.

$\csc\left(\dfrac{3\pi}{2} + A\right) \stackrel{?}{=} -\sec A$

$\dfrac{1}{\sin\left(\frac{3\pi}{2} + A\right)} \stackrel{?}{=} -\sec A$ — *Reciprocal identity: $\csc x = \dfrac{1}{\sin x}$*

$\dfrac{1}{\sin\frac{3\pi}{2}\cos A + \cos\frac{3\pi}{2}\sin A} \stackrel{?}{=} -\sec A$ — *Sum identity for sine*

$\dfrac{1}{(-1)\cos A + (0)\sin A} \stackrel{?}{=} -\sec A$ — *$\sin\frac{3\pi}{2} = -1$; $\cos\frac{3\pi}{2} = 0$*

$-\dfrac{1}{\cos A} \stackrel{?}{=} -\sec A$ — *Simplify.*

$-\sec A = -\sec A$ — *Reciprocal identity*

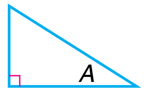
CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** how you would convince a friend that $\sin(x + y) \neq \sin x + \sin y$.
2. **Explain** how to use the sum and difference identities to find values for the secant, cosecant, and cotangent functions of a sum or difference.

3. **Write** an interpretation of the identity $\sin(90° - A) = \cos A$ in terms of a right triangle.



4. **Derive** a formula for $\cot(\alpha + \beta)$ in terms of $\cot \alpha$ and $\cot \beta$.

Guided Practice

Use sum or difference identities to find the exact value of each trigonometric function.

5. $\cos 165°$
6. $\tan \frac{\pi}{12}$
7. $\sec 795°$

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$.

8. $\sin(x - y)$ if $\sin x = \frac{4}{9}$ and $\sin y = \frac{1}{4}$

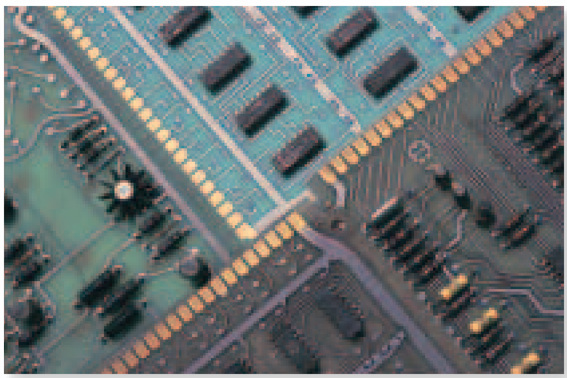
9. $\tan(x + y)$ if $\csc x = \frac{5}{3}$ and $\cos y = \frac{5}{13}$

Verify that each equation is an identity.

10. $\sin(90° + A) = \cos A$
11. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot \theta$
12. $\sin(x - y) = \frac{1 - \cot x \tan y}{\csc x \sec y}$

ω is the Greek letter omega.

13. **Electrical Engineering** Analysis of the voltage in certain types of circuits involves terms of the form $\sin(n\omega_0 t - 90°)$, where n is a positive integer, ω_0 is the frequency of the voltage, and t is time. Use an identity to simplify this expression.



EXERCISES

Practice

Use sum or difference identities to find the exact value of each trigonometric function.

14. $\cos 105°$
15. $\sin 165°$
16. $\cos \frac{7\pi}{12}$
17. $\sin \frac{\pi}{12}$
18. $\tan 195°$
19. $\cos\left(-\frac{\pi}{12}\right)$
20. $\tan 165°$
21. $\tan \frac{23\pi}{12}$
22. $\sin 735°$
23. $\sec 1275°$
24. $\csc \frac{5\pi}{12}$
25. $\cot \frac{113\pi}{12}$

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$.

26. $\sin(x + y)$ if $\cos x = \frac{8}{17}$ and $\sin y = \frac{12}{37}$

27. $\cos(x - y)$ if $\cos x = \frac{3}{5}$ and $\cos y = \frac{4}{5}$

28. $\tan(x - y)$ if $\sin x = \frac{8}{17}$ and $\cos y = \frac{3}{5}$

29. $\cos(x + y)$ if $\tan x = \frac{5}{3}$ and $\sin y = \frac{1}{3}$

30. $\tan(x + y)$ if $\cot x = \frac{6}{5}$ and $\sec y = \frac{3}{2}$

31. $\sec(x - y)$ if $\csc x = \frac{5}{3}$ and $\tan y = \frac{12}{5}$

32. If α and β are two angles in Quadrant I such that $\sin\alpha = \frac{1}{5}$ and $\cos\beta = \frac{2}{7}$, find $\sin(\alpha - \beta)$.

33. If x and y are acute angles such that $\cos x = \frac{1}{3}$ and $\cos y = \frac{3}{4}$, what is the value of $\cos(x + y)$?

Verify that each equation is an identity.

34. $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

35. $\cos(60° + A) = \sin(30° - A)$

36. $\sin(A + \pi) = -\sin A$

37. $\cos(180° + x) = -\cos x$

38. $\tan(x + 45°) = \frac{1 + \tan x}{1 - \tan x}$

39. $\sin(A + B) = \frac{\tan A + \tan B}{\sec A \sec B}$

40. $\cos(A + B) = \frac{1 - \tan A \tan B}{\sec A \sec B}$

41. $\sec(A - B) = \frac{\sec A \sec B}{1 + \tan A \tan B}$

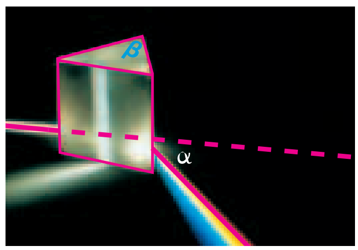
42. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$

Applications and Problem Solving

Real World Application

43. Electronics In an electric circuit containing a capacitor, inductor, and resistor the voltage drop across the inductor is given by $V_L = I_0 \omega L \cos\left(\omega t + \frac{\pi}{2}\right)$, where I_0 is the peak current, ω is the frequency, L is the inductance, and t is time. Use the sum identity for cosine to express V_L as a function of $\sin \omega t$.

44. Optics The index of refraction for a medium through which light is passing is the ratio of the velocity of light in free space to the velocity of light in the medium. For light passing symmetrically through a glass prism, the index of refraction n is given by the equation $n = \dfrac{\sin\left[\frac{1}{2}(\alpha + \beta)\right]}{\sin\frac{\beta}{2}}$, where α is the deviation angle and β is the angle of the apex of the prism as shown in the diagram. If $\beta = 60°$, show that $n = \sqrt{3}\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}$.



45. Critical Thinking Simplify the following expression without expanding any of the sums or differences.

$$\sin\left(\frac{\pi}{3} - A\right)\cos\left(\frac{\pi}{3} + A\right) - \cos\left(\frac{\pi}{3} - A\right)\sin\left(\frac{\pi}{3} + A\right)$$

46. Calculus In calculus, you will explore the *difference quotient* $\frac{f(x+h) - f(x)}{h}$.

a. Let $f(x) = \sin x$. Write and expand an expression for the difference quotient.

b. Set your answer from part a equal to y. Let $h = 0.1$ and graph.

c. What function has a graph similar to the graph in part b?

47. Critical Thinking Derive the sum and difference identities for the tangent function.

48. Critical Thinking Consider the following theorem.
If A, B, and C are the angles of a nonright triangle, then $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

a. Choose values for A, B, and C. Verify that the conclusion is true for your specific values.

b. Prove the theorem.

Mixed Review

49. Verify the identity $\sec^2 x = \frac{1 - \cos^2 x}{1 - \sin^2 x} + \csc^2 x - \cot^2 x$. *(Lesson 7-2)*

50. If $\sin \theta = -\frac{1}{8}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\tan \theta$. *(Lesson 7-1)*

51. Find $\sin\left(\text{Arctan } \sqrt{3}\right)$. *(Lesson 6-8)*

52. Find the values of θ for which $\csc \theta$ is undefined. *(Lesson 6-7)*

53. Weather The average seasonal high temperatures for Greensboro, North Carolina, are given in the table. Write a sinusoidal function that models the temperatures, using $t = 1$ to represent winter. *(Lesson 6-6)*

Winter	Spring	Summer	Fall
50°	70°	86°	71°

Source: Rand McNally & Company

54. State the amplitude, period, and phase shift for the function $y = 8 \cos (\theta - 30°)$. *(Lesson 6-5)*

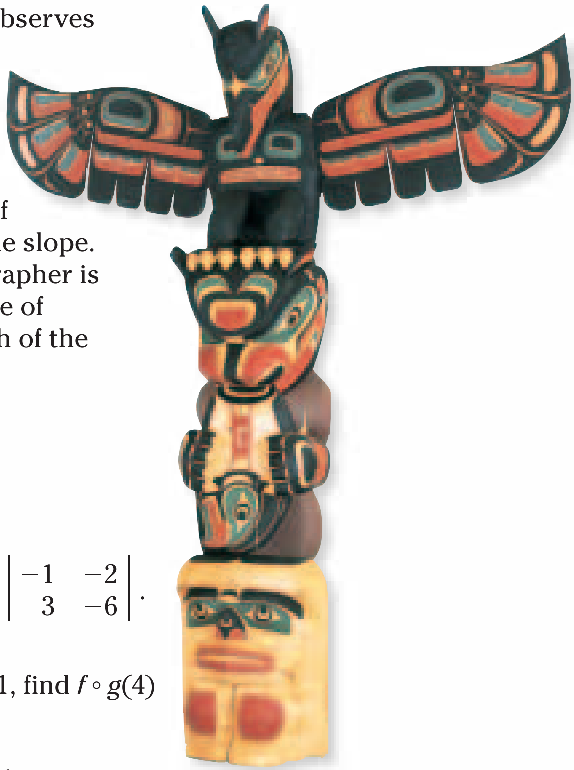
55. Find the value of $\sin (-540°)$. *(Lesson 6-3)*

56. Geometry A sector has arc length of 18 feet and a central angle measuring 2.9 radians. Find the radius and the area of the sector. *(Lesson 6-1)*

57. Navigation A ship at sea is 70 miles from one radio transmitter and 130 miles from another. The angle formed by the rays from the ship to the transmitters measures 130°. How far apart are the transmitters? *(Lesson 5-8)*

58. Determine the number of possible solutions for a triangle if $A = 120°$, $b = 12$, and $a = 4$. *(Lesson 5-7)*

59. **Photography** A photographer observes a 35-foot totem pole that stands vertically on a uniformly-sloped hillside and the shadow cast by it at different times of day. At a time when the angle of elevation of the sun is $37°12'$, the shadow of the pole extends directly down the slope. This is the effect that the photographer is seeking. If the hillside has an angle of inclination of $6°40'$, find the length of the shadow. *(Lesson 5-6)*



60. Find the roots of the equation $4x^3 + 3x^2 - x = 0$. *(Lesson 4-1)*

61. Solve $|x + 1| > 4$. *(Lesson 3-3)*

62. Find the value of the determinant $\begin{vmatrix} -1 & -2 \\ 3 & -6 \end{vmatrix}$. *(Lesson 2-5)*

63. If $f(x) = 3x^2 - 4$ and $g(x) = 5x + 1$, find $f \circ g(4)$ and $g \circ f(4)$. *(Lesson 1-2)*

64. **SAT Practice** What is the value of $(-8)^{62} \div 8^{62}$?
   A 1
   B 0
   C −1
   D −8
   E −62

MID-CHAPTER QUIZ

Use the given information to determine the exact trigonometric value. (Lesson 7-1)

1. $\sin\theta = \frac{2}{7}$, $0 < \theta < \frac{\pi}{2}$; $\cot\theta$

2. $\tan\theta = -\frac{4}{3}$, $90° < \theta < 180°$; $\cos\theta$

3. Express $\cos\frac{19\pi}{4}$ as a trigonometric function of an angle in Quadrant I. (Lesson 7-1)

Verify that each equation is an identity. (Lesson 7-2)

4. $\frac{1}{1 + \tan^2 x} + \frac{1}{1 + \cot^2 x} = 1$

5. $\frac{\csc^2\theta + \sec^2\theta}{\sec^2\theta} = \csc^2\theta$

Verify that each equation is an identity. (Lessons 7-2 and 7-3)

6. $\cot x \sec x \sin x = 2 - \tan x \cos x \csc x$

7. $\tan(\alpha - \beta) = \frac{1 - \cot\alpha\tan\beta}{\cot\alpha + \tan\beta}$

8. Use a sum or difference identity to find the exact value of $\cos 75°$. (Lesson 7-3)

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$. (Lesson 7-3)

9. $\cos(x + y)$ if $\sin x = \frac{2}{3}$ and $\sin y = \frac{3}{4}$

10. $\tan(x - y)$ if $\tan x = \frac{5}{4}$ and $\sec y = 2$

Extra Practice See p. A38.

GRAPHING CALCULATOR EXPLORATION

7-3B Reduction Identities

An Extension of Lesson 7-3

OBJECTIVE
- Identify equivalent values for trigonometric functions involving quadrantal angles.

In Chapter 5, you learned that any trigonometric function of an acute angle is equal to the *cofunction* of the complement of the angle. For example, $\sin \alpha = \cos (90° - \alpha)$. This is a part of a large family of identities called the **reduction identities.** These identities involve adding and subtracting the quandrantal angles, 90°, 180°, and 270°, from the angle measure to find equivalent values of the trigonometric function. You can use your knowledge of phase shifts and reflections to find the components of these identities.

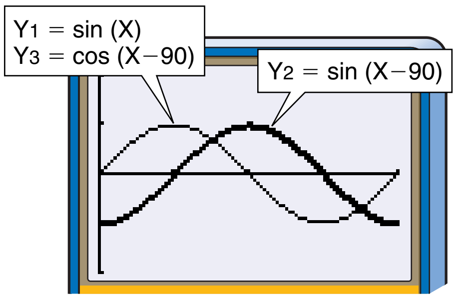
Example

Find the values of the sine and cosine functions for $\alpha - 90°$, $\alpha - 180°$, and $\alpha - 270°$ that are equivalent to sin α.

α − 90°

Graph $y = \sin \alpha$, $y = \sin (\alpha - 90°)$, and $y = \cos (\alpha - 90°)$, letting X in degree mode represent α. *You can select different display formats to help you distinguish the three graphs.*

You may recall from Chapter 6 that a phase shift of 90° right for the cosine function results in the sine function.



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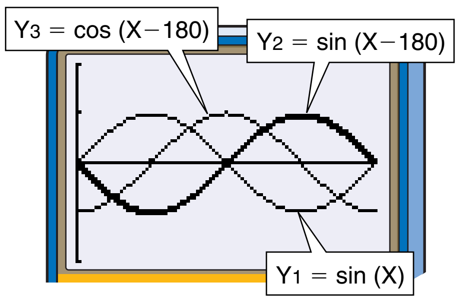
Note that the graph of $y = \cos (X - 90)$ is the same as the graph of $y = \sin X$. This suggests that $\sin \alpha = \cos (\alpha - 90°)$. *Remember that an identity must be proved algebraically. A graph does not prove an identity.*

α − 180°

Graph $y = \sin \alpha$, $y = \sin (\alpha - 180°)$, and $y = \cos (\alpha - 180°)$ using X to represent α.

Discount $y = \cos (\alpha - 180°)$ as a possible equivalence because it would involve a phase shift, which would change the actual value of the angle being considered.



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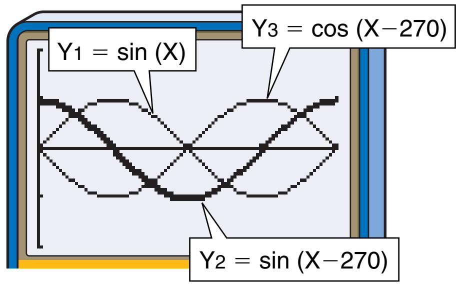
Note that the graph of $\sin (\alpha - 180°)$ is a mirror reflection of $\sin \alpha$. Remember that a reflection over the x-axis results in the mapping $(x, y) \rightarrow (x, -y)$. So to obtain a graph that is identical to $y = \sin \alpha$, we need the reflection of $y = \sin (\alpha - 180°)$ over the x-axis, or $y = -\sin (\alpha - 180°)$. Thus, $\sin \alpha = -\sin (\alpha - 180°)$. Graph the two equations to investigate this equality.

α − 270°

In this case, sin (α − 270°) is a phase shift, so ignore it. The graph of cos (α − 270°) is a reflection of sin α over the *x*-axis. So, sin α = − cos (α − 270°).



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The family of reduction identities also contains the relationships among the other cofunctions of tangent and cotangent and secant and cosecant. In addition to α − 90°, α − 180°, and α − 270° angle measures, the reduction identities address other measures such as 90° ± α, 180° ± α, 270° ± α, and 360° ± α.

TRY THESE

Copy and complete each statement with the proper trigonometric functions.

1. cos α = ___?___ (α − 90°) = ___?___ (α − 180°) = ___?___ (α − 270°)

2. tan α = ___?___ (α − 90°) = ___?___ (α − 180°) = ___?___ (α − 270°)

3. cot α = ___?___ (α − 90°) = ___?___ (α − 180°) = ___?___ (α − 270°)

4. sec α = ___?___ (α − 90°) = ___?___ (α − 180°) = ___?___ (α − 270°)

5. csc α = ___?___ (α − 90°) = ___?___ (α − 180°) = ___?___ (α − 270°)

WHAT DO YOU THINK?

6. Suppose the expressions involving subtraction in Exercises 1-5 were changed to sums.

a. Copy and complete each statement with the proper trigonometric functions.

(1) sin α = ___?___ (α + 90°) = ___?___ (α + 180°) = ___?___ (α + 270°)

(2) cos α = ___?___ (α + 90°) = ___?___ (α + 180°) = ___?___ (α + 270°)

(3) tan α = ___?___ (α + 90°) = ___?___ (α + 180°) = ___?___ (α + 270°)

(4) cot α = ___?___ (α + 90°) = ___?___ (α + 180°) = ___?___ (α + 270°)

(5) sec α = ___?___ (α + 90°) = ___?___ (α + 180°) = ___?___ (α + 270°)

(6) csc α = ___?___ (α + 90°) = ___?___ (α + 180°) = ___?___ (α + 270°)

b. How do the identities in part a compare to those in Exercises 1-5?

7. a. Copy and complete each statement with the proper trigonometric functions.

(1) sin α = ___?___ (90° − α) = ___?___ (180° − α) = ___?___ (270° − α)

(2) cos α = ___?___ (90° − α) = ___?___ (180° − α) = ___?___ (270° − α)

(3) tan α = ___?___ (90° − α) = ___?___ (180° − α) = ___?___ (270° − α)

(4) cot α = ___?___ (90° − α) = ___?___ (180° − α) = ___?___ (270° − α)

(5) sec α = ___?___ (90° − α) = ___?___ (180° − α) = ___?___ (270° − α)

(6) csc α = ___?___ (90° − α) = ___?___ (180° − α) = ___?___ (270° − α)

b. How do the identities in part a compare to those in Exercise 6a?

8. a. How did reduction identities get their name?

b. If you needed one of these identities, but could not remember it, what other type(s) of identities could you use to derive it?

7-4

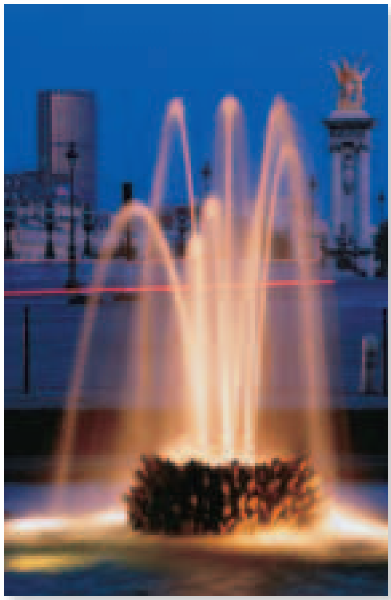
Double-Angle and Half-Angle Identities

OBJECTIVE

- Use the double- and half-angle identities for the sine, cosine, and tangent functions.

Real World Application

ARCHITECTURE Mike MacDonald is an architect who designs water fountains. One part of his job is determining the placement of the water jets that shoot the water into the air to create arcs. These arcs are modeled by parabolic functions. When a stream of water is shot into the air with velocity v at an angle of θ with the horizontal, the model predicts that the water will travel a horizontal distance of $D = \frac{v^2}{g} \sin 2\theta$ and reach a maximum height of $H = \frac{v^2}{2g} \sin^2 \theta$, where g is the acceleration due to gravity. The ratio of H to D helps determine the total height and width of the fountain. Express $\frac{H}{D}$ as a function of θ. *This problem will be solved in Example 3.*



It is sometimes useful to have identities to find the value of a function of twice an angle or half an angle. We can substitute θ for both α and β in $\sin(\alpha + \beta)$ to find an identity for $\sin 2\theta$.

$$\begin{aligned} \sin 2\theta &= \sin(\theta + \theta) \\ &= \sin\theta\cos\theta + \cos\theta\sin\theta && \textit{Sum identity for sine} \\ &= 2\sin\theta\cos\theta \end{aligned}$$

The same method can be used to find an identity for $\cos 2\theta$.

$$\begin{aligned} \cos 2\theta &= \cos(\theta + \theta) \\ &= \cos\theta\cos\theta - \sin\theta\sin\theta && \textit{Sum identity for cosine} \\ &= \cos^2\theta - \sin^2\theta \end{aligned}$$

If we substitute $1 - \cos^2\theta$ for $\sin^2\theta$ or $1 - \sin^2\theta$ for $\cos^2\theta$, we will have two alternate identities for $\cos 2\theta$.

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

These identities may be used if θ is measured in degrees or radians. So, θ may represent either a degree measure or a real number.

The tangent of a double angle can be found by substituting θ for both α and β in $\tan(\alpha + \beta)$.

$$\tan 2\theta = \tan(\theta + \theta)$$

$$= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} \quad \textit{Sum identity for tangent}$$

$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$

Double-Angle Identities

If θ represents the measure of an angle, then the following identities hold for all values of θ.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Example 1 **If $\sin\theta = \frac{2}{3}$ and θ has its terminal side in the first quadrant, find the exact value of each function.**

a. $\sin 2\theta$

To use the double-angle identity for $\sin 2\theta$, we must first find $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\left(\frac{2}{3}\right)^2 + \cos^2\theta = 1 \quad \sin\theta = \frac{2}{3}$$

$$\cos^2\theta = \frac{5}{9}$$

$$\cos\theta = \frac{\sqrt{5}}{3}$$

Then find $\sin 2\theta$.

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) \quad \sin\theta = \frac{2}{3}; \cos\theta = \frac{\sqrt{5}}{3}$$

$$= \frac{4\sqrt{5}}{9}$$

b. $\cos 2\theta$

Since we know the values of $\cos\theta$ and $\sin\theta$, we can use any of the double-angle identities for cosine.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \left(\frac{\sqrt{5}}{3}\right)^2 - \left(\frac{2}{3}\right)^2 \quad \cos\theta = \frac{\sqrt{5}}{3}; \sin\theta = \frac{2}{3}$$

$$= \frac{1}{9}$$

c. tan 2θ

We must find $\tan \theta$ to use the double-angle identity for $\tan 2\theta$.

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}} \qquad \textit{\sin \theta = \frac{2}{3}, \cos \theta = \frac{\sqrt{5}}{3}}$$

$$= \frac{2}{\sqrt{5}} \text{ or } \frac{2\sqrt{5}}{5}$$

Then find $\tan 2\theta$.

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$= \frac{2\left(\frac{2\sqrt{5}}{5}\right)}{1 - \left(\frac{2\sqrt{5}}{5}\right)^2} \qquad \tan \theta = \frac{2\sqrt{5}}{5}$$

$$= \frac{\frac{4\sqrt{5}}{5}}{\frac{1}{5}} \text{ or } 4\sqrt{5}$$

d. cos 4θ

Since $4\theta = 2(2\theta)$, use a double-angle identity for cosine again.

$$\cos 4\theta = \cos 2(2\theta)$$

$$= \cos^2(2\theta) - \sin^2(2\theta) \qquad \textit{Double-angle identity}$$

$$= \left(\frac{1}{9}\right)^2 - \left(\frac{4\sqrt{5}}{9}\right)^2 \qquad \cos 2\theta = \frac{1}{9}, \sin 2\theta = \frac{4\sqrt{5}}{9} \text{ (parts a and b)}$$

$$= -\frac{79}{81}$$

We can solve two of the forms of the identity for $\cos 2\theta$ for $\cos \theta$ and $\sin \theta$, respectively, and the following equations result.

$$\cos 2\theta = 2\cos^2 \theta - 1 \quad \xrightarrow{\text{Solve for } \cos \theta.} \quad \cos \theta = \pm\sqrt{\frac{1 + \cos 2\theta}{2}}$$

$$\cos 2\theta = 1 - 2\sin^2 \theta \quad \xrightarrow{\text{Solve for } \sin \theta.} \quad \sin \theta = \pm\sqrt{\frac{1 - \cos 2\theta}{2}}$$

We can replace 2θ with α and θ with $\frac{\alpha}{2}$ to derive the half-angle identities.

$$\tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}$$

$$= \frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}} \text{ or } \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Half-Angle Identities

If α represents the measure of an angle, then the following identities hold for all values of α.

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}},\ \cos\alpha \neq -1$$

Unlike with the double-angles identities, you must determine the sign.

Example 2 **Use a half-angle identity to find the exact value of each function.**

a. $\sin\frac{7\pi}{12}$

$$\sin\frac{7\pi}{12} = \sin\frac{\frac{7\pi}{6}}{2}$$

$$= \sqrt{\frac{1-\cos\frac{7\pi}{6}}{2}}$$

Use $\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$. *Since* $\frac{7\pi}{12}$ *is in Quadrant II, choose the positive sine value.*

$$= \sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}$$

$$= \frac{\sqrt{2+\sqrt{3}}}{2}$$

b. $\cos 67.5°$

$$\cos 67.5° = \cos\frac{135°}{2}$$

$$= \sqrt{\frac{1+\cos 135°}{2}}$$

Use $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$. *Since 67.5° is in Quadrant I, choose the positive cosine value.*

$$= \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

Double- and half-angle identities can be used to simplify trigonometric expressions.

Example 3 **ARCHITECTURE** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Find and simplify $\frac{H}{D}$.

b. What is the ratio of the maximum height of the water to the horizontal distance it travels for an angle of 27°?

a. $\frac{H}{D} = \dfrac{\frac{v^2}{2g}\sin^2\theta}{\frac{v^2}{g}\sin 2\theta}$

$= \frac{\sin^2\theta}{2\sin 2\theta}$ *Simplify.*

$= \frac{\sin^2\theta}{4\sin\theta\cos\theta}$ *$\sin 2\theta = 2\sin\theta\cos\theta$*

$= \frac{1}{4}\cdot\frac{\sin\theta}{\cos\theta}$ *Simplify.*

$= \frac{1}{4}\tan\theta$ *Quotient identity: $\frac{\sin\theta}{\cos\theta} = \tan\theta$*

Therefore, the ratio of the maximum height of the water to the horizontal distance it travels is $\frac{1}{4}\tan\theta$.

b. When $\theta = 27°$, $\frac{H}{D} = \frac{1}{4}\tan 27°$, or about 0.13.

For an angle of 27°, the ratio of the maximum height of the water to the horizontal distance it travels is about 0.13.

The double- and half-angle identities can also be used to verify other identities.

Example 4 **Verify that $\frac{\cos 2\theta}{1 + \sin 2\theta} = \frac{\cot\theta - 1}{\cot\theta + 1}$ is an identity.**

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cot\theta - 1}{\cot\theta + 1}$

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \dfrac{\frac{\cos\theta}{\sin\theta} - 1}{\frac{\cos\theta}{\sin\theta} + 1}$ *Reciprocal identity: $\cot\theta = \frac{\cos\theta}{\sin\theta}$*

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$ *Multiply numerator and denominator by $\sin\theta$.*

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\cdot\frac{\cos\theta + \sin\theta}{\cos\theta + \sin\theta}$ *Multiply each side by 1.*

$\frac{\cos 2\theta}{1+\sin 2\theta} \stackrel{?}{=} \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta}$ *Multiply.*

$$\frac{\cos 2\theta}{1 + \sin 2\theta} \stackrel{?}{=} \frac{\cos^2 \theta - \sin^2 \theta}{1 + 2\cos\theta \sin\theta}$$ *Simplify.*

$$\frac{\cos 2\theta}{1 + \sin 2\theta} = \frac{\cos 2\theta}{1 + \sin 2\theta}$$ *Double-angle identities:* $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$, $2 \cos \theta \sin \theta = \sin 2\theta$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** a paragraph about the conditions under which you would use each of the three identities for $\cos 2\theta$.
2. **Derive** the identity $\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$ from $\cos 2\theta = 1 - 2\sin^2 \theta$.
3. **Name** the quadrant in which the terminal side lies.
   a. x is a second quadrant angle. In which quadrant does $2x$ lie?
   b. $\frac{x}{2}$ is a first quadrant angle. In which quadrant does x lie?
   c. $2x$ is a second quadrant angle. In which quadrant does $\frac{x}{2}$ lie?
4. **Provide a counterexample** to show that $\sin 2\theta = 2 \sin \theta$ is not an identity.
5. **You Decide** Tamika calculated the exact value of $\sin 15°$ in two different ways. Using the difference identity for sine, $\sin 15°$ was $\frac{\sqrt{6} - \sqrt{2}}{4}$. When she used the half-angle identity, $\sin 15°$ equaled $\frac{\sqrt{2 - \sqrt{3}}}{2}$. Which answer is correct? Explain.

Guided Practice

Use a half-angle identity to find the exact value of each function.

6. $\sin \frac{\pi}{8}$
7. $\tan 165°$

Use the given information to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

8. $\sin \theta = \frac{2}{5}$, $0° < \theta < 90°$
9. $\tan \theta = \frac{4}{3}$, $\pi < \theta < \frac{3\pi}{2}$

Verify that each equation is an identity.

10. $\tan 2\theta = \frac{2}{\cot \theta - \tan \theta}$
11. $1 + \frac{1}{2} \sin 2A = \frac{\sec A + \sin A}{\sec A}$
12. $\sin \frac{x}{2} \cos \frac{x}{2} = \frac{\sin x}{2}$
13. **Electronics** Consider an AC circuit consisting of a power supply and a resistor. If the current in the circuit at time t is $I_0 \sin \omega t$, then the power delivered to the resistor is $P = I_0^2 R \sin^2 \omega t$, where R is the resistance. Express the power in terms of $\cos 2\omega t$.

EXERCISES

Practice

Use a half-angle identity to find the exact value of each function.

14. $\cos 15°$

15. $\sin 75°$

16. $\tan \frac{5\pi}{12}$

17. $\sin \frac{3\pi}{8}$

18. $\cos \frac{7\pi}{12}$

19. $\tan 22.5°$

20. If θ is an angle in the first quadrant and $\cos \theta = \frac{1}{4}$, find $\tan \frac{\theta}{2}$.

Use the given information to find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

21. $\cos \theta = \frac{4}{5}, 0° < \theta < 90°$

22. $\sin \theta = \frac{1}{3}, 0 < \theta < \frac{\pi}{2}$

23. $\tan \theta = -2, \frac{\pi}{2} < \theta < \pi$

24. $\sec \theta = -\frac{4}{3}, 90° < \theta < 180°$

25. $\cot \theta = \frac{3}{2}, 180° < \theta < 270°$

26. $\csc \theta = -\frac{5}{2}, \frac{3\pi}{2} < \theta < 2\pi$

27. If α is an angle in the second quadrant and $\cos \alpha = -\frac{\sqrt{2}}{3}$, find $\tan 2\alpha$.

Verify that each equation is an identity.

28. $\csc 2\theta = \frac{1}{2} \sec \theta \csc \theta$

29. $\cos A - \sin A = \frac{\cos 2A}{\cos A + \sin A}$

30. $(\sin \theta + \cos \theta)^2 - 1 = \sin 2\theta$

31. $\cos x - 1 = \frac{\cos 2x - 1}{2(\cos x + 1)}$

32. $\sec 2\theta = \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta - \sin^2 \theta}$

33. $\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$

34. $\sin 3x = 3 \sin x - 4 \sin^3 x$

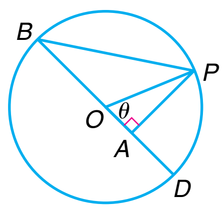
35. $\cos 3x = 4 \cos^3 x - 3 \cos x$

Applications and Problem Solving

Real World Application

36. Architecture Refer to the application at the beginning of the lesson. If the angle of the water is doubled, what is the ratio of the new maximum height to the original maximum height?

37. Critical Thinking Circle O is a unit circle. Use the figure to prove that $\tan \frac{1}{2}\theta = \frac{\sin \theta}{1 + \cos \theta}$.



38. Physics Suppose a projectile is launched with velocity v at an angle θ to the horizontal from the base of a hill that makes an angle α with the horizontal ($\theta > \alpha$). Then the range of the projectile, measured along the slope of the hill, is given by $R = \frac{2v^2 \cos \theta \sin (\theta - \alpha)}{g \cos^2 \alpha}$. Show that if $\alpha = 45°$, then $R = \frac{v^2\sqrt{2}}{g}(\sin 2\theta - \cos 2\theta - 1)$.

interNET CONNECTION

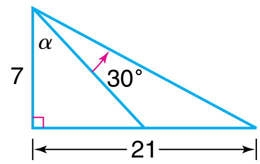
Research
For the latitude and longitude of world cities, and the distance between them, visit: **www.amc.glencoe.com**

39. Geography The Mercator projection of the globe is a projection on which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection involves the expression $\tan\left(45° + \frac{L}{2}\right)$, where L is the latitude of the point.

a. Write this expression in terms of a trigonometric function of L.

b. Find the value of this expression if $L = 60°$.

40. Critical Thinking Determine the tangent of angle α in the figure.



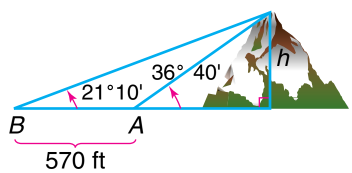
Mixed Review

41. Find the exact value of $\sec \frac{\pi}{12}$. *(Lesson 7-3)*

42. Show that $\sin x^2 + \cos x^2 = 1$ is not an identity. *(Lesson 7-1)*

43. Find the degree measure to the nearest tenth of the central angle of a circle of radius 10 centimeters if the measure of the subtended arc is 17 centimeters. *(Lesson 6-1)*

44. Surveying To find the height of a mountain peak, points A and B were located on a plain in line with the peak, and the angle of elevation was measured from each point. The angle at A was $36°40'$, and the angle at B was $21°10'$. The distance from A to B was 570 feet. How high is the peak above the level of the plain? *(Lesson 5-4)*



45. Write a polynomial equation of least degree with roots −3, 0.5, 6, and 2. *(Lesson 4-1)*

46. Graph $y = 2x + 5$ and its inverse. *(Lesson 3-4)*

47. Solve the system of equations. *(Lesson 2-1)*
$x + 2y = 11$
$3x - 5y = 11$

48. SAT Practice **Grid-In** If $(a - b)^2 = 64$, and $ab = 3$, find $a^2 + b^2$.

Extra Practice See p. A39.

CONTENTS

7-5

Solving Trigonometric Equations

OBJECTIVE

- Solve trigonometric equations and inequalities.

Real World Application

ENTERTAINMENT When you ride a Ferris wheel that has a diameter of 40 meters and turns at a rate of 1.5 revolutions per minute, the height above the ground, in meters, of your seat after t minutes can be modeled by the equation $h = 21 - 20 \cos 3\pi t$. How long after the ride starts will your seat first be 31 meters above the ground? *This problem will be solved in Example 4.*

So far in this chapter, we have studied a special type of trigonometric equation called an identity. Trigonometric identities are equations that are true for all values of the variable for which both sides are defined. In this lesson, we will examine another type of trigonometric equation. These equations are true for only certain values of the variable. Solving these equations resembles solving algebraic equations.

Most trigonometric equations have more than one solution. If the variable is not restricted, the periodic nature of trigonometric functions will result in an infinite number of solutions. Also, many trigonometric expressions will take on a given value twice every period.

If the variable is restricted to two adjacent quadrants, a trigonometric equation will have fewer solutions. These solutions are called **principal values.** For sin x and tan x, the principal values are in Quadrants I and IV. So x is in the interval $-90° \le x \le 90°$. For cos x, the principal values are in Quadrants I and II, so x is in the interval $0° \le x \le 180°$.

Example 1 **Solve $\sin x \cos x - \frac{1}{2}\cos x = 0$ for principal values of x. Express solutions in degrees.**

$$\sin x \cos x - \frac{1}{2}\cos x = 0$$

$$\cos x\left(\sin x - \frac{1}{2}\right) = 0 \quad \textit{Factor.}$$

$\cos x = 0$ or $\sin x - \frac{1}{2} = 0$ *Set each factor equal to 0.*

$x = 90°$

$\sin x = \frac{1}{2}$

$x = 30°$

The principal values are 30° and 90°.

If an equation cannot be solved easily by factoring, try writing the expressions in terms of only one trigonometric function. Remember to use your knowledge of identities.

Example 2 **Solve $\cos^2 x - \cos x + 1 = \sin^2 x$ for $0 \le x < 2\pi$.**

This equation can be written in terms of $\cos x$ only.

$\cos^2 x - \cos x + 1 = \sin^2 x$

$\cos^2 x - \cos x + 1 = 1 - \cos^2 x$ *Pythagorean identity: $\sin^2 x = 1 - \cos^2 x$*

$2\cos^2 x - \cos x = 0$

$\cos x(2\cos x - 1) = 0$ *Factor.*

$\cos x = 0$ or $2\cos x - 1 = 0$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ $\quad$ $\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$

The solutions are $\frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$, and $\frac{5\pi}{3}$.

As indicated earlier, most trigonometric equations have infinitely many solutions. When all of the values of x are required, the solution should be represented as $x + 360k°$ or $x + 2\pi k$ for $\sin x$ and $\cos x$ and $x + 180k°$ or $x + \pi k$ for $\tan x$, where k is any integer.

Example 3 **Solve $2\sec^2 x - \tan^4 x = -1$ for all real values of x.**

A Pythagorean identity can be used to write this equation in terms of $\tan x$ only.

$2\sec^2 x - \tan^4 x = -1$

$2(1 + \tan^2 x) - \tan^4 x = -1$ *Pythagorean identity: $\sec^2 x = 1 + \tan^2 x$*

$2 + 2\tan^2 x - \tan^4 x = -1$ *Simplify.*

$\tan^4 x - 2\tan^2 x - 3 = 0$

$(\tan^2 x - 3)(\tan^2 x + 1) = 0$ *Factor.*

$\tan^2 x - 3 = 0$ or $\tan^2 x + 1 = 0$

$\tan^2 x = 3$ $\quad$ $\tan^2 x = -1$ *This part gives no solutions since $\tan^2 x \ge 0$.*

$\tan x = \pm\sqrt{3}$

$x = \frac{\pi}{3} + \pi k$ or $x = -\frac{\pi}{3} + \pi k$, where k is any integer.

When a problem asks for real values of x, use radians.

The solutions are $\frac{\pi}{3} + \pi k$ and $-\frac{\pi}{3} + \pi k$.

There are times when a general expression for all of the solutions is helpful for determining a specific solution.

Example 4 **ENTERTAINMENT** **Refer to the application at the beginning of the lesson. How long after the Ferris wheel starts will your seat first be 31 meters above the ground?**

Real World Application

$h = 21 - 20 \cos 3\pi t$

$31 = 21 - 20 \cos 3\pi t$ *Replace h with 31.*

$-\frac{1}{2} = \cos 3\pi t$

$\frac{2\pi}{3} + 2\pi k = 3\pi t$ or $\frac{4\pi}{3} + 2\pi k = 3\pi t$ *where k is any integer*

$\frac{2}{9} + \frac{2}{3}k = t$ or $\frac{4}{9} + \frac{2}{3}k = t$

The least positive value for t is obtained by letting $k = 0$ in the first expression. Therefore, $t = \frac{2}{9}$ of a minute or about 13 seconds.

You can solve some trigonometric inequalities using the same techniques as for algebraic inequalities. The unit circle can be useful when deciding which angles to include in the answer.

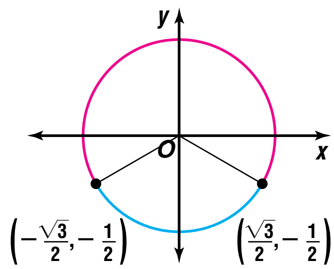
Example 5 **Solve $2 \sin \theta + 1 > 0$ for $0 \leq \theta < 2\pi$.**

$2 \sin \theta + 1 > 0$

$\sin \theta > -\frac{1}{2}$ *Solve for sin θ.*

In terms of the unit circle, we need to find points with y-coordinates greater than $-\frac{1}{2}$.

The values of θ for which $\sin \theta = -\frac{1}{2}$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The figure shows that the solution of the inequality is $0 \leq \theta < \frac{7\pi}{6}$ or $\frac{11\pi}{6} < \theta < 2\pi$.



GRAPHING CALCULATOR EXPLORATION

Some trigonometric equations and inequalities are difficult or impossible to solve with only algebraic methods. A graphing calculator is helpful in such cases.

TRY THESE **Graph each side of the equation as a separate function.**

1. $\sin x = 2 \cos x$ for $0 \leq x \leq 2\pi$

2. $\tan 0.5x = \cos x$ for $-2\pi \leq x \leq 2\pi$

3. Use the **CALC** menu to find the intersection point(s) of the graphs in Exercises 1 and 2.

WHAT DO YOU THINK?

4. What do the values in Exercise 3 represent? How could you verify this conjecture?

5. Graph $y = 2 \cos x - \sin x$ for $0 \leq x \leq 2\pi$.

a. How could you use the graph to solve the equation $\sin x = 2 \cos x$? How does this solution compare with those found in Exercise 3?

b. What equation would you use to apply this method to $\tan 0.5x = \cos x$?

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** the difference between a trigonometric identity and a trigonometric equation that is not an identity.
2. **Explain** why many trigonometric equations have infinitely many solutions.
3. **Write** all the solutions to a trigonometric equation in terms of sin x, given that the solutions between 0° and 360° are 45° and 135°.
4. *Math Journal* **Compare and contrast** solving trigonometric equations with solving linear and quadratic equations. What techniques are the same? What techniques are different? How many solutions do you expect? Do you use a graphing calculator in a similar manner?

Guided Practice

Solve each equation for principal values of x. Express solutions in degrees.

5. $2 \sin x + 1 = 0$
6. $2 \cos x - \sqrt{3} = 0$

Solve each equation for $0° \le x < 360°$.

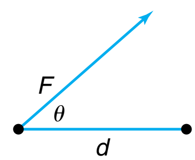
7. $\sin x \cot x = \frac{\sqrt{3}}{2}$
8. $\cos 2x = \sin^2 x - 2$

Solve each equation for $0 \le x < 2\pi$.

9. $3 \tan^2 x - 1 = 0$
10. $2 \sin^2 x = 5 \sin x + 3$

Solve each equation for all real values of x.

11. $\sin^2 2x + \cos^2 x = 0$
12. $\tan^2 x + 2 \tan x + 1 = 0$
13. $\cos^2 x + 3 \cos x = -2$
14. $\sin 2x - \cos x = 0$
15. Solve $2 \cos \theta + 1 < 0$ for $0 \le \theta < 2\pi$.
16. **Physics** The work done in moving an object through a displacement of d meters is given by $W = Fd \cos \theta$, where θ is the angle between the displacement and the force F exerted. If Lisa does 1500 joules of work while exerting a 100-newton force over 20 meters, at what angle was she exerting the force?



EXERCISES

Practice

Solve each equation for principal values of x. Express solutions in degrees.

17. $\sqrt{2} \sin x - 1 = 0$
18. $2 \cos x + 1 = 0$
19. $\sin 2x - 1 = 0$
20. $\tan 2x - \sqrt{3} = 0$
21. $\cos^2 x = \cos x$
22. $\sin x = 1 + \cos^2 x$

Solve each equation for $0° \le x < 360°$.

23. $\sqrt{2} \cos x + 1 = 0$
24. $\cos x \tan x = \frac{1}{2}$
25. $\sin x \tan x - \sin x = 0$
26. $2 \cos^2 x + 3 \cos x - 2 = 0$
27. $\sin 2x = -\sin x$
28. $\cos (x + 45°) + \cos (x - 45°) = \sqrt{2}$

29. Find all solutions to $2 \sin \theta \cos \theta + \sqrt{3} \sin \theta = 0$ in the interval $0° \le \theta < 360°$.

Solve each equation for $0 \le x < 2\pi$.

30. $(2 \sin x - 1)(2 \cos^2 x - 1) = 0$

31. $4 \sin^2 x + 1 = -4 \sin x$

32. $\sqrt{2} \tan x = 2 \sin x$

33. $\sin x = \cos 2x - 1$

34. $\cot^2 x - \csc x = 1$

35. $\sin x + \cos x = 0$

36. Find all values of θ between 0 and 2π that satisfy $-1 - 3 \sin \theta = \cos 2\theta$.

Solve each equation for all real values of *x*.

37. $\sin x = -\frac{1}{2}$

38. $\cos x \tan x - 2 \cos^2 x = -1$

39. $3 \tan^2 x = \sqrt{3} \tan x$

40. $2 \cos^2 x = 3 \sin x$

41. $\frac{1}{\cos x - \sin x} = \cos x + \sin x$

42. $2 \tan^2 x - 3 \sec x = 0$

43. $\sin x \cos x = \frac{1}{2}$

44. $\cos^2 x - \sin^2 x = \frac{\sqrt{3}}{2}$

45. $\sin^4 x - 1 = 0$

46. $\sec^2 x + 2 \sec x = 0$

47. $\sin x + \cos x = 1$

48. $2 \sin x + \csc x = 3$

Solve each inequality for $0 \le \theta < 2\pi$.

49. $\cos \theta \le -\frac{\sqrt{3}}{2}$

50. $\cos \theta - \frac{1}{2} > 0$

51. $\sqrt{2} \sin \theta - 1 < 0$

Graphing Calculator

Solve each equation graphically on the interval $0 \le x < 2\pi$.

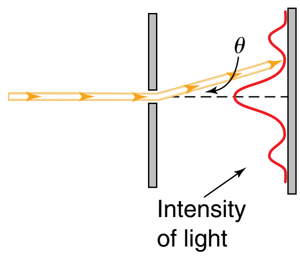
52. $\tan x = 0.5$

53. $\sin x - \frac{x}{2} = 0$

54. $\cos x = 3 \sin x$

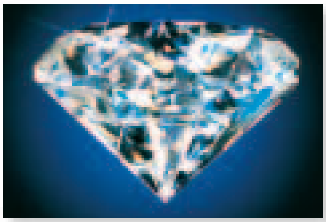
Applications and Problem Solving

55. Optics When light passes through a small slit, it is diffracted. The angle θ subtended by the first diffraction minimum is related to the wavelength λ of the light and the width D of the slit by the equation $\sin \theta = \frac{\lambda}{D}$. Consider light of wavelength 550 nanometers (5.5×10^{-7} m). What is the angle subtended by the first diffraction minimum when the light passes through a slit of width 3 millimeters?



56. Critical Thinking Solve the inequality $\sin 2x < \sin x$ for $0 \le x < 2\pi$ without a calculator.

57. Physics The range of a projectile that is launched with an initial velocity v at an angle of θ with the horizontal is given by $R = \frac{v^2}{g} \sin 2\theta$, where g is the acceleration due to gravity or 9.8 meters per second squared. If a projectile is launched with an initial velocity of 15 meters per second, what angle is required to achieve a range of 20 meters?

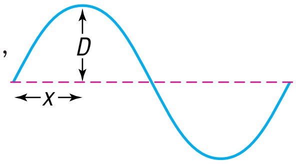


58. **Gemology** The sparkle of a diamond is created by *refracted* light. Light travels at different speeds in different mediums. When light rays pass from one medium to another in which they travel at a different velocity, the light is bent, or refracted. According to Snell's Law, $n_1 \sin i = n_2 \sin r$, where n_1 is the index of refraction of the medium the light is exiting, n_2 is the index of refraction of the medium the light is entering, i is the angle of incidence, and r is the angle of refraction.

   a. The index of refraction of a diamond is 2.42, and the index of refraction of air is 1.00. If a beam of light strikes a diamond at an angle of 35°, what is the angle of refraction?

   b. Explain how a gemologist might use Snell's Law to determine if a diamond is genuine.

59. **Music** A wave traveling in a guitar string can be modeled by the equation $D = 0.5 \sin(6.5x) \sin(2500t)$, where D is the displacement in millimeters at the position x meters from the left end of the string at time t seconds. Find the first positive time when the point 0.5 meter from the left end has a displacement of 0.01 millimeter.



60. **Critical Thinking** How many solutions in the interval $0° \le x < 360°$ should you expect for the equation $a \sin(bx + c) + d = d + \frac{a}{2}$, if $a \neq 0$ and b is a positive integer?

61. **Geometry** The point $P(x, y)$ can be rotated θ degrees counterclockwise about the origin by multiplying the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ on the left by the rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Determine the angle required to rotate the point $P(3, 4)$ to the point $P'(\sqrt{17}, 2\sqrt{2})$.

Mixed Review

62. Find the exact value of cot 67.5°. *(Lesson 7-4)*

63. Find a numerical value of one trigonometric function of x if $\frac{\tan x}{\sec x} = \frac{\sqrt{2}}{5}$. *(Lesson 7-2)*

64. Graph $y = \frac{2}{3} \cos \theta$. *(Lesson 6-4)*

65. **Transportation** A boat trailer has wheels with a diameter of 14 inches. If the trailer is being pulled by a car going 45 miles per hour, what is the angular velocity of the wheels in revolutions per second? *(Lesson 6-2)*

66. Use the unit circle to find the value of csc 180°. *(Lesson 5-3)*

67. Determine the binomial factors of $x^3 - 3x - 2$. *(Lesson 4-3)*

68. Graph $y = x^3 - 3x + 5$. Find and classify its extrema. *(Lesson 3-6)*

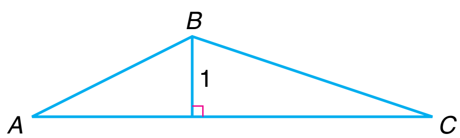
69. Find the values of x and y for which $\begin{bmatrix} 3x + 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 2y \end{bmatrix}$ is true. *(Lesson 2-3)*

70. Solve the system $x - y + z = 1$, $2x + y + 3z = 5$, and $x + y - z = 11$. *(Lesson 2-2)*

71. Graph $g(x) = |x + 3|$. *(Lesson 1-7)*

72. **SAT/ACT Practice** If $AC = 6$, what is the area of triangle ABC?

   A 1 B $\sqrt{6}$ C 3
   D 6 E 12

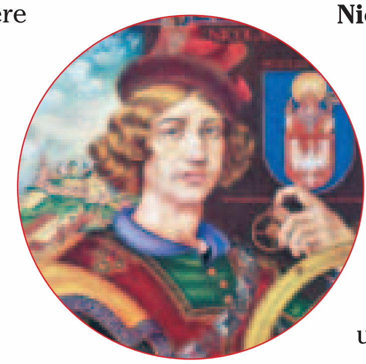


Extra Practice See p. A39.

TRIGONOMETRY

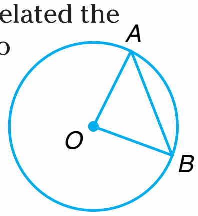
Trigonometry was developed in response to the needs of astronomers. In fact, it was not until the thirteenth century that trigonometry and astronomy were treated as separate disciplines.

Early Evidence The earliest use of trigonometry was to calculate the heights of objects using the lengths of shadows. Egyptian mathematicians produced tables relating the lengths of shadows at particular times of day as early as the thirteenth century B.C.



Copernicus

The Greek mathematician **Hipparchus** (190–120 B.C.), is generally credited with laying the foundations of trigonometry. Hipparchus is believed to have produced a twelve-book treatise on the construction of a table of chords. This table related the lengths of chords of a circle to the angles subtended by those chords. In the diagram, $\angle AOB$ would be compared to the length of chord $\overline{AB}$.



In about 100 A.D., the Greek mathematician **Menelaus**, credited with the first work on spherical trigonometry, also produced a treatise on chords in a circle. **Ptolemy**, a Babylonian mathematician, produced yet another book of chords, believed to have been adapted from Hipparchus' treatise. He used an identity similar to $\sin^2 x + \cos^2 x = 1$, except that it was relative to chords. He also used the formulas $\sin(x + y) = \sin x \cos y + \cos x \sin y$ and $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ as they related to chords.

In about 500 A.D., **Aryabhata**, a Hindu mathematician, was the first person to use the sine function as we know it today. He produced a table of sines and called the sine *jya.* In 628 A.D., another Hindu mathematician, **Brahmagupta**, also produced a table of sines.

The Renaissance Many mathematicians developed theories and applications of trigonometry during this time period. **Nicolas Copernicus** (1473–1543) published a book highlighting all the trigonometry necessary for astronomy at that time. During this period, the sine and versed sine were the most important trigonometric functions. Today, the versed sine, which is defined as $\text{versin } x = 1 - \cos x$, is rarely used.

Modern Era Mathematicians of the 1700s, 1800s, and 1900s worked on more sophisticated trigonometric ideas such as those relating to complex variables and hyperbolic functions. Renowned mathematicians who made contributions to trigonometry during this era were **Bernoulli**, **Cotes**, **DeMoivre**, **Euler**, and **Lambert**.

Today architects, such as Dennis Holloway of Santa Fe, New Mexico, use trigonometry in their daily work. Mr. Holloway is particularly interested in Native American designs. He uses trigonometry to determine the best angles for the walls of his buildings and for finding the correct slopes for landscaping.

ACTIVITIES

1. Draw a circle of radius 5 centimeters. Make a table of chords for angles of measure 10° through 90°(use 10° intervals). The table headings should be "angle measure" and "length of chord." (In the diagram of circle O, you are using $\angle AOB$ and chord $\overline{AB}$.)

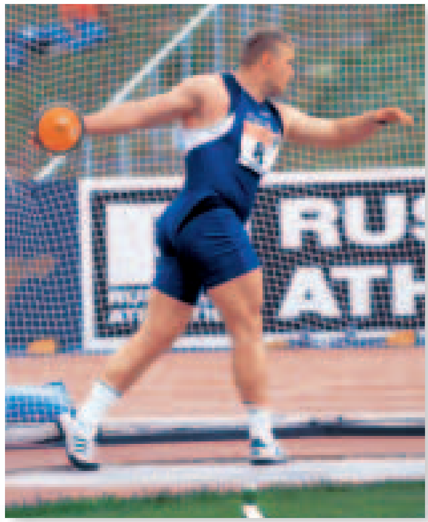
2. Find out more about personalities referenced in this article and others who contributed to the history of trigonometry. Visit **www.amc.glencoe.com**

7-6

Normal Form of a Linear Equation

OBJECTIVES
- Write the standard form of a linear equation given the length of the normal and the angle it makes with the *x*-axis.
- Write linear equations in normal form.

TRACK AND FIELD When a discus thrower releases the discus, it travels in a path that is tangent to the circle traced by the discus while the thrower was spinning around. Suppose the center of motion is the origin in the coordinate system. If the thrower spins so that the discus traces the unit circle and the discus is released at (0.96, 0.28), find an equation of the line followed by the discus. *This problem will be solved in Example 2.*

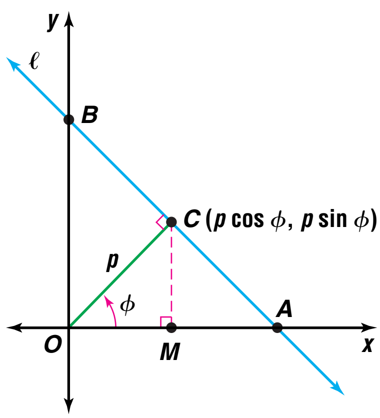


You are familiar with several forms for the equation of a line. These include slope-intercept form, point-slope form, and standard form. The usefulness of each form in a particular situation depends on how much relevant information each form provides upon inspection. In this lesson, you will learn about the **normal form** of a linear equation. The normal form uses trigonometry to provide information about the line.

In general, a **normal line** is a line that is perpendicular to another line, curve, or surface. Given a line in the *xy*-plane, there is a normal line that intersects the given line and passes through the origin. The angle between this normal line and the *x*-axis is denoted by ϕ. The normal form of the equation of the given line is written in terms of ϕ and the length p of the segment of the normal line from the given line to the origin.

ϕ is the Greek letter phi.

Suppose ℓ is a line that does not pass through the origin and p is the length of the normal from the origin. Let C be the point of intersection of ℓ with the normal, and let ϕ be the positive angle formed by the x-axis and $\overline{OC}$. Draw $\overline{MC}$ perpendicular to the x-axis. Since ϕ is in standard position, $\cos \phi = \frac{OM}{p}$ or $OM = p \cos \phi$ and $\sin \phi = \frac{MC}{p}$ or $MC = p \sin \phi$. So $\frac{p \sin \phi}{p \cos \phi}$ or $\frac{\sin \phi}{\cos \phi}$ is the slope of $\overline{OC}$. Since ℓ is perpendicular to $\overline{OC}$, the slope of ℓ is the negative reciprocal of the slope of $\overline{OC}$, or $-\frac{\cos \phi}{\sin \phi}$.



Look Back
You can refer to Lesson 1-4 to review the point-slope form.

Since ℓ contains C, we can use the point-slope form to write an equation of line ℓ.

$$y - y_1 = m(x - x_1)$$

$$y - p \sin \phi = -\frac{\cos \phi}{\sin \phi}(x - p \cos \phi) \quad \textit{Substitute for } m, x_1, \textit{ and } y_1.$$

$$y \sin \phi - p \sin^2 \phi = -x \cos \phi + p \cos^2 \phi \quad \textit{Multiply each side by } \sin \phi.$$

$$x \cos \phi + y \sin \phi = p(\sin^2 \phi + \cos^2 \phi)$$

$$x \cos \phi + y \sin \phi - p = 0 \quad \sin^2 \phi + \cos^2 \phi = 1$$

Normal Form

The normal form of a linear equation is

$$x \cos \phi + y \sin \phi - p = 0,$$

where p is the length of the normal from the line to the origin and ϕ is the positive angle formed by the positive x-axis and the normal.

You can write the standard form of a linear equation if you are given the values of ϕ and p.

Examples

1 **Write the standard form of the equation of a line for which the length of the normal segment to the origin is 6 and the normal makes an angle of 150° with the positive x-axis.**

$$x \cos \phi + y \sin \phi - p = 0 \quad \textit{Normal form}$$

$$x \cos 150° + y \sin 150° - 6 = 0 \quad \phi = 150° \textit{ and } p = 6$$

$$-\frac{\sqrt{3}}{2}x + \frac{1}{2}y - 6 = 0$$

$$\sqrt{3}x - y + 12 = 0 \quad \textit{Multiply each side by } -2.$$

The equation is $\sqrt{3}x - y + 12 = 0$.

2 **TRACK AND FIELD** **Refer to the application at the beginning of the lesson.**

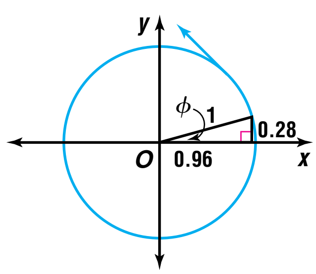
Real World Application

a. Determine an equation of the path of the discus if it is released at (0.96, 0.28).

b. Will the discus strike an official at (-30, 40)? Explain your answer.

a. From the figure, we see that $p = 1$, $\sin \phi = 0.28$ or $\frac{7}{25}$, and $\cos \phi = 0.96$ or $\frac{24}{25}$.

An equation of the line is $\frac{24}{25}x + \frac{7}{25}y - 1 = 0$, or $24x + 7y - 25 = 0$.



b. The y-coordinate of the point on the line with an x-coordinate of -30 is $\frac{745}{7}$ or about 106. So the discus will not strike the official at $(-30, 40)$.

We can transform the standard form of a linear equation, $Ax + By + C = 0$, into normal form if the relationship between the coefficients in the two forms is known. The equations will represent the same line if and only if their corresponding coefficients are proportional. If $\frac{A}{\cos \phi} = \frac{B}{\sin \phi} = \frac{C}{-p}$, then you can solve to find expressions for $\cos \phi$ and $\sin \phi$ in terms of p and the coefficients.

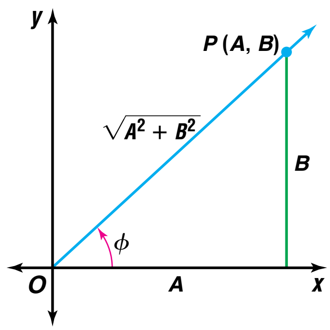
$$\frac{A}{\cos \phi} = \frac{C}{-p} \Rightarrow -\frac{Ap}{C} = \cos \phi \quad \text{or} \quad \cos \phi = -\frac{Ap}{C}$$
$$\frac{B}{\sin \phi} = \frac{C}{-p} \Rightarrow -\frac{Bp}{C} = \sin \phi \quad \text{or} \quad \sin \phi = -\frac{Bp}{C}$$

We can divide $\sin \phi = -\frac{Bp}{C}$ by $\cos \phi = -\frac{Ap}{C}$, where $\cos \phi \neq 0$.

$$\frac{\sin \phi}{\cos \phi} = \frac{-\frac{Bp}{C}}{-\frac{Ap}{C}}$$

$$\tan \phi = \frac{B}{A} \qquad \frac{\sin \phi}{\cos \phi} = \tan \phi$$

Refer to the diagram at the right. Consider an angle ϕ in standard position such that $\tan \phi = \frac{B}{A}$.



The length of $\overline{OP}$ is $\sqrt{A^2 + B^2}$.

So, $\sin \phi = \frac{B}{\pm\sqrt{A^2 + B^2}}$ and $\cos \phi = \frac{A}{\pm\sqrt{A^2 + B^2}}$.

Since we know that $\frac{B}{\sin \phi} = \frac{C}{-p}$, we can substitute to get the result $\frac{B}{\frac{B}{\pm\sqrt{A^2 + B^2}}} = \frac{C}{-p}$.

Therefore, $p = \frac{C}{\pm\sqrt{A^2 + B^2}}$.

If $C = 0$, the sign is chosen so that $\sin \phi$ is positive; that is, the same sign as that of B.

The $\pm$ sign is used since p is a measure and must be positive in the equation $x \cos \phi + y \sin \phi - p = 0$. Therefore, the sign must be chosen as opposite of the sign of C. That is, if C is positive, use $-\sqrt{A^2 + B^2}$, and, if C is negative, use $\sqrt{A^2 + B^2}$.

Substitute the values for $\sin \phi$, $\cos \phi$, and p into the normal form.

$$\frac{Ax}{\pm\sqrt{A^2 + B^2}} + \frac{By}{\pm\sqrt{A^2 + B^2}} - \frac{C}{\pm\sqrt{A^2 + B^2}} = 0$$

Notice that the standard form is closely related to the normal form.

Changing the Standard Form to Normal Form

The standard form of a linear equation, $Ax + By + C = 0$, can be changed to normal form by dividing each term by $\pm\sqrt{A^2 + B^2}$. The sign is chosen opposite the sign of C.

If the equation of a line is in normal form, you can find the length of the normal, p units, directly from the equation. You can find the angle ϕ by using the relation $\tan \phi = \frac{\sin \phi}{\cos \phi}$. However, you must find the quadrant in which the normal lies to find the correct angle for ϕ. When the equation of a line is in normal form, the coefficient of x is equal to $\cos \phi$, and the coefficient of y is equal to $\sin \phi$. Thus, the correct quadrant can be determined by studying the signs of $\cos \phi$ and $\sin \phi$. For example, if $\sin \phi$ is negative and $\cos \phi$ is positive, the normal lies in the fourth quadrant.

Example 3 **Write each equation in normal form. Then find the length of the normal and the angle it makes with the positive *x*-axis.**

a. $6x + 8y + 3 = 0$

Since C is positive, use $-\sqrt{A^2 + B^2}$ to determine the normal form.

$-\sqrt{A^2 + B^2} = -\sqrt{6^2 + 8^2}$ or -10

The normal form is $\frac{6}{-10}x + \frac{8}{-10}y + \frac{3}{-10} = 0$, or $-\frac{3}{5}x - \frac{4}{5}y - \frac{3}{10} = 0$.

Therefore, $\sin \phi = -\frac{4}{5}$, $\cos \phi = -\frac{3}{5}$, and $p = \frac{3}{10}$. Since $\sin \phi$ and $\cos \phi$ are both negative, ϕ must lie in the third quadrant.

$\tan \phi = \frac{-\frac{4}{5}}{-\frac{3}{5}}$ or $\frac{4}{3}$ *$\tan \phi = \frac{\sin \phi}{\cos \phi}$*

$\phi \approx 233°$ *Add 180° to the arctangent to get the angle in Quadrant III.*

The normal segment to the origin has length 0.3 unit and makes an angle of 233° with the positive x-axis.

b. $-x + 4y - 6 = 0$

Since C is negative, use $\sqrt{A^2 + B^2}$ to determine the normal form.

$\sqrt{A^2 + B^2} = \sqrt{(-1)^2 + 4^2}$ or $\sqrt{17}$

The normal form is $-\frac{1}{\sqrt{17}}x + \frac{4}{\sqrt{17}}y - \frac{6}{\sqrt{17}} = 0$ or

$-\frac{\sqrt{17}}{17}x + \frac{4\sqrt{17}}{17}y - \frac{6\sqrt{17}}{17} = 0$. Therefore, $\sin \phi = \frac{4\sqrt{17}}{17}$,

$\cos \phi = -\frac{\sqrt{17}}{17}$, and $p = \frac{6\sqrt{17}}{17}$. Since $\sin \phi > 0$ and $\cos \phi < 0$, ϕ must lie in the second quadrant.

$\tan \phi = \frac{\frac{4\sqrt{17}}{17}}{-\frac{\sqrt{17}}{17}}$ or -4 *$\tan \phi = \frac{\sin \phi}{\cos \phi}$*

$\phi \approx 104°$ *Add 180° to the arctangent to get the angle in Quadrant II.*

The normal segment to the origin has length $\frac{6\sqrt{17}}{17} \approx 1.46$ units and makes an angle of 104° with the positive x-axis.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Define** the geometric meaning of the word *normal.*
2. **Describe** how to write the normal form of the equation of a line when $p = 10$ and $\phi = 30°$.
3. **Refute or defend** the following statement. *Determining the normal form of the equation of a line is like finding the equation of a tangent line to a circle of radius p.*
4. **Write** each form of the equation of a line that you have learned. Compare and contrast the information that each provides upon inspection. Create a sample problem that would require you to use each form.

Guided Practice

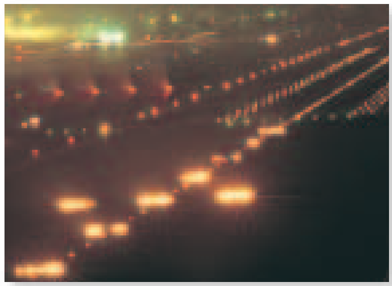
Write the standard form of the equation of each line given *p*, the length of the normal segment, and ϕ, the angle the normal segment makes with the positive *x*-axis.

5. $p = 10$, $\phi = 30°$
6. $p = \sqrt{3}$, $\phi = 150°$
7. $p = 5\sqrt{2}$, $\phi = \frac{7\pi}{4}$

Write each equation in normal form. Then find the length of the normal and the angle it makes with the positive *x*-axis.

8. $4x + 3y = -10$
9. $y = -3x + 2$
10. $\sqrt{2}x - \sqrt{2}y = 6$

11. **Transportation** An airport control tower is located at the origin of a coordinate system where the coordinates are measured in miles. An airplane radios in to report its direction and location. The controller determines that the equation of the plane's path is $3x - 4y = 8$.
    a. Make a drawing to illustrate the problem.
    b. What is the closest the plane will come to the tower?



EXERCISES

Practice

Write the standard form of the equation of each line given *p*, the length of the normal segment, and ϕ, the angle the normal segment makes with the positive *x*-axis.

12. $p = 15$, $\phi = 60°$
13. $p = 12$, $\phi = \frac{\pi}{4}$
14. $p = 3\sqrt{2}$, $\phi = 135°$
15. $p = 2\sqrt{3}$, $\phi = \frac{5\pi}{6}$
16. $p = 2$, $\phi = \frac{\pi}{2}$
17. $p = 5$, $\phi = 210°$
18. $p = 5$, $\phi = \frac{4\pi}{3}$
19. $p = \frac{3}{2}$, $\phi = 300°$
20. $p = 4\sqrt{3}$, $\phi = \frac{11\pi}{6}$

Write each equation in normal form. Then find the length of the normal and the angle it makes with the positive *x*-axis.

21. $5x + 12y + 65 = 0$
22. $x + y = 1$
23. $3x - 4y = 15$
24. $y = 2x - 4$
25. $x = 3$
26. $-\sqrt{3}x - y = 2$
27. $y - 2 = \frac{1}{4}(x + 20)$
28. $\frac{x}{3} = y - 4$
29. $\frac{x}{20} + \frac{y}{24} = 1$

30. Write the standard form of the equation of a line if the point on the line nearest to the origin is at (6, 8).

31. The point nearest to the origin on a line is at (4, −4). Find the standard form of the equation of the line.

Applications and Problem Solving

32. Geometry The three sides of a triangle are tangent to a unique circle called the *incircle*. On the coordinate plane, the incircle of $\triangle ABC$ has its center at the origin. The lines whose equations are $x + 4y = 6\sqrt{17}$, $2x + \sqrt{5}y = -18$, and $2\sqrt{2}x = y + 18$ contain the sides of $\triangle ABC$. What is the length of the radius of the incircle?

33. History Ancient slingshots were made from straps of leather that cradled a rock until it was released. One would spin the slingshot in a circle, and the initial path of the released rock would be a straight line tangent to the circle at the point of release.

The rock will travel the greatest distance if it is released when the angle between the normal to the path and the horizontal is −45°. The center of the circular path is the origin and the radius of the circle measures 1.25 feet.

a. Draw a labeled diagram of the situation.

b. Write an equation of the initial path of the rock in standard form.

34. Critical Thinking Consider a line ℓ with positive x- and y-intercepts. Suppose ℓ makes an angle of θ with the positive x-axis.

a. What is the relationship between θ and ϕ, the angle the normal line makes with the positive x-axis?

b. What is the slope of ℓ in terms of θ?

c. What is the slope of the normal line in terms of θ?

d. What is the slope of ℓ in terms of ϕ?

35. Analytic Geometry Armando was trying to determine how to rotate the graph of a line $\delta°$ about the origin. He hypothesized that the following would be an equation of the new line.

$$x \cos(\phi + \delta) + y \sin(\phi + \delta) - p = 0$$

a. Write the normal form of the line $5x + 12y - 39 = 0$.

b. Determine ϕ. Replace ϕ by $\phi + 90°$ and graph the result.

c. Choose another value for δ, not divisible by 90°, and test Armando's conjecture.

d. Write an argument as to whether Armando is correct. Include a graph in your argument.

36. **Critical Thinking** Suppose two lines intersect to form an acute angle α. Suppose that each line has a positive y-intercept and that the x-intercepts of the lines are on opposite sides of the origin.

a. How are the angles ϕ_1 and ϕ_2 that the respective normals make with the positive x-axis related?

b. Write an equation involving $\tan \alpha$, $\tan \phi_1$, and $\tan \phi_2$.

37. **Engineering** The village of Plentywood must build a new water tower to meet the needs of its residents. This means that each of the water mains must be connected to the new tower. On a map of the village, with units in hundreds of feet, the water tower was placed at the origin. Each of the existing water mains can be described by an equation. These equations are $5x - y = 15$, $3x + 4y = 36$, and $5x - 2y = -20$. The cost of laying new underground pipe is \$500 per 100 feet. Find the lowest possible cost of connecting the existing water mains to the new tower.

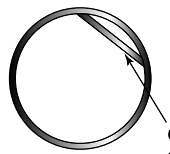
Mixed Review

38. Solve $2\cos^2 x + 7\cos x - 4 = 0$ for $0 \le x < 2\pi$. *(Lesson 7-5)*

39. If x and y are acute angles such that $\cos x = \frac{1}{6}$ and $\cos y = \frac{2}{3}$, find $\sin(x + y)$. *(Lesson 7-3)*

40. Graph $y = \sin 4\theta$. *(Lesson 6-4)*

41. **Engineering** A metallic ring used in a sprinkler system has a diameter of 13.4 centimeters. Find the length of the metallic cross brace if it subtends a central angle of $26°20'$. *(Lesson 5-8)*



42. Solve $\frac{x}{x-5} + \frac{17}{25-x^2} = \frac{1}{x+5}$. *(Lesson 4-6)*

43. **Manufacturing** A porcelain company produces collectible thimble sets that contain 8 thimbles in a box that is 4 inches by 6 inches by 2 inches. To celebrate the company's 100th anniversary, they wish to market a deluxe set of 8 large thimbles. They would like to increase each of the dimensions of the box by the same amount to create a new box with a volume that is 1.5 times the volume of the original box. What should be the dimensions of the new box for the large thimbles? *(Lesson 4-5)*

44. Find the maximum and minimum values of the function $f(x, y) = 3x - y + 4$ for the polygonal convex set determined by the system of inequalities. *(Lesson 2-6)*

$x + y \le 8$
$y \ge 3$
$x \ge 2$

45. Solve $\begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$. *(Lesson 2-5)*

46. **SAT/ACT Practice** If $\frac{a}{b} = \frac{4}{5}$, what is the value of $2a + b$?

A 3 **B** 13 **C** 14 **D** 26

E cannot be determined from the given information

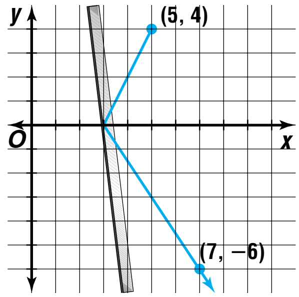
Extra Practice See p. A39.

7-7 Distance From a Point to a Line

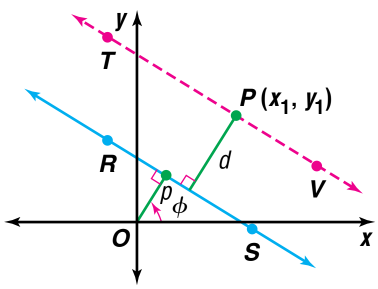
OBJECTIVES
- Find the distance from a point to a line.
- Find the distance between two parallel lines.
- Write equations of lines that bisect angles formed by intersecting lines.

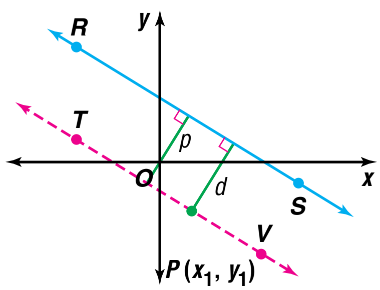
Real World Application

OPTICS When light waves strike a surface, they are reflected in such a way that the angle of incidence equals the angle of reflection. Consider a flat mirror situated in a coordinate system such that light emanating from the point at (5, 4) strikes the mirror at (3, 0) and then passes through the point at (7, −6). Determine an equation of the line on which the mirror lies. Use the equation to determine the angle the mirror makes with the *x*-axis. *This problem will be solved in Example 4.*



The normal form of a linear equation can be used to find the distance from a point to a line. Let $\overleftrightarrow{RS}$ be a line in the coordinate plane, and let $P(x_1, y_1)$ be a point not on $\overleftrightarrow{RS}$. P may lie on the same side of $\overleftrightarrow{RS}$ as the origin does or it may lie on the opposite side. If a line segment joining P to the origin does not intersect $\overleftrightarrow{RS}$, point P is on the same side of the line as the origin. Construct $\overleftrightarrow{TV}$ parallel to $\overleftrightarrow{RS}$ and passing through P. The distance d between the parallel lines is the distance from P to $\overleftrightarrow{RS}$. So that the derivation is valid in both cases, we will use a negative value for d if point P and the origin are on the same side of $\overleftrightarrow{RS}$.





interNET CONNECTION

Graphing Calculator Programs
To download a graphing calculator program that computes the distance from a point to a line, visit: **www.amc.glencoe.com**

Let $x \cos \phi + y \sin \phi - p = 0$ be the equation of $\overleftrightarrow{RS}$ in normal form. Since $\overleftrightarrow{TV}$ is parallel to $\overleftrightarrow{RS}$, they have the same slope. The equation for $\overleftrightarrow{TV}$ can be written as $x \cos \phi + y \sin \phi - (p + d) = 0$. Solve this equation for d.

$$d = x \cos \phi + y \sin \phi - p$$

Since $P(x_1, y_1)$ is on $\overleftrightarrow{TV}$, its coordinates satisfy this equation.

$$d = x_1 \cos \phi + y_1 \sin \phi - p$$

We can use an equivalent form of this expression to find d when the equation of a line is in standard form.

Distance from a Point to a Line

The following formula can be used to find the distance from a point at (x_1, y_1) to a line with equation $Ax + By + C = 0$.

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

The sign of the radical is chosen opposite the sign of C.

The distance will be positive if the point and the origin are on opposite sides of the line. The distance will be treated as negative if the origin is on the same side of the line as the point. If you are solving an application problem, the absolute value of d will probably be required.

Example 1 **Find the distance between $P(4, 5)$ and the line with equation $8x + 5y = 20$.**

First rewrite the equation of the line in standard form.

$$8x + 5y = 20 \Rightarrow 8x + 5y - 20 = 0$$

Then use the formula for the distance from a point to a line.

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

$$d = \frac{8(4) + 5(5) - 20}{\pm\sqrt{8^2 + 5^2}} \qquad A = 8,\ B = 5,\ C = -20,\ x_1 = 4,\ y_1 = 5$$

$$d = \frac{37}{\sqrt{89}} \text{ or about } 3.92 \qquad \text{Since } C \text{ is negative, use } +\sqrt{A^2 + B^2}.$$

Therefore, P is approximately 3.92 units from the line $8x + 5y = 20$. Since d is positive, P is on the opposite side of the line from the origin.

You can use the formula for the distance from a point to a line to find the distance between two parallel lines. To do this, choose any point on one of the lines and use the formula to find the distance from that point to the other line.

Example 2 **Find the distance between the lines with equations $6x - 2y = 7$ and $y = 3x + 4$.**

Since $y = 3x + 4$ is in slope-intercept form, we know that it passes through the point at (0, 4). Use this point to find the distance to the other line.

The standard form of the other equation is $6x - 2y - 7 = 0$.

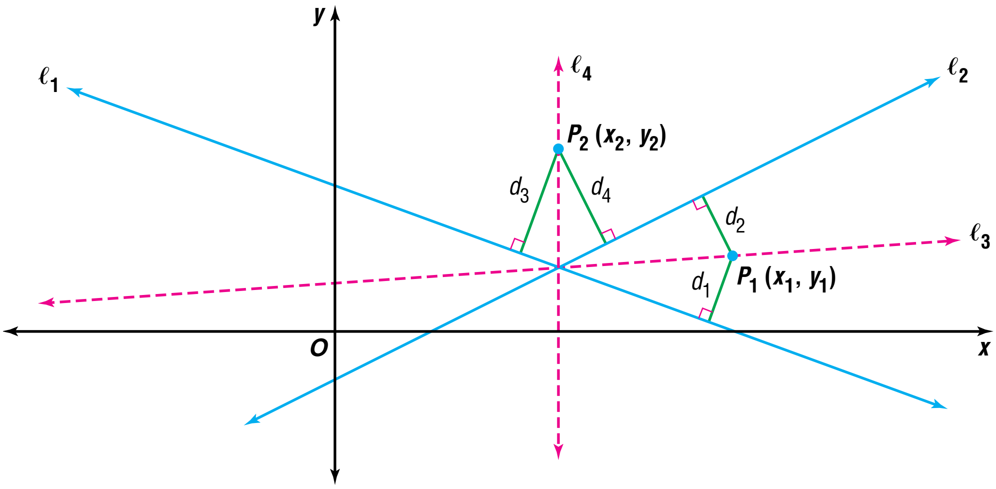
$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$

$$d = \frac{6(0) - 2(4) - 7}{\pm\sqrt{6^2 + (-2)^2}} \qquad A = 6,\ B = -2,\ C = -7,\ x_1 = 0,\ y_1 = 4$$

$$d = -\frac{15}{\sqrt{40}} \text{ or about } -2.37 \qquad \text{Since } C \text{ is negative, use } +\sqrt{A^2 + B^2}.$$

The distance between the lines is about 2.37 units.

An equation of the bisector of an angle formed by two lines in the coordinate plane can also be found using the formula for the distance between a point and a line. The bisector of an angle is the set of all points in the plane equidistant from the sides of the angle. Using this definition, equations of the bisectors of the angles formed by two lines can be found.



In the figure, ℓ_3 and ℓ_4 are the bisectors of the angles formed by ℓ_1 and ℓ_2. $P_1(x_1, y_1)$ is a point on ℓ_3, and $P_2(x_2, y_2)$ is a point on ℓ_4. Let d_1 be the distance from ℓ_1 to P_1, and let d_2 be the distance from ℓ_2 to P_1.

Notice that P_1 and the origin lie on opposite sides of ℓ_1, so d_1 is positive. Since the origin and P_1 are on opposite sides of ℓ_2, d_2 is also positive. Therefore, for any point $P_1(x_1, y_1)$ on ℓ_3, $d_1 = d_2$. However, d_3 is positive and d_4 is negative. *Why?* Therefore, for any point $P_2(x_2, y_2)$ on ℓ_4, $d_3 = -d_4$.

The origin is in the interior of the angle that is bisected by ℓ_3, but in the exterior of the angle bisected by ℓ_4. So, a good way for you to determine whether to equate distances or to let one distance equal the opposite of the other is to observe the position of the origin.

Relative Position of the Origin

If the origin lies within the angle being bisected or the angle vertical to it, the distances from each line to a point on the bisector have the same sign.

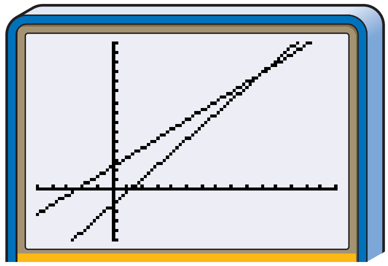
If the origin does not lie within the angle being bisected or the angle vertical to it, the distances have opposite signs.

To find the equation of a specific angle bisector, first graph the lines. Then, determine whether to equate the distances or to let one distance equal the opposite of the other.

Examples

3 **Find equations of the lines that bisect the angles formed by the lines $8x - 6y = 9$ and $20x - 21y = -50$.**

Graph each equation. Note the location of the origin. The origin is in the interior of the acute angle.



[−5, 15] scl:1 by [−5, 15] scl:1

Bisector of the acute angle

$d_1 = d_2$

$d_1 = \dfrac{8x_1 - 6y_1 - 9}{\sqrt{8^2 + (-6)^2}}$

$d_2 = \dfrac{20x_1 - 21y_1 + 50}{-\sqrt{20^2 + (-21)^2}}$

$\dfrac{8x_1 - 6y_1 - 9}{10} = \dfrac{20x_1 - 21y_1 + 50}{-29}$

Bisector of the obtuse angle

$d_1 = -d_2$

$d_1 = \dfrac{8x_1 - 6y_1 - 9}{\sqrt{8^2 + (-6)^2}}$

$d_2 = \dfrac{20x_1 - 21y_1 + 50}{-\sqrt{20^2 + (-21)^2}}$

$\dfrac{8x_1 - 6y_1 - 9}{10} = -\dfrac{20x_1 - 21y_1 + 50}{-29}$

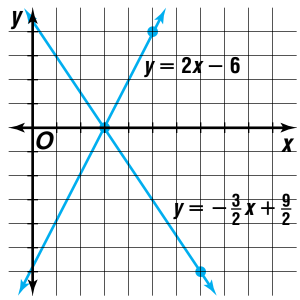
Simplifying and dropping the subscripts yields $432x - 384y = -239$ and $32x + 36y = 761$, respectively.

4 **OPTICS** **Refer to the application at the beginning of the lesson.**

Real World Application

a. Determine an equation of the line on which the mirror lies.

b. Use the equation to determine the angle the mirror makes with the *x*-axis.



a. Using the points at (5, 4), (7, −6), and (3, 0) and the slope-intercept form, you can find that the light travels along lines with equations $y = 2x - 6$ and $y = -\frac{3}{2}x + \frac{9}{2}$. In standard form, these are $2x - y - 6 = 0$ and $3x + 2y - 9 = 0$.

The mirror lies on the angle bisector of the acute angles formed by these lines.

The origin is not contained in the angle we wish to bisect, so we use $d_1 = -d_2$.

$$\frac{2x - y - 6}{\sqrt{5}} = -\frac{3x + 2y - 9}{\sqrt{13}}$$

$$\sqrt{13}(2x - y - 6) = -\sqrt{5}(3x + 2y - 9)$$

$$\left(2\sqrt{13} + 3\sqrt{5}\right)x + \left(2\sqrt{5} - \sqrt{13}\right)y - 6\sqrt{13} - 9\sqrt{5} = 0$$

The mirror lies on the line with equation $\left(2\sqrt{13} + 3\sqrt{5}\right)x + \left(2\sqrt{5} - \sqrt{13}\right)y - 6\sqrt{13} - 9\sqrt{5} = 0$.

Multiply by $\frac{2\sqrt{5} + \sqrt{13}}{2\sqrt{5} + \sqrt{13}}$ to rationalize the denominator.

b. The slope of the line on which the mirror lies is $-\frac{2\sqrt{13} + 3\sqrt{5}}{2\sqrt{5} - \sqrt{13}}$ or $-8 - \sqrt{65}$. Recall that the slope of a line is the tangent of the angle the line makes with the x-axis. Since $\tan \theta = -8 - \sqrt{65}$, the positive value for θ is approximately 93.56°.

The mirror makes an angle of 93.56° with the positive x-axis.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **State** what is meant by the distance from a point to a line.
2. **Tell** how to choose the sign of the radical in the denominator of the formula for the distance from a point to a line.
3. **Explain** why you can choose any point on either line when calculating the distance between two parallel lines.
4. **Investigate** whether the formula for the distance from a point to a line is valid if the line is vertical or horizontal. Test your conjecture with some specific examples. Explain what is happening when you apply the formula in these situations.

Guided Practice

Find the distance between the point with the given coordinates and the line with the given equation.

5. $(1, 2)$, $2x - 3y = -2$
6. $(-2, 3)$, $6x - y = -3$

Find the distance between the parallel lines with the given equations.

7. $3x - 5y = 1$
   $3x - 5y = -3$
8. $y = -\frac{1}{3}x + 3$
   $y = -\frac{1}{3}x - 7$

9. Find equations of the lines that bisect the acute and obtuse angles formed by the graphs of $6x + 8y = -5$ and $2x - 3y = 4$.

10. **Navigation** Juwan drives an ATV due east from the edge of a road into the woods. The ATV breaks down and stops after he has gone 2000 feet. In a coordinate system where the positive y-axis points north and the units are hundreds of feet, the equation of the road is $5x - 3y = 0$. How far is Juwan from the road?

Exercises

Practice

Find the distance between the point with the given coordinates and the line with the given equation.

11. $(2, 0), 3x - 4y + 15 = 0$

12. $(3, 5), 5x - 3y + 10 = 0$

13. $(0, 0), -2x - y = -3$

14. $(-2, -3), y = 4 - \frac{2}{3}x$

15. $(3, 1), y = 2x - 5$

16. $(-1, 2), y = -\frac{4}{3}x + 6$

17. What is the distance from the origin to the graph of $3x - y + 1 = 0$?

Find the distance between the parallel lines with the given equations.

18. $6x - 8y = 3$
$6x - 8y = -5$

19. $4x - 5y = 12$
$4x - 5y = 6$

20. $y = 2x + 1$
$2x - y = 2$

21. $y = -3x + 6$
$3x + y = 4$

22. $y = \frac{8}{5}x - 1$
$8x + 15 = 5y$

23. $y = -\frac{3}{2}x$
$y = -\frac{3}{2}x - 4$

24. What is the distance between the lines with equations $x + y - 1 = 0$ and $y = -x + 6$?

Find equations of the lines that bisect the acute and obtuse angles formed by the lines with the given equations.

25. $3x + 4y = 10$
$5x - 12y = 26$

26. $4x + y = 6$
$-15x + 8y = 68$

27. $y = \frac{2}{3}x + 1$
$y = -3x - 2$

Applications and Problem Solving

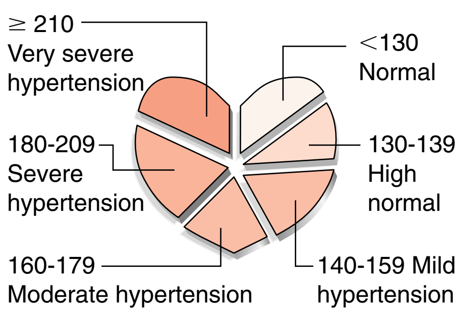
Real World Application

28. Statistics *Prediction equations* are often used to show the relationship between two quantities. A prediction equation relating a person's systolic blood pressure y to their age x is $4x - 3y + 228 = 0$. If an actual data point is close to the graph of the prediction equation, the equation gives a good approximation for the coordinates of that point.

a. Linda is 19 years old, and her systolic blood pressure is 112. Her father, who is 45 years old, has a systolic blood pressure of 120. For whom is the given prediction equation a better predictor?

b. Refer to the graph at the right. At what age does the equation begin to predict mild hypertension?

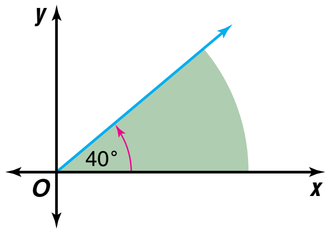
Classification of Adult Systolic Blood Pressure



Source: *Archives of Internal Medicine*

Randy Barnes

29. **Track and Field** In the shot put, the shot must land within a 40° sector. Consider a coordinate system where the vertex of the sector is at the origin, one side lies along the positive x-axis, and the units are meters. If a throw lands at the point with coordinates (16, 12), how far is it from being out-of-bounds?



30. **Critical Thinking** Circle P has its center at $(-5, 6)$ and passes through the point at $(-2, 2)$. Show that the line with equation $5x - 12y + 32 = 0$ is tangent to circle P.

31. **Geometry** Find the lengths of the altitudes in the triangle with vertices $A(-3, 4)$, $B(1, 7)$, and $C(-1, -3)$.

32. **Critical Thinking** In a triangle, the intersection of the angle bisectors is the center of the *inscribed circle* of the triangle. The inscribed circle is tangent to each side of the triangle. Determine the radius of the inscribed circle (the *inradius*) for the triangle with vertices at (0, 0), (10, 0), and (4, 12).

Mixed Review

33. Find the normal form of the equation $-2x + 7y = 5$. *(Lesson 7-6)*

34. Find $\cos 2A$ if $\sin A = \frac{\sqrt{3}}{6}$. *(Lesson 7-4)*

35. Graph $y = \csc(\theta + 60°)$. *(Lesson 6-7)*

36. **Aviation** Two airplanes leave an airport at the same time. Each flies at a speed of 110 mph. One flies in the direction 60° east of north. The other flies in the direction 40° east of south. How far apart are the planes after 3 hours? *(Lesson 5-8)*

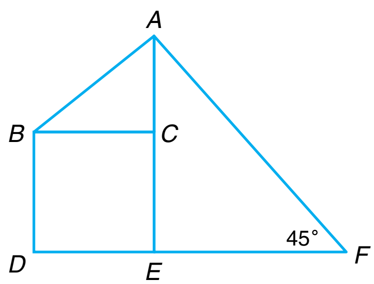
37. **Physics** The period of a pendulum can be determined by the formula $T = 2\pi\sqrt{\frac{\ell}{g}}$, where T represents the period, ℓ represents the length of the pendulum, and g represents acceleration due to gravity. Determine the period of a 2-meter pendulum on Earth if the acceleration due to gravity at Earth's surface is 9.8 meters per second squared. *(Lesson 4-7)*

38. Find the value of k in $(x^2 + 8x + k) \div (x - 2)$ so that the remainder is zero. *(Lesson 4-3)*

39. Solve the system of equations. *(Lesson 2-2)*
$x - 2y + z = -7$
$2x + y - z = -9$
$-x + 3y - 2z = 10$

40. **SAT/ACT Practice** If the area of square $BCED = 16$ and the area of $\triangle ABC = 6$, what is the length of $\overline{EF}$?

A 5 **B** 6
C 7 **D** 8
E 12



Extra Practice See p. A39.

VOCABULARY

counterexample (p. 421)
difference identity (p. 437)
double-angle identity (p. 449)
half-angle identity (p. 451)
identity (p. 421)
normal form (p. 463)
normal line (p. 463)
opposite-angle identity (p. 426)
principal value (p. 456)
Pythagorean identity (p. 423)
quotient identity (p. 422)
reciprocal identity (p. 422)
reduction identity (p. 446)
sum identity (p. 437)
symmetry identity (p. 424)
trigonometric identity (p. 421)

UNDERSTANDING AND USING THE VOCABULARY

Choose the letter of the term that best matches each equation or phrase.

1. $\sin \frac{\alpha}{2} = \pm\sqrt{\frac{1 - \cos \alpha}{2}}$
2. perpendicular to a line, curve, or surface
3. located in Quadrants I and IV for sin x and tan x
4. $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
5. $\cot \theta = \frac{1}{\tan \theta}$
6. $\frac{\sin \theta}{\cos \theta} = \tan \theta$
7. $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
8. $\sin (360k° - A) = -\sin A$
9. $1 + \cot^2 \theta = \csc^2 \theta$
10. uses trigonometry to provide information about a line

a. sum identity
b. half-angle identity
c. normal form
d. principal value
e. Pythagorean identity
f. symmetry identity
g. normal line
h. double-angle identity
i. reciprocal identity
j. quotient identity
k. opposite-angle identity

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

CONTENTS

SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 7-1 Identify and use reciprocal identities, quotient identities, Pythagorean identities, symmetry identities, and opposite-angle identities.

- If θ is in Quadrant I and $\cos\theta = \frac{1}{3}$, find $\sin\theta$.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + \left(\frac{1}{3}\right)^2 = 1$$
$$\sin^2\theta + \frac{1}{9} = 1$$
$$\sin^2\theta = \frac{8}{9}$$
$$\sin\theta = \frac{2\sqrt{2}}{3}$$

REVIEW EXERCISES

Use the given information to determine the trigonometric value. In each case, $0° < \theta < 90°$.

11. If $\sin\theta = \frac{1}{2}$, find $\csc\theta$.

12. If $\tan\theta = 4$, find $\sec\theta$.

13. If $\csc\theta = \frac{5}{3}$, find $\cos\theta$.

14. If $\cos\theta = \frac{4}{5}$, find $\tan\theta$.

15. Simplify $\csc x - \cos^2 x \csc x$.

Lesson 7-2 Use the basic trigonometric identities to verify other identities.

- Verify that $\csc x \sec x = \cot x + \tan x$ is an identity.

$$\csc x \sec x \stackrel{?}{=} \cot x + \tan x$$
$$\frac{1}{\sin x} \cdot \frac{1}{\cos x} \stackrel{?}{=} \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$
$$\frac{1}{\sin x \cos x} \stackrel{?}{=} \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$$
$$\frac{1}{\sin x \cos x} = \frac{1}{\sin x \cos x}$$

Verify that each equation is an identity.

16. $\cos^2 x + \tan^2 x \cos^2 x = 1$

17. $\frac{1 - \cos\theta}{1 + \cos\theta} = (\csc\theta - \cot\theta)^2$

18. $\frac{\sec\theta + 1}{\tan\theta} = \frac{\tan\theta}{\sec\theta - 1}$

19. $\frac{\sin^4 x - \cos^4 x}{\sin^2 x} = 1 - \cot^2 x$

Lesson 7-3 Use the sum and difference identities for the sine, cosine, and tangent functions.

- Find the exact value of $\sin 105°$.

$$\sin 105° = \sin(60° + 45°)$$
$$= \sin 60° \cos 45° + \cos 60° \sin 45°$$
$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

Use sum or difference identities to find the exact value of each trigonometric function.

20. $\cos 195°$

21. $\cos 15°$

22. $\sin\left(-\frac{17\pi}{12}\right)$

23. $\tan\frac{11\pi}{12}$

Find each exact value if $0 < x < \frac{\pi}{2}$ and $0 < y < \frac{\pi}{2}$.

24. $\cos(x - y)$ if $\sin x = \frac{7}{25}$ and $\cos y = \frac{2}{3}$

25. $\tan(x + y)$ if $\tan x = \frac{5}{4}$ and $\sec y = \frac{3}{2}$

OBJECTIVES AND EXAMPLES

Lesson 7-4 Use the double- and half-angle identities for the sine, cosine, and tangent functions.

- If θ is an angle in the first quadrant and $\sin \theta = \frac{3}{4}$, find $\cos 2\theta$.

$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$= 1 - 2\left(\frac{3}{4}\right)^2$$
$$= -\frac{1}{8}$$

REVIEW EXERCISES

Use a half-angle identity to find the exact value of each function.

26. $\cos 75°$ **27.** $\sin \frac{7\pi}{8}$

28. $\sin 22.5°$ **29.** $\tan \frac{\pi}{12}$

If θ is an angle in the first quadrant and $\cos \theta = \frac{3}{5}$, find the exact value of each function.

30. $\sin 2\theta$ **31.** $\cos 2\theta$

32. $\tan 2\theta$ **33.** $\sin 4\theta$

Lesson 7-5 Solve trigonometric equations and inequalities.

- Solve $2\cos^2 x - 1 = 0$ for $0° \leq x < 360°$.

$$2\cos^2 x - 1 = 0$$
$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \pm\frac{\sqrt{2}}{2}$$
$$x = 45°, 135°, 225°, \text{ or } 315°$$

Solve each equation for $0° \leq x < 360°$.

34. $\tan x + 1 = \sec x$

35. $\sin^2 x + \cos 2x - \cos x = 0$

36. $\cos 2x + \sin x = 1$

Solve each equation for all real values of x.

37. $\sin x \tan x - \frac{\sqrt{2}}{2}\tan x = 0$

38. $\sin 2x + \sin x = 0$

39. $\cos^2 x = 2 - \cos x$

Lesson 7-6 Write linear equations in normal form.

- Write $3x + 2y - 6 = 0$ in normal form. Since C is negative, use the positive value of $\sqrt{A^2 + B^2}$. $\sqrt{3^2 + 2^2} = \sqrt{13}$

The normal form is

$$\frac{3}{\sqrt{13}}x + \frac{2}{\sqrt{13}}y - \frac{6}{\sqrt{13}} = 0 \text{ or}$$
$$\frac{3\sqrt{13}}{13}x + \frac{2\sqrt{13}}{13}y - \frac{6\sqrt{13}}{13} = 0.$$

Write the standard form of the equation of each line given p, the length of the normal segment, and ϕ, the angle the normal segment makes with the positive x-axis.

40. $p = 2\sqrt{3}, \phi = \frac{\pi}{3}$ **41.** $p = 5, \phi = 90°$

42. $p = 3, \phi = \frac{2\pi}{3}$ **43.** $p = 4\sqrt{2}, \phi = 225°$

Write each equation in normal form. Then find the length of the normal and the angle it makes with the positive x-axis.

44. $7x + 3y - 8 = 0$ **45.** $6x = 4y - 5$

46. $9x = -5y + 3$ **47.** $x - 7y = -5$

OBJECTIVES AND EXAMPLES

Lesson 7-7 Find the distance from a point to a line.

Find the distance between $P(-1, 3)$ and the line with equation $-3x + 4y = -5$.

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}$$
$$= \frac{-3(-1) + 4(3) + 5}{-\sqrt{(-3)^2 + 4^2}}$$
$$= \frac{20}{-5}$$
$$= -4$$

REVIEW EXERCISES

Find the distance between the point with the given coordinates and the line with the given equation.

48. $(5, 6), 2x - 3y + 2 = 0$

49. $(-3, -4), 2y = -3x + 6$

50. $(-2, 4), 4y = 3x - 1$

51. $(21, 20), y = \frac{1}{3}x + 6$

OBJECTIVES AND EXAMPLES

Lesson 7-7 Write equations of lines that bisect angles formed by intersecting lines.

Find the equations of the lines that bisect the angles formed by the lines with equations $-4x + 3y = 2$ and $x + 2y = 1$.

$$d_1 = \frac{-4x_1 + 3y_1 - 2}{\sqrt{(-4)^2 + 3^2}}$$
$$= \frac{-4x_1 + 3y_1 - 2}{5}$$

$$d_2 = \frac{x_1 + 2y_1 - 1}{\sqrt{1^2 + 2^2}}$$
$$= \frac{x_1 + 2y_1 - 1}{\sqrt{5}}$$

The origin is in the interior of one of the obtuse angles formed by the given lines.

Bisector of the acute angle: $d_1 = -d_2$

$$\frac{-4x + 3y - 2}{5} = -\frac{x + 2y - 1}{\sqrt{5}}$$
$$\left(-4\sqrt{5} + 5\right)x + \left(3\sqrt{5} + 10\right)y - 2\sqrt{5} - 5 = 0$$

Bisector of the obtuse angle: $d_1 = d_2$

$$\frac{-4x + 3y - 2}{5} = \frac{x + 2y - 1}{\sqrt{5}}$$
$$\left(-4\sqrt{5} - 5\right)x + \left(3\sqrt{5} - 10\right)y - 2\sqrt{5} + 5 = 0$$

REVIEW EXERCISES

Find the distance between the parallel lines with the given equations.

52. $y = \frac{x}{3} - 6$
$y = \frac{x}{3} + 2$

53. $y = \frac{3}{4}x + 3$
$y = \frac{3}{4}x - \frac{1}{2}$

54. $x + y = 1$
$x + y = 5$

55. $2x - 3y + 3 = 0$
$y = \frac{2}{3}x - 2$

Find the equations of the lines that bisect the acute and obtuse angles formed by the lines with the given equations.

56. $y = -3x - 2$
$y = -\frac{x}{2} + \frac{3}{2}$

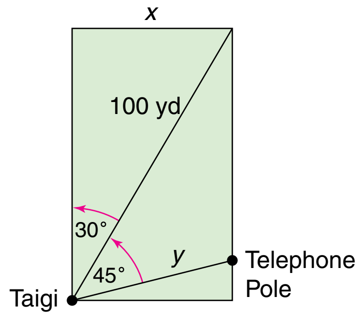
57. $-x + 3y - 2 = 0$
$y = \frac{3}{5}x + 3$

APPLICATIONS AND PROBLEM SOLVING

58. Physics While studying two different physics books, Hector notices that two different formulas are used to find the maximum height of a projectile. One formula is $h = \frac{v_0^2 \sin^2 \theta}{2g}$, and the other is $h = \frac{v_0^2 \tan^2 \theta}{2g \sec^2 \theta}$. Are these two formulas equivalent or is there an error in one of the books? Show your work. *(Lesson 7-2)*

59. Navigation Wanda hikes due east from the edge of the road in front of a lodge into the woods. She stops to eat lunch after she has hiked 1600 feet. In a coordinate system where the positive *y*-axis points north and the units are hundreds of feet, the equation of the road is $4x - 2y = 0$. How far is Wanda from the road? *(Lesson 7-7)*

60. Surveying Taigi is surveying a rectangular lot for a new office building. She measures the angle between one side of the lot and the line from her position to the opposite corner of the lot as 30°. She then measures the angle between that line and the line to a telephone pole on the edge of the lot as 45°. If Taigi stands 100 yards from the opposite corner of the lot, how far is she from the telephone pole? *(Lesson 7-3)*



ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. Give the measure θ of an angle such that trigonometric identities can be used to find exact values for $\sin \theta$, $\cos \theta$, and $\tan \theta$. Find exact values for $\sin \theta$, $\cos \theta$, and $\tan \theta$. Show your work.

2. Write an equation with $\sin x \tan x$ on one side and an expression containing one or more different trigonometric functions on the other side. Verify that your equation is an identity.

PORTFOLIO

Choose one of the identities you studied in this chapter. Explain why it is an important identity and how it is used.

Unit 2 interNET Project

THE CYBERCLASSROOM

That's as clear as mud!

- Search the Internet to find web sites that have lessons involving trigonometric identities. Find at least two types of identities that were not presented in Chapter 7.
- Select one of the types of identities and write at least two of your own examples or sample problems using what you learned on the Internet and in this textbook.
- Design your own web page presenting your examples. Use appropriate software to create the web page. Have another student critique your web page.

Additional Assessment See p. A62 for Chapter 7 practice test.

Geometry Problems — Triangles and Quadrilaterals

THE PRINCETON REVIEW

TEST-TAKING TIP

The third side of a triangle cannot be longer than the sum of the other two sides. The third side cannot be shorter than the difference between the other two sides.

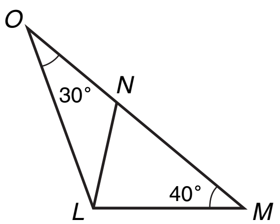
The ACT and SAT contain problems that deal with triangles, quadrilaterals, lengths, and angles. Be sure to review the properties of isosceles and equilateral triangles. Often several geometry concepts are combined in one problem.

Know the number of degrees in various figures.

- A straight angle measures 180°.
- A right angle measures 90°.
- The sum of the measures of the angles in a triangle is 180°.
- The sum of the measures of the angles in a quadrilateral is 360°.

ACT EXAMPLE

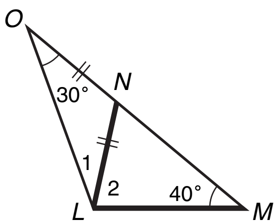
1. In the figure below, O, N, and M are collinear. If the lengths of $\overline{ON}$ and $\overline{NL}$ are the same, the measure of $\angle LON$ is 30°, and $\angle LMN$ is 40°, what is the measure of $\angle NLM$?



A 40° **B** 80° **C** 90° **D** 120° **E** 130°

HINT Look at *all* the triangles in a figure—large and small.

Solution Mark the angle you need to find. You may want to label the missing angles and congruent sides, as shown.



Since two sides of $\triangle ONL$ are the same length, it is isosceles. So $m\angle 1 = 30°$. Since the angles in any triangle total 180°, you can write the following equation for $\triangle OML$.

$180° = 30° + 40° + (30° + m\angle 2)$
$180° = 100° + m\angle 2$
$80° = m\angle 2$

The answer is choice **B.**

SAT EXAMPLE

2. If the average measure of two angles in a parallelogram is $y°$, what is the average degree measure of the other two angles?

A $180 - y$ **B** $180 - \frac{y}{2}$ **C** $360 - 2y$

D $360 - y$ **E** y

HINT Underline important words in the problem and the quantity you must find. Look at the answer choices.

Solution Look for key words in the problem—*average* and *parallelogram.* You need to find the average of two angles. The answer choices are expressions with the variable y. Recall that if the average of two numbers is y, then the sum of the numbers is $2y$.

So the sum of two angle measures is $2y$. Let the sum of the other two angle measures be $2x$. Find x.

The sum of the angle measures in a parallelogram is 360.

$360 = 2y + 2x$
$360 = 2(y + x)$
$180 = y + x$ *Divide each side by 2.*
$x = 180 - y$ *Solve for x.*

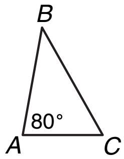
The answer is choice **A.**

SAT AND ACT PRACTICE

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

Multiple Choice

1. In the figure, the measure of $\angle A$ is 80°. If the measure of $\angle B$ is half the measure of $\angle A$, what is the measure of $\angle C$?



A 40° **B** 60° **C** 80°

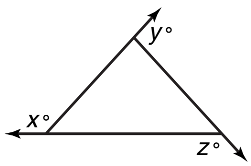
D 100° **E** 120°

2. At what point (x, y) do the two lines with equations $y = 2x - 2$ and $7x - 3y = 11$ intersect?

A $(5, 8)$ **B** $(8, 5)$ **C** $\left(\frac{5}{8}, -1\right)$

D $\left(\frac{5}{8}, 1\right)$ **E** $\left(\frac{25}{16}, \frac{9}{8}\right)$

3. In the figure below, $x + y + z =$



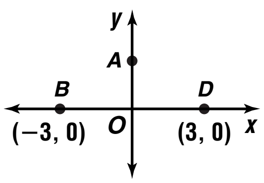
A 90 **B** 180 **C** 270 **D** 360 **E** 450

4. If $x + y = 90°$ and x and y are positive, then $\frac{\sin x}{\cos y} =$

A -1 **B** 0 **C** $\frac{1}{2}$ **D** 1

E It cannot be determined from the information given.

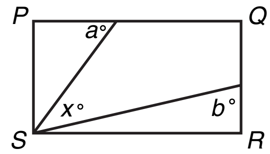
5. In the figure below, what is the sum of the slopes of $\overline{AB}$ and $\overline{AD}$?



A -1 **B** 0 **C** 1 **D** 6

E It cannot be determined from the information given.

6. In the rectangle $PQRS$ below, what is the sum $a + b$ in terms of x?



A $90 + x$

B $190 - x$

C $180 + x$

D $270 - x$

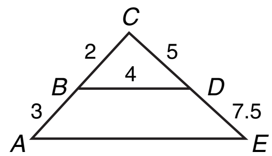
E $360 - x$

7. $$\frac{y - \frac{1}{y}}{1 - \frac{2}{y} + \frac{1}{y^2}} =$$

A $\frac{y^2 + y}{y - 1}$ **B** $\frac{y^2 - y}{(y + 1)^2}$ **C** $\frac{y^2 - y}{y + 1}$

D $\frac{y^2 + y}{(y - 1)^2}$ **E** $\frac{y}{y - 1}$

8. In the figure below, $\triangle ACE$ is similar to $\triangle BCD$. What is the length of $\overline{AE}$?



A 5

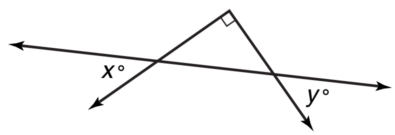
B 6

C 6.5

D 7

E 10

9. The number of degrees in the sum $x° + y°$ would be the same as the number of degrees in which type of angle?



A straight angle

B obtuse angle

C right angle

D acute angle

E It cannot be determined from the information given.

10. Grid-In A triangle has a base of 13 units, and the other two sides are congruent. If the side lengths are integers, what is the length of the shortest possible side?

interNET CONNECTION **SAT/ACT Practice** For additional test practice questions, visit: **www.amc.glencoe.com**