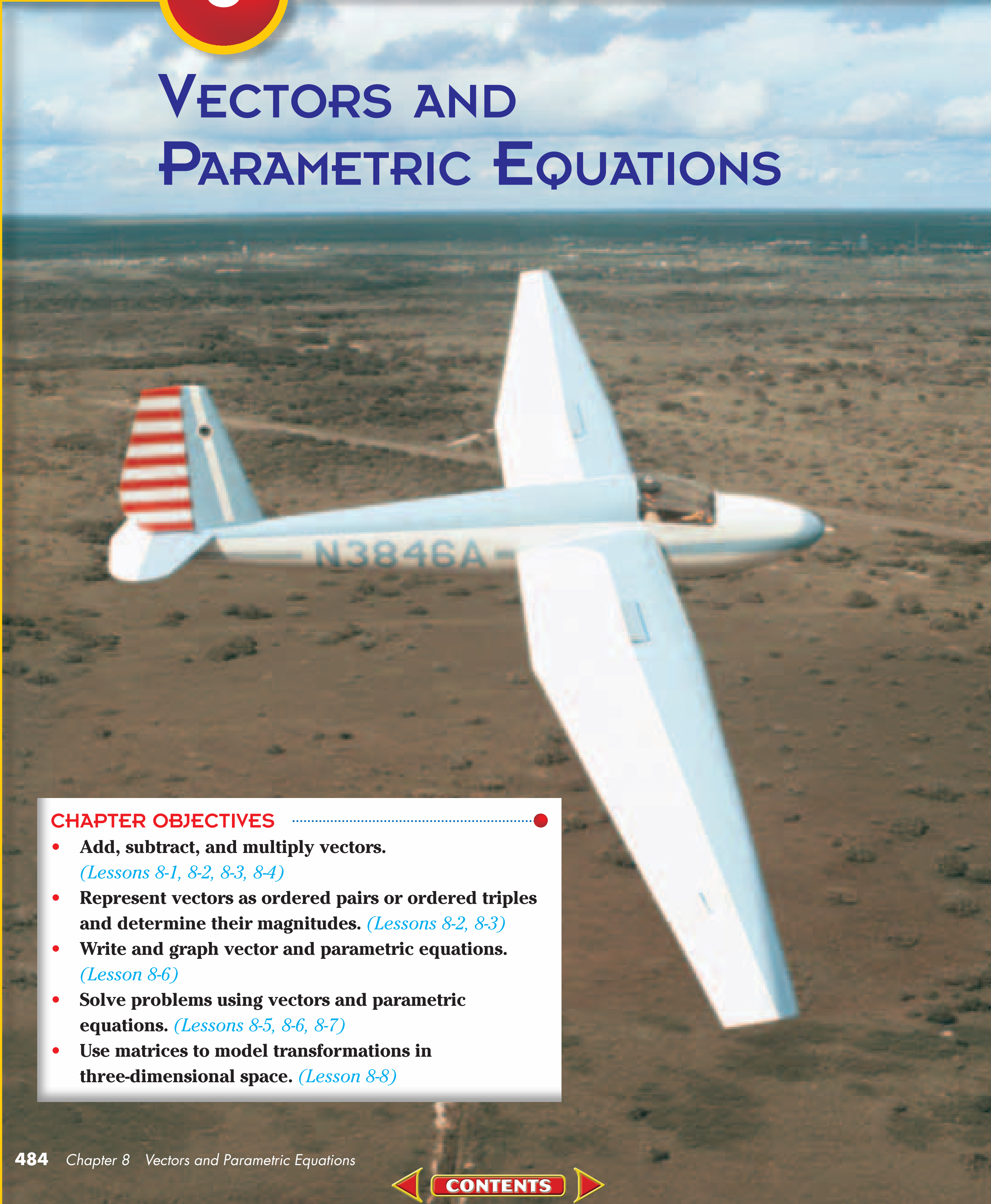
Chapter 8

Vectors and Parametric Equations



CHAPTER OBJECTIVES

- **Add, subtract, and multiply vectors.** *(Lessons 8-1, 8-2, 8-3, 8-4)*
- **Represent vectors as ordered pairs or ordered triples and determine their magnitudes.** *(Lessons 8-2, 8-3)*
- **Write and graph vector and parametric equations.** *(Lesson 8-6)*
- **Solve problems using vectors and parametric equations.** *(Lessons 8-5, 8-6, 8-7)*
- **Use matrices to model transformations in three-dimensional space.** *(Lesson 8-8)*

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8-1

Geometric Vectors

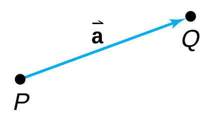
OBJECTIVES
- Find equal, opposite, and parallel vectors.
- Add and subtract vectors geometrically.

Real World Application

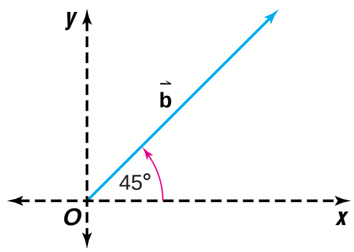
AERONAUTICS An advanced glider known as a sailplane, has high maneuverability and glide capabilities. The Ventus 2B sailplane placed first at the World Soaring Contest in New Zealand. At one competition, a sailplane traveled forward at a rate of 8 m/s, and it descended at a rate of 4 m/s. *A problem involving this situation will be solved in Example 3.*

The velocity of a sailplane can be represented mathematically by a **vector.** A vector is a quantity that has both *magnitude* and *direction*. A vector is represented geometrically by a directed line segment.

A directed line segment with an initial point at P and a terminal point at Q is shown at the right. The length of the line segment represents the **magnitude** of the vector. The direction of the arrowhead indicates the direction of the vector. The vectors can be denoted by $\vec{\mathbf{a}}$ or $\overrightarrow{PQ}$. The magnitude of $\vec{\mathbf{a}}$ is denoted by $|\vec{\mathbf{a}}|$.

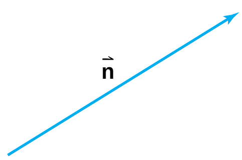


If a vector has its initial point at the origin, it is in **standard position.** The **direction** of the vector is the directed angle between the positive x-axis and the vector. The direction of $\vec{\mathbf{b}}$ is 45°.

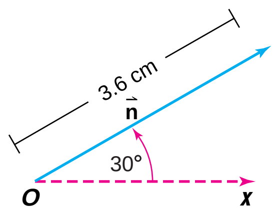


If both the initial point and the terminal point are at the origin, the vector is the **zero vector** and is denoted by $\vec{\mathbf{0}}$. The magnitude of the zero vector is 0, and it can be in any direction.

Example 1 **Use a ruler and protractor to determine the magnitude (in centimeters) and the direction of $\vec{\mathbf{n}}$.**



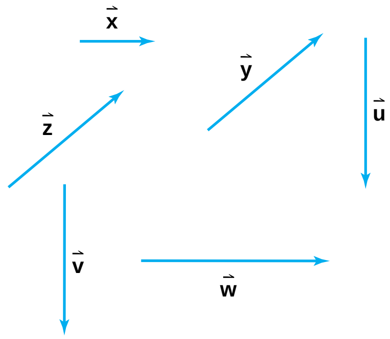
Sketch the vector in standard position and measure the magnitude and direction. The magnitude is 3.6 centimeters, and the direction is 30°.



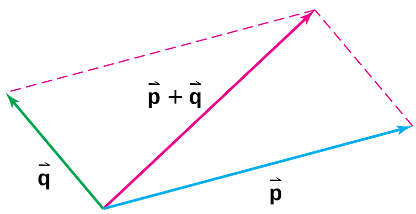
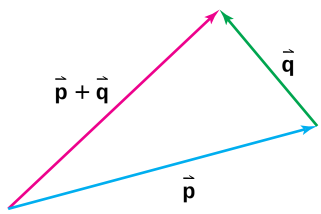
Two vectors are **equal** if and only if they have the same direction and the same magnitude.

Six vectors are shown at the right.

- $\vec{z}$ and $\vec{y}$ are equal since they have the same direction and $|\vec{z}| = |\vec{y}|$.
- $\vec{v}$ and $\vec{u}$ are equal.
- $\vec{x}$ and $\vec{w}$ have the same direction but $|\vec{x}| \neq |\vec{w}|$, so $\vec{x} \neq \vec{w}$.
- $|\vec{v}| = |\vec{y}|$, but they have different directions, so they are not equal.



The sum of two or more vectors is called the **resultant** of the vectors. The resultant can be found using either the *parallelogram method* or the *triangle method.*

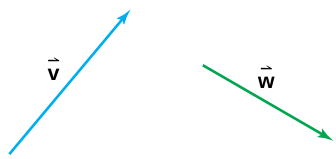
Parallelogram Method	Triangle Method
Draw the vectors so that their initial points coincide. Then draw lines to form a complete parallelogram. The diagonal from the initial point to the opposite vertex of the parallelogram is the resultant.  *The parallelogram method cannot be used to find the sum of a vector and itself.*	Draw the vectors one after another, placing the initial point of each successive vector at the terminal point of the previous vector. Then draw the resultant from the initial point of the first vector to the terminal point of the last vector.  *This method is also called the tip-to-tail method.*

Example 2 **Find the sum of $\vec{v}$ and $\vec{w}$ using:**

a. the parallelogram method.

b. the triangle method.

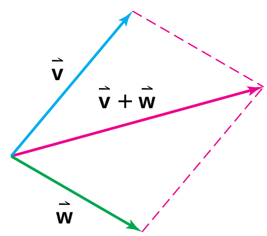
c. Compare the resultants found in both methods.



a. Copy $\vec{v}$ then copy $\vec{w}$ placing the initial points together.

Form a parallelogram that has $\vec{v}$ and $\vec{w}$ as two of its sides. Draw dashed lines to represent the other two sides.

The resultant is the vector from the vertex of $\vec{v}$ and $\vec{w}$ to the opposite vertex of the parallelogram.



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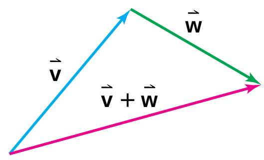
b. Copy $\vec{\mathbf{v}}$, then copy $\vec{\mathbf{w}}$ so that the initial point of $\vec{\mathbf{w}}$ is on the terminal point of $\vec{\mathbf{v}}$. (The tail of $\vec{\mathbf{v}}$ connects to the tip of $\vec{\mathbf{w}}$.)

The resultant is the vector from the initial point of $\vec{\mathbf{v}}$ to the terminal point of $\vec{\mathbf{w}}$.



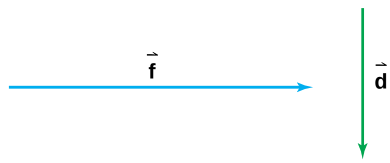
c. Use a ruler and protractor to measure the magnitude and direction of each resultant. The resultants of both methods have magnitudes of 3.5 centimeters and directions of 20°. So, the resultants found in both methods are equal.

Vectors can be used to solve real-world applications.

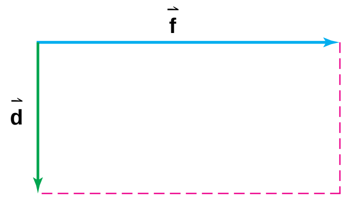
Example 3 Real World Application

AERONAUTICS **Refer to the application at the beginning of the lesson. At the European Championships, a sailplane traveled forward at 8 m/s and descended at 4 m/s. Determine the magnitude of the resultant velocity of the sailplane.**

Let 1 centimeter represent 2 m/s. Draw two vectors, $\vec{\mathbf{f}}$ and $\vec{\mathbf{d}}$, to represent the forward velocity and the descending velocity of the sailplane, respectively.

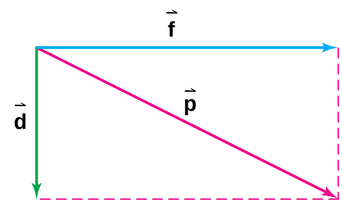


Use the parallelogram method. Copy $\vec{\mathbf{f}}$. Then copy $\vec{\mathbf{d}}$, placing the initial point at the initial point of $\vec{\mathbf{f}}$. Draw dashed lines to represent the other two sides of the parallelogram.



The resultant velocity of the sailplane is the vector $\vec{\mathbf{p}}$ from the vertex of $\vec{\mathbf{f}}$ and $\vec{\mathbf{d}}$ to the opposite vertex of the parallelogram.

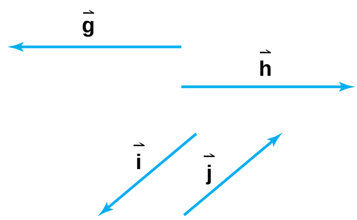
Measure the resultant, 4.5 centimeters. Multiply the resultant's magnitude by 2 to determine the magnitude of the resultant velocity of the sailplane. *Why?*



4.5 cm × 2 m/s = 9 m/s

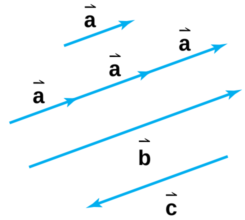
The sailplane is moving at 9 m/s.

Two vectors are **opposites** if they have the same magnitude and opposite directions. In the diagram, $\vec{\mathbf{g}}$ and $\vec{\mathbf{h}}$ are opposites, as are $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$. The opposite of $\vec{\mathbf{g}}$ is denoted by $-\vec{\mathbf{g}}$. You can use opposite vectors to subtract vectors. To find $\vec{\mathbf{g}} - \vec{\mathbf{h}}$, find $\vec{\mathbf{g}} + (-\vec{\mathbf{h}})$.

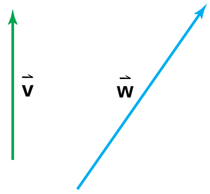


A quantity with only magnitude is called a **scalar quantity.** Examples of scalars include mass, length, time, and temperature. The numbers used to measure scalar quantities are called **scalars.**

The product of a scalar k and a vector $\vec{\mathbf{a}}$ is a vector with the same direction as $\vec{\mathbf{a}}$ and a magnitude of $k|\vec{\mathbf{a}}|$, if $k > 0$. If $k < 0$, the vector has the opposite direction of $\vec{\mathbf{a}}$ and a magnitude of $k|\vec{\mathbf{a}}|$. In the figure at the right, $\vec{\mathbf{b}} = 3\vec{\mathbf{a}}$ and $\vec{\mathbf{c}} = -2\vec{\mathbf{a}}$.



Example 4 **Use the triangle method to find $2\vec{\mathbf{v}} - \frac{1}{2}\vec{\mathbf{w}}$.**



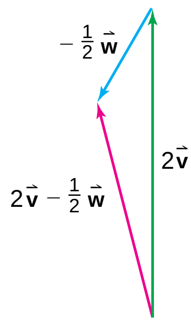
Rewrite the expression as a sum.

$$2\vec{\mathbf{v}} - \frac{1}{2}\vec{\mathbf{w}} = 2\vec{\mathbf{v}} + \left(-\frac{1}{2}\vec{\mathbf{w}}\right)$$

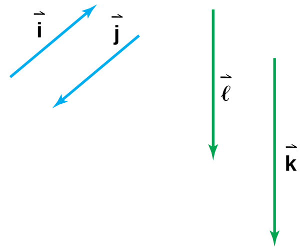
Draw a vector twice the magnitude of $\vec{\mathbf{v}}$ to represent $2\vec{\mathbf{v}}$. Draw a vector with the opposite direction to $\vec{\mathbf{w}}$ and half its magnitude to represent $-\frac{1}{2}\vec{\mathbf{w}}$.

Place the initial point of $-\frac{1}{2}\vec{\mathbf{w}}$ on the terminal point of $2\vec{\mathbf{v}}$. *(Tip-to-tail method)*

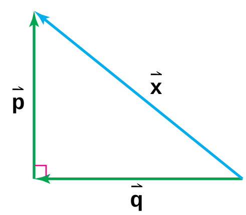
Then $2\vec{\mathbf{v}} - \frac{1}{2}\vec{\mathbf{w}}$ has the initial point of $2\vec{\mathbf{v}}$ and the terminal point of $-\frac{1}{2}\vec{\mathbf{w}}$.



Two or more vectors are **parallel** if and only if they have the same or opposite directions. Vectors $\vec{\mathbf{i}}$ and $\vec{\mathbf{j}}$ have opposite directions and they are parallel. Vectors $\vec{\boldsymbol{\ell}}$ and $\vec{\mathbf{k}}$ are parallel and have the same direction.



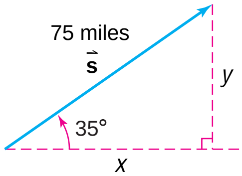
Two or more vectors whose sum is a given vector are called **components** of the given vector. Components can have any direction. Often it is useful to express a vector as the sum of two perpendicular components. In the figure at the right, $\vec{\mathbf{p}}$ is the vertical component of $\vec{\mathbf{x}}$, and $\vec{\mathbf{q}}$ is the horizontal component of $\vec{\mathbf{x}}$.



Example 5 Real World Application

NAVIGATION **A ship leaving port sails for 75 miles in a direction 35° north of due east. Find the magnitude of the vertical and horizontal components.**

Draw $\vec{s}$. Then draw a horizontal vector through the initial point of $\vec{s}$. Label the resulting angle 35°. Draw a vertical vector through the terminal point of $\vec{s}$. The vectors will form a right triangle, so you can use the sine and cosine ratios to find the magnitude of the components.



$$\sin 35° = \frac{y}{75} \qquad \cos 35° = \frac{x}{75}$$

$$y = 75 \sin 35° \qquad x = 75 \cos 35°$$

$$y \approx 43 \qquad x \approx 61$$

The magnitude of the vertical component is approximately 43 miles, and the magnitude of the horizontal component is approximately 61 miles.

Vectors are used in physics to represent velocity, acceleration, and forces acting upon objects.

Example 6 Real World Application

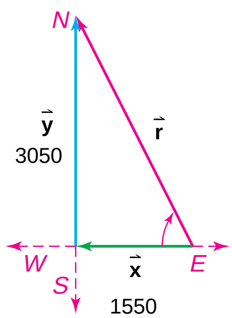
CONSTRUCTION **A piling for a high-rise building is pushed by two bulldozers at exactly the same time. One bulldozer exerts a force of 1550 pounds in a westerly direction. The other bulldozer pushes the piling with a force of 3050 pounds in a northerly direction.**

a. What is the magnitude of the resultant force upon the piling, to the nearest ten pounds?

b. What is the direction of the resulting force upon the piling, to the nearest ten pounds?

a. Let $\vec{x}$ represent the force for bulldozer 1.
Let $\vec{y}$ represent the force for bulldozer 2.

Draw the resultant $\vec{r}$. This represents the total force acting upon the piling. Use the Pythagorean Theorem to find the magnitude of the resultant.



$$c^2 = a^2 + b^2$$

$$\vec{r}^{\,2} = 1550^2 + 3050^2$$

$$|\vec{r}| = \sqrt{(1550)^2 + (3050)^2}$$

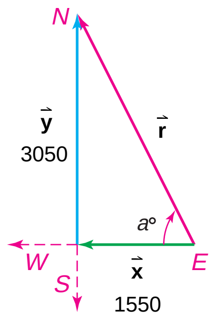
$$|\vec{r}| \approx 3421$$

The magnitude of the resultant force upon the piling is about 3420 pounds.

(continued on the next page)

b. Let a be the measure of the angle $\vec{r}$ makes with $\vec{x}$.

The direction of the resultant can be found by using the tangent ratio.



$$\tan a = \frac{|\vec{y}|}{|\vec{x}|}$$

$$\tan a = \frac{3050}{1550}$$

$$a = 63°$$ *Take $\tan^{-1}$ of each side.*

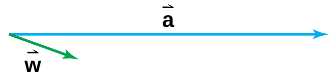
The resultant makes an angle of 63° with $\vec{x}$. The direction of the resultant force upon the piling is 90 − 63 or 27° west of north. *Direction of vectors is often expressed in terms of position relative to due north, due south, due east, or due west, or as a navigational angle measures clockwise from due north.*

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

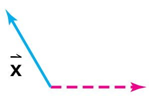
1. **Draw** a diagram showing the resultant of two vectors, and describe how you obtained it.
2. **Compare** a line segment and a vector.
3. **Describe** a real-world situation involving vectors that could be represented by the diagram at the right.

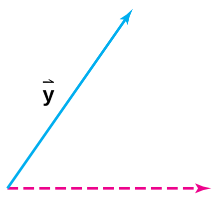


4. **Tell** whether $\overrightarrow{RS}$ is the same as $\overrightarrow{SR}$. Explain.

Guided Practice

Use a ruler and a protractor to determine the magnitude (in centimeters) and direction of each vector.

5. 

6. 

7. 

Use $\vec{x}$, $\vec{y}$, and $\vec{z}$ above to find the magnitude and direction of each resultant.

8. $\vec{x} + \vec{y}$
9. $\vec{x} - \vec{y}$
10. $4\vec{y} + \vec{z}$
11. the difference of a vector twice as long as $\vec{z}$ and a vector one third the magnitude of $\vec{x}$
12. Find the magnitude of the horizontal and vertical components of $\vec{y}$.
13. **Aviation** An airplane is flying due west at a velocity of 100 m/s. The wind is blowing out of the south at 5 m/s.
    a. Draw a labeled diagram of the situation.
    b. What is the magnitude of airplane's resultant velocity?

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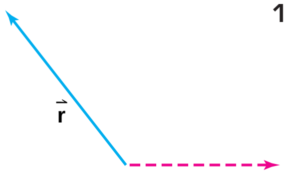
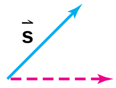
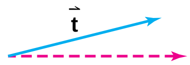
EXERCISES

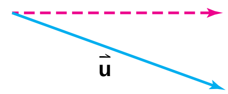
Practice

Use a ruler and a protractor to determine the magnitude (in centimeters) and direction of each vector.

14. 

15. 

16. 

17. 

Use $\vec{r}$, $\vec{s}$, $\vec{t}$, and $\vec{u}$ above to find the magnitude and direction of each resultant.

18. $\vec{r} + \vec{s}$	19. $\vec{s} + \vec{t}$	20. $\vec{s} + \vec{u}$
21. $\vec{u} - \vec{r}$	22. $\vec{r} - \vec{t}$	23. $2\vec{r}$
24. $3\vec{s}$	25. $3\vec{u} - 2\vec{s}$	26. $\vec{r} + \vec{t} + \vec{u}$
27. $\vec{r} + \vec{s} - \vec{u}$	28. $2\vec{s} + \vec{u} - \frac{1}{2}\vec{r}$	29. $\vec{r} - 2\vec{t} - \vec{s} + 3\vec{u}$

30. three times $\vec{t}$ and twice $\vec{u}$

Find the magnitude of the horizontal and vertical components of each vector shown for Exercises 14–17.

31. $\vec{r}$	32. $\vec{s}$	33. $\vec{t}$	34. $\vec{u}$

35. The magnitude of $\vec{m}$ is 29.2 meters, and the magnitude of $\vec{n}$ is 35.2 meters. If $\vec{m}$ and $\vec{n}$ are perpendicular, what is the magnitude of their sum?

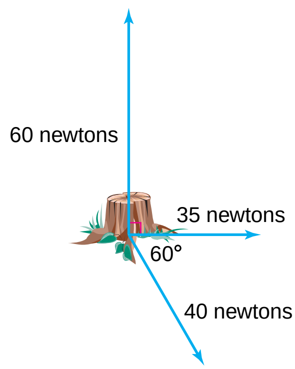
36. In the parallelogram drawn for the parallelogram method, what does the diagonal between the two terminal points of the vectors represent? Explain your answer.

37. Is addition of vectors commutative? Justify your answer. (*Hint:* Find the sum of two vectors, $\vec{r} + \vec{s}$ and $\vec{s} + \vec{r}$, using the triangle method.)

Applications and Problem Solving

Real World Application

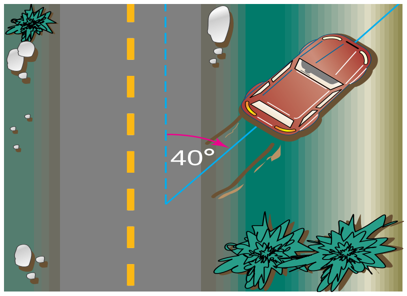
38. **Physics** Three workers are pulling on ropes attached to a tree stump as shown in the diagram. Find the magnitude and direction of the resultant force on the tree. *A newton (N) is a unit of force used in physics. A force of one newton will accelerate a one-kilogram mass at a rate of one meter per second squared.*



39. **Critical Thinking** Does $|\vec{a} + \vec{b}|$ always, sometimes, or never equal $|\vec{a}| + |\vec{b}|$? Draw a diagram to justify your answer.

40. **Toys** Belkis is pulling a toy by exerting a force of 1.5 newtons on a string attached to the toy.
    a. The string makes an angle of 52° with the floor. Find the vertical and horizontal components of the force.
    b. If Belkis raises the string so that it makes a 78° angle with the floor, what are the magnitudes of the horizontal and vertical components of the force?

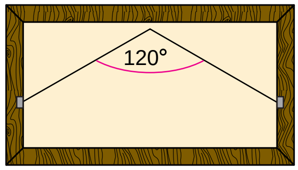
41. **Police Investigation** Police officer Patricia Malloy is investigating an automobile accident. A car slid off an icy road at a 40° angle with the center line of the road with an initial velocity of 47 miles per hour. Use the drawing to determine the initial horizontal and vertical components of the car's velocity.



42. **Critical Thinking** If $\vec{\mathbf{a}}$ is a vector and k is a scalar, is it possible that $\vec{\mathbf{a}} = k\vec{\mathbf{a}}$? Under what conditions is this true?

43. **Ranching** Matsuko has attached two wires to a corner fence post to complete a horse paddock. The wires make an angle of 90° with each other. If each wire pulls on the post with a force of 50 pounds, what is the resultant force acting on the pole?

44. **Physics** Mrs. Keaton is the director of a local art gallery. She needs to hang a painting weighing 24 pounds with a wire whose parts form a 120° angle with each other. Find the pull on each part of the wire.



Mixed Review

45. Find the equation of the line that bisects the acute angle formed by the lines $x - y + 2 = 0$ and $y - 5 = 0$. *(Lesson 7-7)*

46. Verify that $\csc \theta \cos \theta \tan \theta = 1$ is an identity. *(Lesson 7-2)*

47. Find the values of θ for which $\tan \theta = 1$ is true. *(Lesson 6-7)*

48. Use the graphs of the sine and cosine functions to find the values of x for which $\sin x + \cos x = -1$. *(Lesson 6-5)*

49. **Geometry** The base angles of an isosceles triangle measure 18°29′, and the altitude to the base is 5 centimeters long. Find the length of the base and the lengths of the congruent sides. *(Lesson 5-2)*

50. **Manufacturing** A manufacturer produces packaging boxes for other companies. The largest box they currently produce has a height two times its width and a length one more that its width. They wish to produce a larger packaging box where the height will be increased by 2 feet and the width and length will be increased by 1 foot. The volume of the new box is 160 cubic feet. Find the dimensions of the original box. *(Lesson 4-4)*

51. Determine the equations of the vertical and horizontal asymptotes, if any, of $g(x) = \dfrac{x + 2}{(x - 1)(x + 3)}$. *(Lesson 3-7)*

52. **SAT/ACT Practice** **Grid-In** Three times the least of three consecutive odd integers is three greater than two times the greatest. Find the greatest of the three integers.

Extra Practice See p. A40.

8-2 Algebraic Vectors

OBJECTIVES
- Find ordered pairs that represent vectors.
- Add, subtract, multiply, and find the magnitude of vectors algebraically.

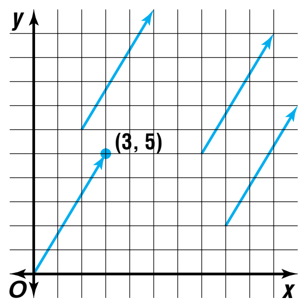
Real World Application

EMERGENCY MEDICINE Paramedics Paquita Gonzalez and Trevor Howard are moving a person on a stretcher. Ms. Gonzalez is pushing the stretcher with a force of 135 newtons at 58° with the horizontal, while Mr. Howard is pulling the stretcher with a force of 214 newtons at 43° with the horizontal. What is the magnitude of the force exerted on the stretcher? *This problem will be solved in Example 3.*

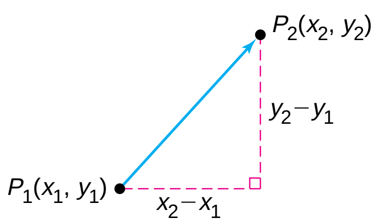
We can find the magnitude and direction of the resultant by drawing vectors to represent the forces on the stretcher. However, drawings can be inaccurate and quite confusing for more complex systems of vectors. In cases where more accuracy is necessary or where the system is more complicated, we can perform operations on vectors algebraically.

Vectors can be represented algebraically using ordered pairs of real numbers. For example, the ordered pair $\langle 3, 5\rangle$ can represent a vector in standard position. That is, its initial point is at the origin and its terminal point is at (3, 5). You can think of this vector as the resultant of a horizontal vector with a magnitude of 3 units and a vertical vector with magnitude of 5 units.

Since vectors with the same magnitude and direction are equal, many vectors can be represented by the same ordered pair. Each vector on the graph can be represented by the ordered pair $\langle 3, 5\rangle$. The initial point of a vector can be any point in the plane. In other words, a vector does not have to be in standard position to be expressed algebraically.



Assume that P_1 and P_2 are any two distinct points in the coordinate plane. Drawing the horizontal and vertical components of $\overrightarrow{P_1P_2}$ yields a right triangle. So, the magnitude of $\overrightarrow{P_1P_2}$ can be found by using the Pythagorean Theorem.



Representation of a Vector as an Ordered Pair	Let $P_1(x_1, y_1)$ be the initial point of a vector and $P_2(x_2, y_2)$ be the terminal point. The ordered pair that represents $\overrightarrow{P_1P_2}$ is $\langle x_2 - x_1, y_2 - y_1 \rangle$. Its magnitude is given by $\lvert \overrightarrow{P_1P_2} \rvert = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

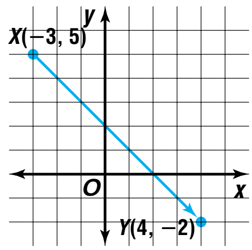
Example 1 **Write the ordered pair that represents the vector from $X(-3, 5)$ to $Y(4, -2)$. Then find the magnitude of $\overrightarrow{XY}$.**

First, represent $\overrightarrow{XY}$ as an ordered pair.

$\overrightarrow{XY} = \langle 4 - (-3), -2 - 5 \rangle$ or $\langle 7, -7 \rangle$

Then, determine the magnitude of $\overrightarrow{XY}$.

$$\lvert \overrightarrow{XY} \rvert = \sqrt{[4 - (-3)]^2 + (-2 - 5)^2}$$
$$= \sqrt{7^2 + 7^2}$$
$$= \sqrt{98} \text{ or } 7\sqrt{2}$$



$\overrightarrow{XY}$ is represented by the ordered pair $\langle 7, -7 \rangle$ and has a magnitude of $7\sqrt{2}$ units.

When vectors are represented by ordered pairs, they can be easily added, subtracted, or multiplied by a scalar. The rules for these operations on vectors are similar to those for matrices. In fact, vectors in a plane can be represented as row matrices of dimension 1×2.

Vector Operations	The following operations are defined for $\vec{a}\ \langle a_1, a_2 \rangle$, $\vec{b}\ \langle b_1, b_2 \rangle$, and any real number k. Addition: $\vec{a} + \vec{b} = \langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle = \langle a_1 + b_1, a_2 + b_2 \rangle$ Subtraction: $\vec{a} - \vec{b} = \langle a_1, a_2 \rangle - \langle b_1, b_2 \rangle = \langle a_1 - b_1, a_2 - b_2 \rangle$ Scalar multiplication: $k\vec{a} = k\langle a_1, a_2 \rangle = \langle ka_1, ka_2 \rangle$

Example 2 **Let $\vec{m} = \langle 5, -7 \rangle$, $\vec{n} = \langle 0, 4 \rangle$, and $\vec{p} = \langle -1, 3 \rangle$. Find each of the following.**

a. $\vec{m} + \vec{p}$

$$\vec{m} + \vec{p} = \langle 5, -7 \rangle + \langle -1, 3 \rangle$$
$$= \langle 5 + (-1), -7 + 3 \rangle$$
$$= \langle 4, -4 \rangle$$

b. $\vec{m} - \vec{n}$

$$\vec{m} - \vec{n} = \langle 5, -7 \rangle - \langle 0, 4 \rangle$$
$$= \langle 5 - 0, -7 - 4 \rangle$$
$$= \langle 5, -11 \rangle$$

c. $7\vec{p}$

$$7\vec{p} = 7\langle -1, 3 \rangle$$
$$= \langle 7 \cdot (-1), 7 \cdot 3 \rangle$$
$$= \langle -7, 21 \rangle$$

d. $2\vec{m} + 3\vec{n} - \vec{p}$

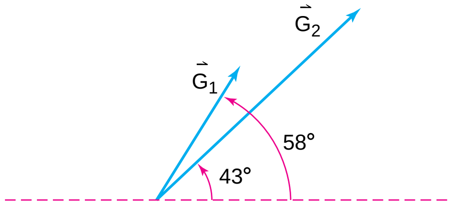
$$2\vec{m} + 3\vec{n} - \vec{p} = 2\langle 5, -7 \rangle + 3\langle 0, 4 \rangle - \langle -1, 3 \rangle$$
$$= \langle 10, -14 \rangle + \langle 0, 12 \rangle - \langle -1, 3 \rangle$$
$$= \langle 11, -5 \rangle$$

If we write the vectors described in the application at the beginning of the lesson as ordered pairs, we can find the resultant vector easily using vector addition. Using ordered pairs to represent vectors allows for a more accurate solution than using geometric representations.

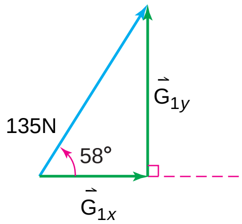
Example 3 Real World Application

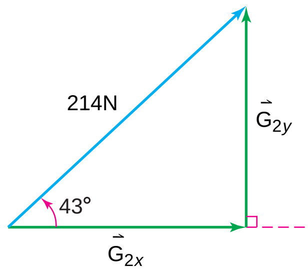
EMERGENCY MEDICINE **Refer to the application at the beginning of the lesson. What is the magnitude of the force exerted on the stretcher?**

Draw a diagram of the situation. Let $\vec{\mathbf{G}}_1$ represent the force Ms. Gonzalez exerts, and let $\vec{\mathbf{G}}_2$ represent the force Mr. Howard exerts.



Write each vector as an ordered pair by finding its horizontal and vertical components. Let $\vec{\mathbf{G}}_{1x}$ and $\vec{\mathbf{G}}_{1y}$ represent the x- and y-components of $\vec{\mathbf{G}}_1$. Let $\vec{\mathbf{G}}_{2x}$ and $\vec{\mathbf{G}}_{2y}$ represent the x- and y-components of $\vec{\mathbf{G}}_2$.





$$\cos 58° = \frac{|\vec{\mathbf{G}}_{1x}|}{135}$$
$$|\vec{\mathbf{G}}_{1x}| = 135 \cos 58°$$
$$|\vec{\mathbf{G}}_{1x}| \approx 71.5$$
$$\sin 58° = \frac{|\vec{\mathbf{G}}_{1y}|}{135}$$
$$|\vec{\mathbf{G}}_{1y}| = 135 \sin 58°$$
$$|\vec{\mathbf{G}}_{1y}| \approx 114.5$$
$$\vec{\mathbf{G}}_1 \approx \langle 71.5, 114.5 \rangle$$

$$\cos 43° = \frac{|\vec{\mathbf{G}}_{2x}|}{214}$$
$$|\vec{\mathbf{G}}_{2x}| = 214 \cos 43°$$
$$|\vec{\mathbf{G}}_{2x}| \approx 156.5$$
$$\sin 43° = \frac{|\vec{\mathbf{G}}_{2y}|}{214}$$
$$|\vec{\mathbf{G}}_{2y}| = 214 \sin 43°$$
$$|\vec{\mathbf{G}}_{2y}| \approx 145.9$$
$$\vec{\mathbf{G}}_2 \approx \langle 156.5, 145.9 \rangle$$

Find the sum of the vectors.

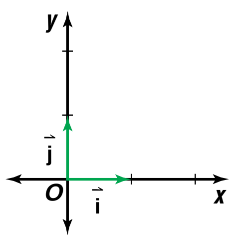
$$\vec{\mathbf{G}}_1 + \vec{\mathbf{G}}_2 \approx \langle 71.5, 114.5 \rangle + \langle 156.5, 145.9 \rangle$$
$$\approx \langle 228.0, 260.4 \rangle$$

The net force on the stretcher is the magnitude of the sum.

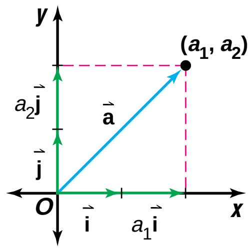
$$|\vec{\mathbf{G}}_1 + \vec{\mathbf{G}}_2| \approx \sqrt{(228.0)^2 + (260.4)^2} \text{ or about } 346$$

The net force on the stretcher is about 346 newtons.
To convert newtons to pounds, divide the number of newtons by 4.45. So, the net force on the stretcher can also be expressed as 346 ÷ 4.45 or 77.8 pounds.

A vector that has a magnitude of one unit is called a **unit vector.** A unit vector in the direction of the positive x-axis is represented by $\vec{\mathbf{i}}$, and a unit vector in the direction of the positive y-axis is represented by $\vec{\mathbf{j}}$. So, $\vec{\mathbf{i}} = \langle 1, 0 \rangle$ and $\vec{\mathbf{j}} = \langle 0, 1 \rangle$.



Any vector $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ can be expressed as $a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}}$.



$$
\begin{aligned}
a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} &= a_1\langle 1, 0\rangle + a_2\langle 0, 1\rangle && \textit{$\vec{i} = \langle 1, 0\rangle$ and $\vec{j} = \langle 0, 1\rangle$}\\
&= \langle a_1, 0\rangle + \langle 0, a_2\rangle && \textit{Scalar product}\\
&= \langle a_1 + 0, 0 + a_2\rangle && \textit{Addition of vectors}\\
&= \langle a_1, a_2\rangle
\end{aligned}
$$

Since $\langle a_1, a_2\rangle = \vec{\mathbf{a}}$, $a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} = \vec{\mathbf{a}}$. Therefore, any vector that is represented by an ordered pair can also be written as the sum of unit vectors. The zero vector can be represented as $\vec{\mathbf{0}} = \langle 0, 0\rangle = 0\vec{\mathbf{i}} + 0\vec{\mathbf{j}}$.

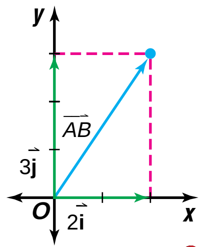
Example 4 **Write $\overrightarrow{AB}$ as the sum of unit vectors for $A(4, -1)$ and $B(6, 2)$.**

First, write $\overrightarrow{AB}$ as an ordered pair.

$$
\begin{aligned}
\overrightarrow{AB} &= \langle 6 - 4, 2 - (-1)\rangle\\
&= \langle 2, 3\rangle
\end{aligned}
$$

Then, write $\overrightarrow{AB}$ as the sum of unit vectors.

$$\overrightarrow{AB} = 2\vec{\mathbf{i}} + 3\vec{\mathbf{j}}$$

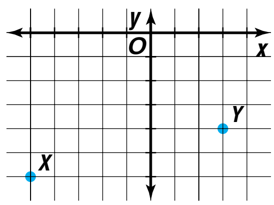


CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Find a counterexample** If $|\vec{\mathbf{a}}| = 10$ and $|\vec{\mathbf{b}}| = 10$, then $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are equal vectors.
2. **Describe** how to find $|\overrightarrow{XY}|$ using the graph at the right.



3. **You Decide** Lina showed Jacqui the representation of $\langle 2, -5\rangle$ as a unit vector as follows.

$$
\begin{aligned}
\langle 2, -5\rangle &= \langle 2, 0\rangle + \langle 0, -5\rangle\\
&= (5)\langle 1, 0\rangle + (-2)\langle 0, 1\rangle\\
&= 5\vec{\mathbf{i}} + (-2)\vec{\mathbf{j}}\\
&= 5\vec{\mathbf{i}} - 2\vec{\mathbf{j}}
\end{aligned}
$$

Jacqui said that her work was incorrect. Who is right? Explain.

Guided Practice

Write the ordered pair that represents $\overrightarrow{MP}$. Then find the magnitude of $\overrightarrow{MP}$.

4. $M(2, -1)$, $P(-3, 4)$
5. $M(5, 6)$, $P(0, 5)$
6. $M(-19, 4)$, $P(4, 0)$

Find an ordered pair to represent $\vec{t}$ in each equation if $\vec{u} = \langle -1, 4\rangle$ and $\vec{v} = \langle 3, -2\rangle$.

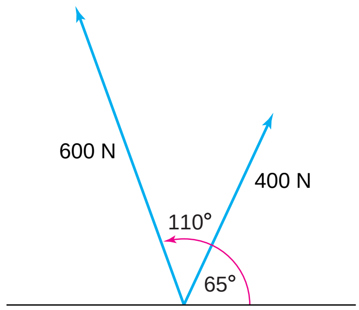
7. $\vec{\mathbf{t}} = \vec{\mathbf{u}} + \vec{\mathbf{v}}$
8. $\vec{\mathbf{t}} = \frac{1}{2}\vec{\mathbf{u}} - \vec{\mathbf{v}}$
9. $\vec{\mathbf{t}} = 4\vec{\mathbf{u}} + 6\vec{\mathbf{v}}$
10. $\vec{\mathbf{t}} = -8\vec{\mathbf{u}}$

Find the magnitude of each vector. Then write each vector as the sum of unit vectors.

11. $\langle 8, -6\rangle$

12. $\langle -7, -5\rangle$

13. Construction The Walker family is building a cabin for vacationing. Mr. Walker and his son Terrell have erected a scaffold to stand on while they build the walls of the cabin. As they stand on the scaffold Terrell pulls on a rope attached to a support beam with a force of 400 newtons (N) at an angle of 65° with the horizontal. Mr. Walker pulls with a force of 600 newtons at an angle of 110° with the horizontal. What is the magnitude of the combined force they exert on the log?



EXERCISES

Practice

Write the ordered pair that represents $\overrightarrow{YZ}$. Then find the magnitude of $\overrightarrow{YZ}$.

14. $Y(4, 2), Z(2, 8)$

15. $Y(-5, 7), Z(-1, 2)$

16. $Y(-2, 5), Z(1, 3)$

17. $Y(5, 4), Z(0, -3)$

18. $Y(3, 1), Z(0, 4)$

19. $Y(-4, 12), Z(1, 19)$

20. $Y(5, 0), Z(7, 6)$

21. $Y(14, -23), Z(23, -14)$

22. Find an ordered pair that represents the vector from $A(31, -33)$ to $B(36, -45)$. Then find the magnitude of $\overrightarrow{AB}$.

Find an ordered pair to represent $\vec{a}$ in each equation if $\vec{b} = \langle 6, 3\rangle$ and $\vec{c} = \langle -4, 8\rangle$.

23. $\vec{\mathbf{a}} = \vec{\mathbf{b}} + \vec{\mathbf{c}}$

24. $\vec{\mathbf{a}} = 2\vec{\mathbf{b}} + \vec{\mathbf{c}}$

25. $\vec{\mathbf{a}} = \vec{\mathbf{b}} + 2\vec{\mathbf{c}}$

26. $\vec{\mathbf{a}} = 2\vec{\mathbf{b}} + 3\vec{\mathbf{c}}$

27. $\vec{\mathbf{a}} = -\vec{\mathbf{b}} + 4\vec{\mathbf{c}}$

28. $\vec{\mathbf{a}} = \vec{\mathbf{b}} - 2\vec{\mathbf{c}}$

29. $\vec{\mathbf{a}} = 3\vec{\mathbf{b}}$

30. $\vec{\mathbf{a}} = -\frac{1}{2}\vec{\mathbf{c}}$

31. $\vec{\mathbf{a}} = 6\vec{\mathbf{b}} + 4\vec{\mathbf{c}}$

32. $\vec{\mathbf{a}} = 0.4\vec{\mathbf{b}} - 1.2\vec{\mathbf{c}}$

33. $\vec{\mathbf{a}} = \frac{1}{3}\left(2\vec{\mathbf{b}} - 5\vec{\mathbf{c}}\right)$

34. $\vec{\mathbf{a}} = \left(3\vec{\mathbf{b}} + \vec{\mathbf{c}}\right) + 5\vec{\mathbf{b}}$

35. For $\vec{\mathbf{m}} = \langle -5, -6\rangle$ and $\vec{\mathbf{n}} = \langle 6, -9\rangle$ find the sum of the vector three times the magnitude of $\vec{\mathbf{m}}$ and the vector two and one half times the magnitude of the opposite of $\vec{\mathbf{n}}$.

Find the magnitude of each vector. Then write each vector as the sum of unit vectors.

36. $\langle 3, 4\rangle$

37. $\langle 2, -3\rangle$

38. $\langle -6, -11\rangle$

39. $\langle 3.5, 12\rangle$

40. $\langle -4, 1\rangle$

41. $\langle -16, -34\rangle$

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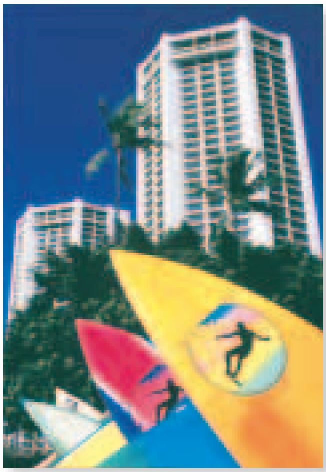
42. Write $\overrightarrow{ST}$ as the sum of unit vectors for points $S(-9, 2)$ and $T(-4, -3)$.

43. Prove that addition of vectors is associative.

Applications and Problem Solving

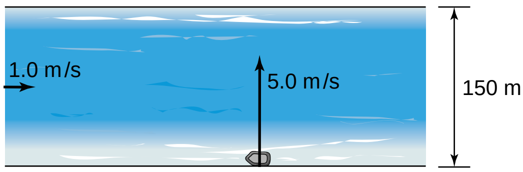
44. **Recreation** In the 12th Bristol International Kite Festival in September 1997 in England, Peter Lynn set a record for flying the world's biggest kite, which had a lifting surface area of 630 square meters. Suppose the wind is blowing against the kite with a force of 100 newtons at an angle 20° above the horizontal.
   a. Draw a diagram representing the situation.
   b. How much force is lifting the kite?



45. **Surfing** During a weekend surfboard competition, Kiyoshi moves at a 30° angle toward the shore. The velocity component toward the shore is 15 mph.
   a. Make a labeled diagram to show Kiyoshi's velocity and the velocity components.
   b. What is Kiyoshi's velocity?

46. **Critical Thinking** Suppose the points Q, R, S, and T are noncollinear, and $\overrightarrow{QR} + \overrightarrow{ST} = \vec{\mathbf{0}}$.
   a. What is the relationship between $\overrightarrow{QR}$ and $\overrightarrow{ST}$?
   b. What is true of the quadrilateral with vertices Q, R, S, and T?

47. **River Rafting** The Soto family is rafting on the Colorado River. Suppose that they are on a stretch of the river that is 150 meters wide, flowing south at a rate of 1.0 m/s. In still water their raft travels 5.0 m/s.



   a. How long does it take them to travel from one bank to the other if they head for a point directly across the river?
   b. How far downriver will the raft land?
   c. What is the velocity of the raft relative to the shore?

48. **Critical Thinking** Show that any vector $\vec{\mathbf{v}}$ can be written as $\langle |\vec{\mathbf{v}}| \cos \theta, |\vec{\mathbf{v}}| \sin \theta \rangle$.

Mixed Review

49. State whether $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are *equal*, *opposite*, *parallel*, or *none of these* for points $P(8, -7)$, $Q(-2, 5)$, $R(8, -7)$, and $S(7, 0)$. *(Lesson 8-1)*

50. Find the distance from the graph of $3x - 7y - 1 = 0$ to the point at $(-1, 4)$. *(Lesson 7-7)*

51. Use the sum or difference identities to find the exact value of sin 255°. *(Lesson 7-3)*

52. Write an equation of the sine function that has an amplitude of 17, a period of 45°, and a phase shift of −60°. *(Lesson 6-5)*

53. **Geometry** Two sides of a triangle are 400 feet and 600 feet long, and the included angle measures 46°20′. Find the perimeter and area of the triangle. *(Lesson 5-8)*

54. Use the Upper Bound Theorem to find an integral upper bound and the Lower Bound Theorem to find the lower bound of the zeros of the function $f(x) = 3x^2 - 2x + 1$. *(Lesson 4-5)*

Extra Practice See p. A40.
CONTENTS

55. Using a graphing calculator to graph $y = x^3 - x^2 + 3$. Determine and classify its extrema. *(Lesson 3-6)*

56. Describe the end behavior of $f(x) = x^2 + 3x + 1$. *(Lesson 3-5)*

57. **SAT Practice** For which values of x is $7x + 1$ greater than $7x - 1$?

A all real numbers
B only positive real numbers
C only $x = 0$
D only negative real numbers
E no real numbers

CAREER CHOICES

Aerospace Engineering

Would you like to design aircraft or even spacecraft? An aerospace engineer works with other specialists to design, build, and test any vehicles that fly. As an aerospace engineer, you might concentrate on one type of air or spacecraft, or you might work on specific components of these crafts.

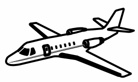
There are also a number of specialties within the field of aerospace engineering including analytical engineering, stress engineering, materials aerospace engineering, and marketing and sales aerospace engineering. One of the aspects of aerospace engineering is the analysis of airline accidents to determine if structural defects in design were the cause of the accident.

Career Overview

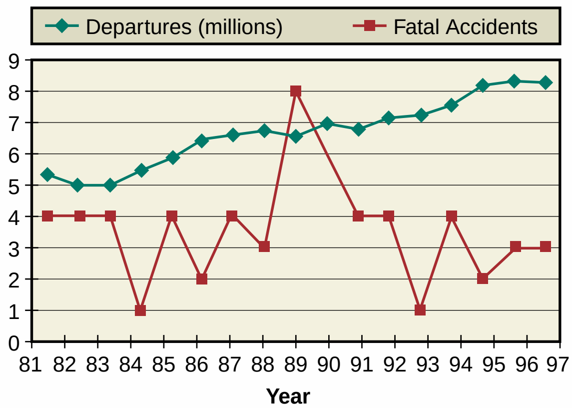
Degree Preferred:
bachelor's degree in aeronautical or aerospace engineering

Related Courses:
mathematics, science, computer science, mechanical drawing

Outlook:
slower than average through the year 2006



Airline Departures and Fatal Accidents



internet CONNECTION For more information on careers in aerospace engineering, visit: **www.amc.glencoe.com**

8-3

Vectors in Three-Dimensional Space

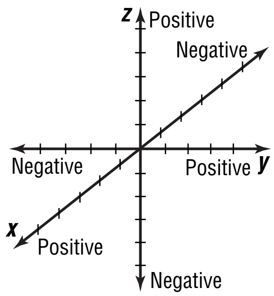
OBJECTIVES

- Add and subtract vectors in three-dimensional space.
- Find the magnitude of vectors in three-dimensional space.

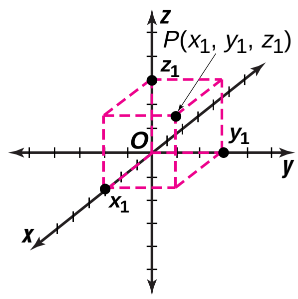
Real World Application

ENTOMOLOGY Entomology (en tuh MAHL uh jee) is the study of insects. Entomologists often use time-lapse photography to study the flying behavior of bees. They have discovered that worker honeybees let other workers know about new sources of food by rapidly vibrating their wings and performing a dance. Researchers have been able to create three-dimensional models using vectors to represent the flying behavior of bees. *You will use vectors in Example 3 to model a bee's path.*

Vectors in three-dimensional space can be described by coordinates in a way similar to the way we describe vectors in a plane. Imagine three real number lines intersecting at the zero point of each, so that each line is perpendicular to the plane determined by the other two. To show this arrangement on paper, a figure with the x-axis appearing to come out of the page is used to convey the feeling of depth. The axes are named the x-axis, y-axis, and z-axis.



Each point in space corresponds to an ordered triple of real numbers. To locate a point P with coordinates (x_1, y_1, z_1), first find x_1 on the x-axis, y_1 on the y-axis, and z_1 on the z-axis. Then imagine a plane perpendicular to the x-axis at x_1 and planes perpendicular to the y- and z-axes at y_1 and z_1, respectively. The three planes will intersect at point P.

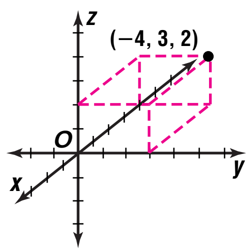


Example 1 **Locate the point at (−4, 3, 2).**

Locate −4 on the x-axis, 3 on the y-axis, and 2 on the z-axis.

Now draw broken lines for parallelograms to represent the three planes.

The planes intersect at (−4, 3, 2).



Ordered triples, like ordered pairs, can be used to represent vectors. The geometric interpretation is the same for a vector in space as it is for a vector in a plane. A directed line segment from the origin O to $P(x, y, z)$ is called vector $\overrightarrow{OP}$, corresponding to vector $\langle x, y, z \rangle$.

An extension of the formula for the distance between two points in a plane allows us to find the distance between two points in space. The distance from the origin to a point (x, y, z) is $\sqrt{x^2 + y^2 + z^2}$. So the magnitude of vector $\langle x, y, z \rangle$ is $\sqrt{x^2 + y^2 + z^2}$. This can be adapted to represent any vector in space.

Representation of a Vector as an Ordered Triple

Suppose $P_1(x_1, y_1, z_1)$ is the initial point of a vector in space and $P_2(x_2, y_2, z_2)$ is the terminal point. The ordered triple that represents $\overrightarrow{P_1P_2}$ is $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. Its magnitude is given by $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Examples

2 **Write the ordered triple that represents the vector from $X(5, -3, 2)$ to $Y(4, -5, 6)$.**

$$\overrightarrow{XY} = (4, -5, 6) - (5, -3, 2)$$
$$= \langle 4 - 5, -5 - (-3), 6 - 2 \rangle$$
$$= \langle -1, -2, 4 \rangle$$

Real World Application

3 **ENTOMOLOGY** **Refer to the application at the beginning of the lesson. Suppose the flight of a honeybee passed through points at (0, 3, 3) and (5, 0, 4), in which each unit represents a meter. What is the magnitude of the displacement the bee experienced in traveling between these two points?**

$$\text{magnitude} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{(5 - 0)^2 + (0 - 3)^2 + (4 - 3)^2} \qquad \langle x_1, y_1, z_1 \rangle = \langle 0, 3, 3 \rangle, \ \langle x_2, y_2, z_2 \rangle = \langle 5, 0, 4 \rangle$$
$$= \sqrt{25 + 9 + 1}$$
$$= 5.9$$

The magnitude of the displacement is about 5.9 meters.

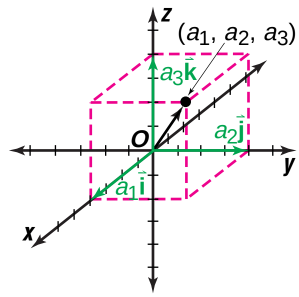
Operations on vectors represented by ordered triples are similar to those on vectors represented by ordered pairs.

Example

4 **Find an ordered triple that represents $3\vec{\mathbf{p}} - 2\vec{\mathbf{q}}$ if $\vec{\mathbf{p}} = \langle 3, 0, 4 \rangle$ and $\vec{\mathbf{q}} = \langle 2, 1, -1 \rangle$.**

$$3\vec{\mathbf{p}} - 2\vec{\mathbf{q}} = 3\langle 3, 0, 4 \rangle - 2\langle 2, 1, -1 \rangle \qquad \vec{p} = \langle 3, 0, 4 \rangle, \vec{q} = \langle 2, 1, -1 \rangle$$
$$= \langle 9, 0, 12 \rangle - \langle 4, 2, -2 \rangle$$
$$= \langle 5, -2, 14 \rangle$$

Three unit vectors are used as components of vectors in space. The unit vectors on the x-, y-, and z-axes are $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$ respectively, where $\vec{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\vec{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\vec{\mathbf{k}} = \langle 0, 0, 1 \rangle$. The vector $\vec{\mathbf{a}} = \langle a_1, a_2, a_3 \rangle$ is shown on the graph at the right. The component vectors of $\vec{\mathbf{a}}$ along the three axes are $a_1\vec{\mathbf{i}}$, $a_2\vec{\mathbf{j}}$, and $a_3\vec{\mathbf{k}}$. Vector $\vec{\mathbf{a}}$ can be written as the sum of unit vectors; that is, $\vec{\mathbf{a}} = a_1\vec{\mathbf{i}} + a_2\vec{\mathbf{j}} + a_3\vec{\mathbf{k}}$.



Example 5 **Write $\overrightarrow{AB}$ as the sum of unit vectors for $A(5, 10, -3)$ and $B(-1, 4, -2)$.**

First, express $\overrightarrow{AB}$ as an ordered triple. Then, write the sum of the unit vectors $\vec{\mathbf{i}}$, $\vec{\mathbf{j}}$, and $\vec{\mathbf{k}}$.

$$\begin{aligned}\overrightarrow{AB} &= (-1, 4, -2) - (5, 10, -3)\\ &= \langle -1 - 5, 4 - 10, -2 - (-3) \rangle\\ &= \langle -6, -6, 1 \rangle\\ &= -6\vec{\mathbf{i}} - 6\vec{\mathbf{j}} + \vec{\mathbf{k}}\end{aligned}$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** the process you would use to locate $\overrightarrow{AB} = 2\vec{\mathbf{i}} + 3\vec{\mathbf{j}} + 4\vec{\mathbf{k}}$ in space. Then sketch the vector on a coordinate system.
2. **Describe** the information you need to find the components of a three-dimensional vector from its given magnitude.
3. **You Decide** Marshall wrote the vector $\vec{\mathbf{v}} = \langle 1, -4, 0 \rangle$ as the sum of unit vectors $\vec{\mathbf{i}} + 4\vec{\mathbf{j}} + \vec{\mathbf{k}}$. Denise said that $\vec{\mathbf{v}}$ should be written as $\vec{\mathbf{i}} - 4\vec{\mathbf{j}} + \vec{\mathbf{k}}$. Who is correct? Explain.

Guided Practice

4. Locate point $G(4, -1, 7)$ in space. Then find the magnitude of a vector from the origin to G.

Write the ordered triple that represents $\overrightarrow{RS}$. Then find the magnitude of $\overrightarrow{RS}$.

5. $R(-2, 5, 8)$, $S(3, 9, -3)$
6. $R(3, 7, -1)$, $S(10, -4, 0)$

Find an ordered triple to represent $\vec{\mathbf{a}}$ in each equation if $\vec{\mathbf{f}} = \langle 1, -3, -8 \rangle$ and $\vec{\mathbf{g}} = \langle 3, 9, -1 \rangle$.

7. $\vec{\mathbf{a}} = 3\vec{\mathbf{f}} + \vec{\mathbf{g}}$
8. $\vec{\mathbf{a}} = 2\vec{\mathbf{g}} - 5\vec{\mathbf{f}}$

Write $\overrightarrow{EF}$ as the sum of unit vectors.

9. $E(-5, -2, 4)$, $F(6, -6, 6)$
10. $E(-12, 15, -9)$, $F(-12, 17, -22)$

11. **Physics** Suppose that during a storm the force of the wind blowing against a skyscraper can be expressed by the vector $\langle 132, 3454, 0\rangle$, where each measure in the ordered triple represents the force in newtons. What is the magnitude of this force?

EXERCISES

Practice

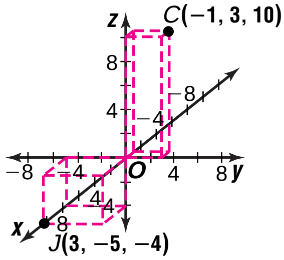
Locate point *B* in space. Then find the magnitude of a vector from the origin to *B*.

12. $B(4, 1, -3)$
13. $B(7, 2, 4)$
14. $B(10, -3, 15)$

Write the ordered triple that represents $\overrightarrow{TM}$. Then find the magnitude of $\overrightarrow{TM}$.

15. $T(2, 5, 4), M(3, 1, -4)$
16. $T(-2, 4, 7), M(-3, 5, 2)$
17. $T(2, 5, 4), M(3, 1, 0)$
18. $T(3, -5, 6), M(-1, 1, 2)$
19. $T(-5, 8, 3), M(-2, -1, -6)$
20. $T(0, 6, 3), M(1, 4, -3)$

21. Write the ordered triple to represent $\overrightarrow{CJ}$. Then find the magnitude of $\overrightarrow{CJ}$.



Find an ordered triple to represent $\vec{u}$ in each equation if $\vec{v} = \langle 4, -3, 5\rangle$, $\vec{w} = \langle 2, 6, -1\rangle$, and $\vec{z} = \langle 3, 0, 4\rangle$.

22. $\vec{u} = 6\vec{w} + 2\vec{z}$
23. $\vec{u} = \frac{1}{2}\vec{v} - \vec{w} + 2\vec{z}$
24. $\vec{u} = \frac{3}{4}\vec{v} - \vec{w}$
25. $\vec{u} = 3\vec{v} - \frac{2}{3}\vec{w} + 2\vec{z}$
26. $\vec{u} = 0.75\vec{v} + 0.25\vec{w}$
27. $\vec{u} = -4\vec{w} + \vec{z}$

28. Find an ordered triple to represent the sum $\frac{2}{3}\vec{f} + 3\vec{g} - \frac{2}{5}\vec{h}$, if $\vec{f} = \langle -3, 4.5, -1\rangle$, $\vec{g} = \langle -2, 1, 6\rangle$, and $\vec{h} = \langle 6, -3, -3\rangle$.

Write $\overrightarrow{LB}$ as the sum of unit vectors.

29. $L(2, 2, 7), B(5, -6, 2)$
30. $L(-6, 1, 0), B(-4, 5, -1)$
31. $L(9, 7, -11), B(7, 3, -2)$
32. $L(12, 2, 6), B(-8, 7, -5)$
33. $L(-1, 2, -4), B(-8, 5, -10)$
34. $L(-9, 12, -5), B(6, 5, -5)$

35. Show that $|\overrightarrow{G_1G_2}| = |\overrightarrow{G_2G_1}|$.

36. If $\vec{m} = \langle m_1, m_2, m_3\rangle$, then $-\vec{m}$ is defined as $\langle -m_1, -m_2, -m_3\rangle$. Show that $|-\vec{m}| = |\vec{m}|$

Applications and Problem Solving

37. **Physics** An object is in equilibrium if the magnitude of the resultant force on it is zero. Two forces on an object are represented by $\langle 3, -2, 4\rangle$ and $\langle 6, 2, 5\rangle$. Find a third vector that will place the object in equilibrium.

38. **Critical Thinking** Find the midpoint of $\vec{\mathbf{v}}$ that extends from $V(2, 3, 6)$ to $W(4, 5, 2)$.

39. **Computer Games** Nate Rollins is designing a computer game. In the game, a knight is standing at point (1, 4, 0) watching a wizard sitting at the top of a tree. In the computer screen, the tree is one unit high, and its base is at (2, 4, 0). Find the displacement vector for each situation.

    a. from the origin to the knight
    b. from the bottom of the tree to the knight

40. **Critical Thinking** Find the vector $\vec{\mathbf{c}}$ that must be added to $\vec{\mathbf{a}} = \langle 1, 3, 1\rangle$ to obtain $\vec{\mathbf{b}} = \langle 3, 1, 5\rangle$.

Dr. Chiaki Mukai

41. **Aeronautics** Dr. Chiaki Mukai is Japan's first female astronaut. Suppose she is working inside a compartment shaped like a cube with sides 15 feet long. She realizes that the tool she needs is diagonally in the opposite corner of the compartment.

    a. Draw a diagram of the situation described above.
    b. What is the minimum distance she has to glide to secure the tool?
    c. At what angle to the floor must she launch herself?

42. **Chemistry** Dr. Alicia Sanchez is a researcher for a pharmaceutical firm. She has graphed the structure of a molecule with atoms having positions $A = (2, 0, 0)$, $B = \left(1, \sqrt{3}, 0\right)$, and $C = \left(1, \frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$. She needs to have every atom in this molecule equidistant from each other. Has she achieved this goal? Explain why or why not.

Mixed Review

43. Find the sum of the vectors $\langle 3, 5\rangle$ and $\langle -1, 2\rangle$ algebraically. *(Lesson 8-2)*

44. Find the coordinates of point D such that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal vectors for points $A(5, 2)$, $B(-3, 3)$, and $C(0, 0)$. *(Lesson 8-1)*

45. Verify that $\cot X = (\sin 2X) \div (1 - \cos 2X)$ is an identity. *(Lesson 7-4)*

46. If $\cos \theta = \frac{2}{3}$ and $0° \le \theta \le 90°$, find $\sin \theta$. *(Lesson 7-1)*

47. State the amplitude and period for the function $y = 6 \sin \frac{\theta}{2}$. *(Lesson 6-4)*

48. **Physics** If a pulley is rotating at 16 revolutions per minute, what is its rate in radians per second? *(Lesson 6-2)*

49. Determine if $(7, -2)$ is a solution for $y < 4x^2 - 3x + 5$. Explain. *(Lesson 3-3)*

50. **SAT/ACT Practice** You have added the same positive quantity to the numerator and denominator of a fraction. The result is

    A greater than the original fraction.
    B less than the original fraction.
    C equal to the original fraction.
    D one-half the original fraction.
    E cannot be determined from the information given.

Extra Practice See p. A40.

8-4

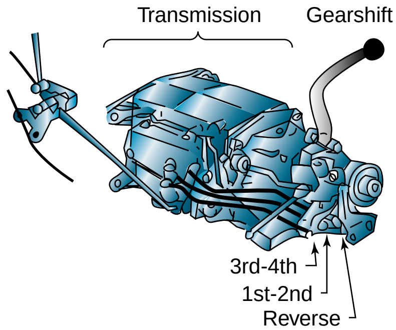
Perpendicular Vectors

OBJECTIVES

- Find the inner and cross products of two vectors.
- Determine whether two vectors are perpendicular.

Real World Application

AUTO RACING In addition to being an actor, Emilio Estevez is an avid stock car driver. Drivers in stock car races must constantly shift gears to maneuver their cars into competitive positions. A gearshift rotates about the connection to the transmission. The rate of change of the rotation of the gearshift depends on the magnitude of the force exerted by the driver and on the perpendicular distance of its line of action from the center of rotation. *You will use vectors in Example 4 to determine the force on a gearshift.*

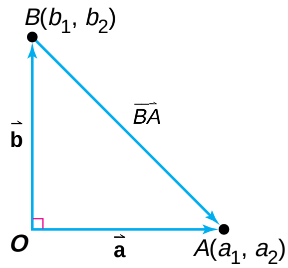


In physics, *torque* is the measure of the effectiveness of a force in turning an object about a pivot point. We can use perpendicular vectors to find the torque of a force.

Let $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ be perpendicular vectors, and let $\overrightarrow{BA}$ be a vector between their terminal points as shown. Then the magnitudes of $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\overrightarrow{BA}$ must satisfy the Pythagorean Theorem.

$$|\overrightarrow{BA}|^2 = |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2$$

Now use the definition of magnitude of a vector to evaluate $|\overrightarrow{BA}|^2$.



$$|\overrightarrow{BA}| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \quad \textit{Definition of magnitude}$$

$$|\overrightarrow{BA}|^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 \quad \textit{Square each side.}$$

$$|\overrightarrow{BA}|^2 = a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 \quad \textit{Simplify.}$$

$$|\overrightarrow{BA}|^2 = (a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2(a_1b_1 + a_2b_2) \quad \textit{Group the squared terms.}$$

$$|\overrightarrow{BA}|^2 = |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2 - 2(a_1b_1 + a_2b_2) \quad |\vec{\mathbf{a}}|^2 = a_1^2 + a_2^2, \; |\vec{\mathbf{b}}|^2 = b_1^2 + b_2^2$$

Compare the resulting equation with the original one. $|\overrightarrow{BA}|^2 = |\vec{\mathbf{a}}|^2 + |\vec{\mathbf{b}}|^2$ if and only if $a_1b_1 + a_2b_2 = 0$.

The expression $a_1b_1 + a_2b_2$ is frequently used in the study of vectors. It is called the **inner product** of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Inner Product of Vectors in a Plane

If $\vec{a}$ and $\vec{b}$ are two vectors, $\langle a_1, a_2\rangle$ and $\langle b_1, b_2\rangle$, the inner product of $\vec{a}$ and $\vec{b}$ is defined as $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$.

$\vec{a} \cdot \vec{b}$ is read "a dot b" and is often called the dot product.

Two vectors are perpendicular if and only if their inner product is zero. That is, vectors $\vec{a} = \langle a_1, a_2\rangle$ and $\vec{b} = \langle b_1, b_2\rangle$ are perpendicular if $\vec{a} \cdot \vec{b} = 0$.

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$$

Let $a_1b_1 + a_2b_2 = 0$ if

$$a_1b_1 = -a_2b_2$$

$$\frac{a_1}{a_2} = -\frac{b_2}{b_1} \qquad a_2 \neq 0, b_1 \neq 0$$

Since the ratio of the components can also be thought of as the slopes of the line on which the vectors lie, the slopes are opposite reciprocals. Thus, the lines and the vectors are perpendicular.

Example 1 **Find each inner product if $\vec{p} = \langle 7, 14\rangle$, $\vec{q} = \langle 2, -1\rangle$ and $\vec{m} = \langle 3, 5\rangle$. Are any pair of vectors perpendicular?**

a. $\vec{p} \cdot \vec{q}$

$$\vec{p} \cdot \vec{q} = 7(2) + 14(-1) = 14 - 14 = 0$$

$\vec{p}$ and $\vec{q}$ are perpendicular.

b. $\vec{p} \cdot \vec{m}$

$$\vec{p} \cdot \vec{m} = 7(3) + 14(5) = 21 + 70 = 91$$

$\vec{p}$ and $\vec{m}$ are not perpendicular.

c. $\vec{q} \cdot \vec{m}$

$$\vec{q} \cdot \vec{m} = 2(3) + (-1)(5) = 6 - 5 = 1$$

$\vec{q}$ and $\vec{m}$ are not perpendicular.

The inner product of vectors in space is similar to that of vectors in a plane.

Inner Product of Vectors in Space

If $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

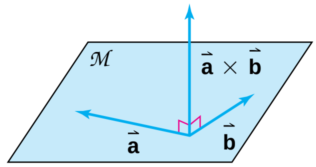
Just as in a plane, two vectors in space are perpendicular if and only if their inner product is zero.

Example 2 **Find the inner product of $\vec{a}$ and $\vec{b}$ if $\vec{a} = \langle -3, 1, 1\rangle$ and $\vec{b} = \langle 2, 8, -2\rangle$. Are $\vec{a}$ and $\vec{b}$ perpendicular?**

$$\vec{a} \cdot \vec{b} = (-3)(2) + (1)(8) + (1)(-2) = -6 + 8 + (-2) = 0$$

$\vec{a}$ and $\vec{b}$ are perpendicular since their inner product is zero.

Another important product involving vectors in space is the **cross product.** Unlike the inner product, the cross product of two vectors is a vector. This vector does not lie in the plane of the given vectors, but is perpendicular to the plane containing the two vectors. In other words, the cross product of two vectors is perpendicular to both vectors. The cross product of $\vec{a}$ and $\vec{b}$ is written $\vec{a} \times \vec{b}$.



$\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.
$\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

Cross Product of Vectors in Space

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\vec{a}$ and $\vec{b}$ is defined as follows.

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

Look Back
Refer to Lesson 2-5 to review determinants and expansion by minors.

An easy way to remember the coefficients of $\vec{i}$, $\vec{j}$, $\vec{k}$ is to write a determinant as shown and expand by minors using the first row.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 3 **Find the cross product of $\vec{v}$ and $\vec{w}$ if $\vec{v} = \langle 0, 3, 1 \rangle$ and $\vec{w} = \langle 0, 1, 2 \rangle$. Verify that the resulting vector is perpendicular to $\vec{v}$ and $\vec{w}$.**

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 3 \\ 0 & 1 \end{vmatrix} \vec{k} \quad \textit{Expand by minors.}$$

$$= 5\vec{i} - 0\vec{j} + 0\vec{k}$$
$$= 5\vec{i} \text{ or } \langle 5, 0, 0 \rangle$$

Find the inner products $\langle 5, 0, 0 \rangle \cdot \langle 0, 3, 1 \rangle$ and $\langle 5, 0, 0 \rangle \cdot \langle 0, 1, 2 \rangle$.

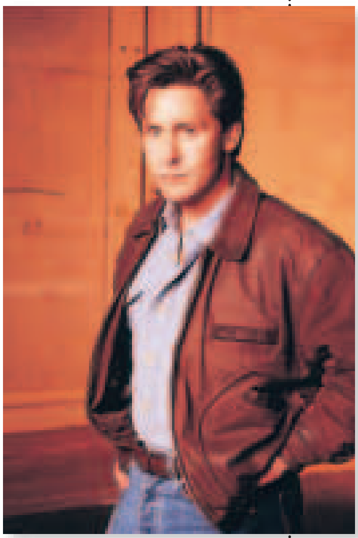
$5(0) + 0(3) + 0(1) = 0 \qquad 5(0) + 0(1) + 0(2) = 0$

Since the inner products are zero, the cross product $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

In physics, the torque $\vec{T}$ about a point A created by a force $\vec{F}$ at a point B is given by $\vec{T} = \overrightarrow{AB} \times \vec{F}$. The magnitude of $\vec{T}$ represents the torque in foot-pounds.

Example 4 **AUTO RACING** **Refer to the application at the beginning of the lesson. Suppose Emilio Estevez is applying a force of 25 pounds along the positive z-axis to the gearshift of his car. If the center of the connection of the gearshift is at the origin, the force is applied at the point (0.75, 0, 0.27). Find the torque.**

Real World Application



Emilio Estevez

We need to find $|\vec{\mathbf{T}}|$, the torque of the force at (0.75, 0, 0.27) where each value is the distance from the origin in feet and $\vec{\mathbf{F}}$ represents the force in pounds.

To find the magnitude of $\vec{\mathbf{T}}$, we must first find $\overrightarrow{AB}$ and $\vec{\mathbf{F}}$.

$$\overrightarrow{AB} = (0.75, 0, 0.27) - (0, 0, 0)$$
$$= \langle 0.75 - 0, 0 - 0, 0.27 - 0\rangle \text{ or } \langle 0.75, 0, 0.27\rangle$$

Any upward force is measured along the z-axis, so $\vec{\mathbf{F}} = 25\vec{\mathbf{k}}$ or $\langle 0, 0, 25\rangle$.

Now, find $\vec{\mathbf{T}}$.

$$\vec{\mathbf{T}} = \overrightarrow{AB} \times \vec{\mathbf{F}}$$

$$= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 0.75 & 0 & 0.27 \\ 0 & 0 & 25 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0.27 \\ 0 & 25 \end{vmatrix}\vec{\mathbf{i}} - \begin{vmatrix} 0.75 & 0.27 \\ 0 & 25 \end{vmatrix}\vec{\mathbf{j}} + \begin{vmatrix} 0.75 & 0 \\ 0 & 0 \end{vmatrix}\vec{\mathbf{k}}$$

$$= 0\vec{\mathbf{i}} - 18.75\vec{\mathbf{j}} + 0\vec{\mathbf{k}} \text{ or } \langle 0, -18.75, 0\rangle$$

Find the magnitude of $\vec{\mathbf{T}}$.

$$|\vec{\mathbf{T}}| = \sqrt{0^2 + (-18.75)^2 + 0^2}$$
$$= \sqrt{(-18.75)^2} \text{ or } 18.75$$

The torque is 18.75 foot-pounds.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Compare** $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$ and $\vec{\mathbf{w}} \times \vec{\mathbf{v}}$ for $\vec{\mathbf{v}} = \langle -1, 0, 3\rangle$ and $\vec{\mathbf{w}} = \langle 1, 2, 4\rangle$.
2. **Show** that the cross product of a three-dimensional vector with itself is the zero vector.
3. *Math Journal* Could the inner product of a nonzero vector and itself ever be zero? Explain why or why not.

Guided Practice

Find each inner product and state whether the vectors are perpendicular. Write *yes* or *no*.

4. $\langle 5, 2\rangle \cdot \langle -3, 7\rangle$
5. $\langle -8, 2\rangle \cdot \langle 4.5, 18\rangle$
6. $\langle -4, 9, 8\rangle \cdot \langle 3, 2, -2\rangle$

Find each cross product. Then verify that the resulting vector is perpendicular to the given vectors.

7. $\langle 1, -3, 2\rangle \times \langle -2, 1, -5\rangle$

8. $\langle 6, 2, 10\rangle \times \langle 4, 1, 9\rangle$

9. Find a vector perpendicular to the plane containing the points $(0, 1, 2)$, $(-2, 2, 4)$, and $(-1, -1, -1)$.

10. **Mechanics** Tikiro places a wrench on a nut and applies a downward force of 32 pounds to tighten the nut. If the center of the nut is at the origin, the force is applied at the point $(0.65, 0, 0.3)$. Find the torque.

EXERCISES

Practice

Find each inner product and state whether the vectors are perpendicular. Write *yes* or *no*.

11. $\langle 4, 8\rangle \cdot \langle 6, -3\rangle$

12. $\langle 3, 5\rangle \cdot \langle 4, -2\rangle$

13. $\langle 5, -1\rangle \cdot \langle -3, 6\rangle$

14. $\langle 7, 2\rangle \cdot \langle 0, -2\rangle$

15. $\langle 8, 4\rangle \cdot \langle 2, 4\rangle$

16. $\langle 4, 9, -3\rangle \cdot \langle -6, 7, 5\rangle$

17. $\langle 3, 1, 4\rangle \cdot \langle 2, 8, -2\rangle$

18. $\langle -2, 4, 8\rangle \cdot \langle 16, 4, 2\rangle$

19. $\langle 7, -2, 4\rangle \cdot \langle 3, 8, 1\rangle$

20. Find the inner product of $\vec{a}$ and $\vec{b}$, $\vec{b}$ and $\vec{c}$, and $\vec{a}$ and $\vec{c}$ if $\vec{a} = \langle 3, 12\rangle$, $\vec{b} = \langle 8, -2\rangle$, and $\vec{c} = \langle 3, -2\rangle$. Are any of the pairs perpendicular? If so, which one(s)?

Find each cross product. Then verify that the resulting vector is perpendicular to the given vectors.

21. $\langle 0, 1, 2\rangle \times \langle 1, 1, 4\rangle$

22. $\langle 5, 2, 3\rangle \times \langle -2, 5, 0\rangle$

23. $\langle 3, 2, 0\rangle \times \langle 1, 4, 0\rangle$

24. $\langle 1, -3, 2\rangle \times \langle 5, 1, -2\rangle$

25. $\langle -3, -1, 2\rangle \times \langle 4, -4, 0\rangle$

26. $\langle 4, 0, -2\rangle \times \langle -7, 1, 0\rangle$

27. Prove that for any vector $\vec{a}$, $\vec{a} \times (-\vec{a}) = 0$.

28. Use the definition of cross products to prove that for any vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$.

Find a vector perpendicular to the plane containing the given points.

29. $(0, -2, 2)$, $(1, 2, -3)$, and $(4, 0, -1)$

30. $(-2, 1, 0)$, $(-3, 0, 0)$, and $(5, 2, 0)$

31. $(0, 0, 1)$, $(1, 0, 1)$, and $(-1, -1, -1)$

32. Explain whether the equation $\vec{m} \times \vec{n} = \vec{n} \times \vec{m}$ is true. Discuss your reasoning.

Applications and Problem Solving

Real World Application

33. **Physiology** Whenever we lift an object, a torque is applied by the biceps muscle on the lower arm. The elbow acts as the axis of rotation through the joint. Suppose the muscle is attached 4 centimeters from the joint and you exert a force of 600 N lifting an object 30° to the horizontal.

a. Make a labeled diagram showing this situation.

b. What is the torque about the elbow?

34. **Critical Thinking** Let $\vec{\mathbf{x}} = \langle 2, 3, 0 \rangle$ and $\vec{\mathbf{y}} = \langle -1, 1, 4 \rangle$. Find the area of the triangle whose vertices are the origin and the endpoints of $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$.

35. **Business Management** Mr. Toshiro manages a company that supplies a variety of domestic and imported nuts to supermarkets. He received an order for 120 bags of cashews, 310 bags of walnuts, and 60 bags of Brazil nuts. The prices per bag for each type are \$29, \$18, and \$21, respectively.

a. Represent the number of bags ordered and the cost as vectors.

b. Using what you have learned in the lesson about vectors, compute the value of the order.

36. **Physics** The work done by a force $\vec{\mathbf{F}}$, that displaces an object through a distance d is defined as $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$. This can also be expressed as $|\vec{\mathbf{F}}|\,|\vec{\mathbf{d}}| \cos \theta$. Alexa is pushing a construction barrel up a ramp 4 feet long into the back of a truck. She is using a force of 120 pounds in a horizontal direction and the ramp is 45° from the horizontal.

a. Sketch a drawing of the situation.

b. How much work is Alexa doing?

37. **Architecture** Steve Herr is an architect in Minneapolis, Minnesota. His latest project is designing a park. On the blueprint, the park is determined by a plane which contains the points at (1, 0, 3), (2, 5, 0), and (3, 1, 4). One of the features of the park is a monument that must be perpendicular to the ground.

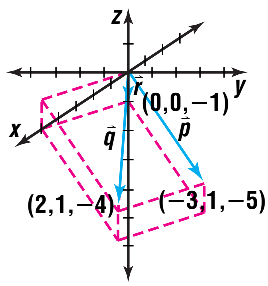
a. Find a nonzero vector, representing the monument, perpendicular to the plane defined by the given points.

b. Explain how you know that the vector is perpendicular to the plane defining the park.

38. **Geometry** A parallelepiped is a prism whose opposite faces are all parallelograms.

a. Determine the volume of the parallelepiped using the expression $\vec{\mathbf{p}} \cdot (\vec{\mathbf{q}} \times \vec{\mathbf{r}})$.

b. Write a 3×3 matrix using the three vectors and calculate its determinant. How does this value compare to your answer in part a?



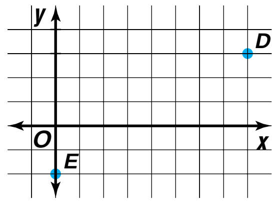
39. **Critical Thinking** If $\vec{\mathbf{v}} = \langle 1, 2 \rangle$, $\vec{\mathbf{w}} = \langle -1, 2 \rangle$, and $\vec{\mathbf{u}} = \langle 5, 12 \rangle$, for what scalar k will the vector $k\vec{\mathbf{v}} + \vec{\mathbf{w}}$ be perpendicular to $\vec{\mathbf{u}}$?

40. **Proof** Let $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ and $\vec{\mathbf{b}} = \langle b_1, b_2 \rangle$. Use the Law of Cosines to show that if the measure of the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, θ, is any value, then the inner product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}| \cos \theta$. (*Hint:* Refer to the proof of inner product using the Pythagorean Theorem on page 505.)

Mixed Review

41. Given points $A(3, 3, -1)$ and $B(5, 3, 2)$, find an ordered triple that represents $\overrightarrow{AB}$. *(Lesson 8-3)*

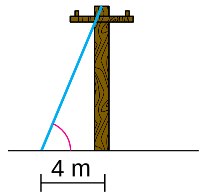
42. Write the ordered pair that represents $\overrightarrow{DE}$ for the points on the graph. Then find the magnitude of $\overrightarrow{DE}$. *(Lesson 8-2)*



43. Write the equation $4x + y = 6$ in normal form. Then find p, the measure of the normal, and ϕ, the angle it makes with the positive x-axis. *(Lesson 7-6)*

44. Solve $\triangle ABC$ if $A = 36°$, $b = 13$, and $c = 6$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-8)*

45. Utilities A utility pole is braced by a cable attached to it at the top and anchored in a concrete block at ground level, a distance of 4 meters from the base of the pole. If the angle between the cable and the ground is 73°, find the height of the pole and the length of the cable to the nearest tenth of a meter. *(Lesson 5-4)*



46. Solve $3+\sqrt{3x-4} \geq 10$. *(Lesson 4-7)*

47. SAT/ACT Practice Let x be an integer greater than 1. What is the least value of x for which $a^2 = b^3 = x$ for some integers a and b?

A 81 **B** 64 **C** 4 **D** 2 **E** 9

MID-CHAPTER QUIZ

Use a ruler and a protractor to draw a vector with the given magnitude and direction. Then find the magnitude of the horizontal and vertical components of the vector. (Lesson 8-1)

1. 2.3 centimeters, 46°

2. 27 millimeters, 245°

Write the ordered pair or ordered triple that represents $\overrightarrow{CD}$. Then find the magnitude of $\overrightarrow{CD}$. (Lessons 8-2 and 8-3)

3. $C(-9, 2)$, $D(-4, -3)$

4. $C(3, 7, -1)$, $D(5, 7, 2)$

Find an ordered pair or ordered triple to represent $\vec{\mathbf{r}}$ in each equation if $\vec{\mathbf{s}} = \langle 4, -3\rangle$, $\vec{\mathbf{t}} = \langle -6, 2\rangle$, $\vec{\mathbf{u}} = \langle 1, -3, -8\rangle$, and $\vec{\mathbf{v}} = \langle 3, 9, -1\rangle$. (Lessons 8-2 and 8-3)

5. $\vec{\mathbf{r}} = \vec{\mathbf{t}} - 2\vec{\mathbf{s}}$

6. $\vec{\mathbf{r}} = 3\vec{\mathbf{u}} + \vec{\mathbf{v}}$

Find each inner product and state whether the vectors are perpendicular. Write *yes* or *no*. (Lesson 8-4)

7. $\langle 3, 6\rangle \cdot \langle -4, 2\rangle$

8. $\langle 3, -2, 4\rangle \cdot \langle 1, -4, 0\rangle$

9. Find the cross product $\langle 1, 3, 2\rangle \times \langle 2, -1, -1\rangle$. Then verify that the resulting vector is perpendicular to the given vectors. (Lesson 8-4)

10. Entomology Suppose the flight of a housefly passed through points at (2, 0, 4) and (7, 4, 6), in which each unit represents a meter. What is the magnitude of the displacement the housefly experienced in traveling between these points? (Lesson 8-3)

Extra Practice See p. A41.

CONTENTS

8-4B Finding Cross Products

An Extension of Lesson 8-4

OBJECTIVE

- Use a graphing calculator program to obtain the components of the cross product of two vectors in space.

The following graphing calculator program allows you to input the components of two vectors $\vec{\mathbf{p}} = \langle a, b, c\rangle$ and $\vec{\mathbf{q}} = \langle x, y, z\rangle$ in space and obtain the components of their cross product.

```
PROGRAM: CROSSP
:Disp "SPECIFY (A, B, C)"
:Input A
:Input B
:Input C
:Disp "SPECIFY (X, Y, Z)"
:Input X
:Input Y
:Input Z
:Disp "CROSS PROD OF"
:Disp "(A, B, C) AND"
:Disp "(X, Y, Z) IS"
:Disp BZ - CY
:Disp CX - AZ
:Disp AY - BX
:Stop
```

interNET CONNECTION

Graphing Calculator Programs
To download this program, visit **www.amc.glencoe.com**

Enter this program on your calculator and use it as needed to complete the following exercises.

TRY THESE

Use a graphing calculator to determine each cross product.

1. $\langle 7, 9, -1\rangle \times \langle 3, -4, -5\rangle$

2. $\langle 8, 14, 0\rangle \times \langle -2, 6, 12\rangle$

3. $\langle 5, 5, 5\rangle \times \langle 4, 4, 4\rangle$

4. $\langle -3, 2\ -1\rangle \times \langle 6, -3, 7\rangle$

5. $\langle 1, 6, 0\rangle \times \langle 2, 5, 0\rangle$

6. $\langle 4, 0, -2\rangle \times \langle 6, 0, -13\rangle$

It can be proved that if $\vec{u}$ and $\vec{v}$ are adjacent sides of a parallelogram, then the area of the parallelogram is $|\vec{u} \times \vec{v}|$. Find the area of the parallelogram having the given vectors as adjacent sides.

7. $\vec{\mathbf{u}} = \langle -2, 4, 1\rangle, \vec{\mathbf{v}} = \langle 0, 6, 3\rangle$

8. $\vec{\mathbf{u}} = \langle 1, 3, 2\rangle, \vec{\mathbf{v}} = \langle 9, 7, 5\rangle$

WHAT DO YOU THINK?

9. What instructions could you insert at the end of the program to have the program display the magnitude of the vectors resulting from the cross products of the vectors above?

8-5 Applications with Vectors

OBJECTIVE
- Solve problems using vectors and right triangle trigonometry.

Real World Application

SPORTS Ty Murray was a recent World Championship Bull Rider in the National Finals Rodeo in Las Vegas, Nevada. One of the most dangerous tasks, besides riding the bulls, is the job of getting the bulls back into their pens after each event. Experienced handlers, dressed as clowns, expertly rope the animals without harming them.

Ty Murray has completed his competition ride and the two rodeo clowns are restraining the bull to return it to the paddocks. Suppose one clown is exerting a force of 270 newtons due north and the other is pulling with a force of 360 newtons due east. What is the resultant force on the bull? *This problem will be solved in Example 1.*

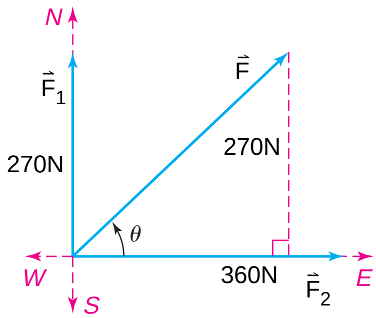
Vectors can represent the forces exerted by the handlers on the bull. Physicists resolve this force into component vectors. Vectors can be used to represent any quantity, such as a force, that has magnitude and direction. Velocity, weight, and gravity are some of the quantities that physicists represent with vectors.

Example 1 Real World Application

SPORTS **Use the information above to describe the resultant force on the bull.**

a. Draw a labeled diagram that represents the forces.

Let $\vec{F}_1$ and $\vec{F}_2$ represent the forces exerted by the clowns. Then $\vec{F}$ represents the resultant. Let θ represent the angle $\vec{F}$ makes with the east-west or x-axis.



b. Determine the resultant force exerted on the bull by the two clowns.

$$|\vec{F}|^2 = |\vec{F}_1|^2 + |\vec{F}_2|^2$$

$$|\vec{F}|^2 = (270)^2 + (360)^2$$

$$|\vec{F}|^2 = 202{,}500$$

$$\sqrt{|\vec{F}|^2} = \sqrt{202{,}500} \text{ or } 450$$

The resultant force on the bull is 450 newtons.

1 pound ≈ 4.45 newtons so 450 N ≈ 101.12 lb

c. Find the angle the resultant force makes with the east-west axis.

Use the tangent ratio.

$\tan \theta = \frac{270}{360}$

$\theta = \tan^{-1} \frac{270}{360}$

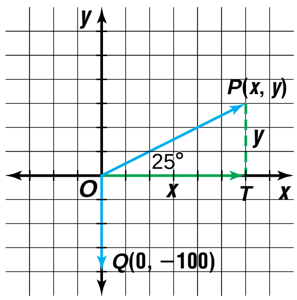
$\theta \approx 36.9°$ north of due east

The resultant force is applied at an angle of 36.9° north of due east.

In physics, if a constant force $\vec{\mathbf{F}}$ displaces an object an amount represented by $\vec{\mathbf{d}}$, it does *work* on the object. The amount of work is given by $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$.

Example 2 **Alvaro works for a package delivery service. Suppose he is pushing a cart full of packages weighing 100 pounds up a ramp 8 feet long at an incline of 25°. Find the work done by gravity as the cart moves the length of the ramp. Assume that friction is not a factor.**

First draw a labeled diagram representing the forces involved. Let $\overrightarrow{OQ}$ represent the force of gravity, or weight. The weight has a magnitude of 100 pounds and its direction is down. The corresponding unit vector is $0\vec{\mathbf{i}} - 100\vec{\mathbf{j}}$. So, $\vec{\mathbf{F}} = 0\vec{\mathbf{i}} - 100\vec{\mathbf{j}}$. The application of the force is $\overrightarrow{OP}$, and it has a magnitude of 8 feet.



Write $\overrightarrow{OP}$ as $\vec{\mathbf{d}} = x\vec{\mathbf{i}} + y\vec{\mathbf{j}}$ and use trigonometry to find x and y.

$\cos 25° = \frac{x}{8}$

$x = 8 \cos 25°$

$x \approx 7.25$

$\sin 25° = \frac{y}{8}$

$y = 8 \sin 25°$

$y \approx 3.38$

Then, $\vec{\mathbf{d}} = 7.25\vec{\mathbf{i}} + 3.38\vec{\mathbf{j}}$

Apply the formula for determining the work done by gravity.

$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$

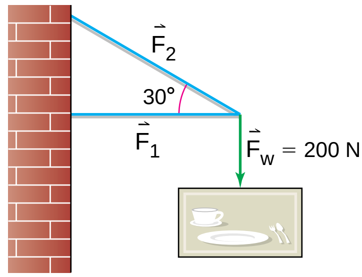
$W = \langle 0\vec{\mathbf{i}} - 100\vec{\mathbf{j}} \rangle \cdot \langle 7.25\vec{\mathbf{i}} + 3.38\vec{\mathbf{j}} \rangle$

$W = 0 - 338$ or -338

Work done by gravity is negative when an object is lifted or raised. As the cart moves the length of the ramp, the work done by gravity is −338 ft-lb.

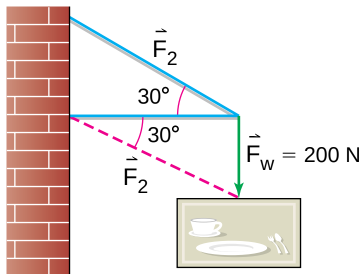
Sometimes there is no motion when several forces are at work on an object. This situation, when the forces balance one another, is called *equilibrium*. Recall that an object is in equilibrium if the resultant force on it is zero.

Example 3 **Ms. Davis is hanging a sign for her restaurant. The sign is supported by two lightweight support bars as shown in the diagram. If the bars make a 30° angle with each other, and the sign weighs 200 pounds, what are the magnitudes of the forces exerted by the sign on each support bar?**



$\vec{\mathbf{F}}_1$ represents the force exerted on bar 1 by the sign, $\vec{\mathbf{F}}_2$ represents the force exerted on bar 2 by the sign, and $\vec{\mathbf{F}}_w$ represents the weight of the sign.

Remember that equal vectors have the same magnitude and direction. So by drawing another vector from the initial point of $\vec{\mathbf{F}}_1$ to the terminal point of $\vec{\mathbf{F}}_w$, we can use the sine and cosine ratios to determine $|\vec{\mathbf{F}}_1|$ and $|\vec{\mathbf{F}}_2|$.



$$\sin 30° = \frac{200}{|\vec{\mathbf{F}}_2|}$$

$$|\vec{\mathbf{F}}_2| = \frac{200}{\sin 30°}$$ *Divide each side by sin 30° and multiply each side by $|\vec{F}_2|$.*

$$|\vec{\mathbf{F}}_2| = 400$$

Likewise, $|\vec{\mathbf{F}}_1|$ can be determined by using cos 30°.

$$\cos 30° = \frac{|\vec{\mathbf{F}}_1|}{400}$$

$$|\vec{\mathbf{F}}_1| = 400 \cos 30°$$

$$|\vec{\mathbf{F}}_1| \approx 346$$

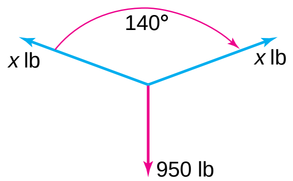
The sign exerts a force of about 346 pounds on bar 1 and a force of 400 pounds on bar 2.

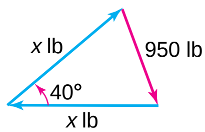
CONTENTS

If the vectors representing forces at equilibrium are drawn tip-to-tail, they will form a polygon.

Example **4** **A lighting system for a theater is supported equally by two cables suspended from the ceiling of the theater. The cables form a 140° angle with each other. If the lighting system weighs 950 pounds, what is the force exerted by each of the cables on the lighting system?**

Draw a diagram of the situation. Then draw the vectors tip-to-tail.





Look Back
Refer to Lessons 5-6 through 5-8 to review Law of Sines and Law of Cosines.

Since the triangle is isosceles, the base angles are congruent. Thus, each base angle measures $\frac{180° - 40°}{2}$ or 70°. We can use the Law of Sines to find the force exerted by the cables.

$$\frac{950}{\sin 40°} = \frac{x}{\sin 70°} \quad \textit{Law of Sines}$$

$$x = \frac{950 \sin 70°}{\sin 40°}$$

$$x \approx 1388.81$$

The force exerted by each cable is about 1389 pounds.

CHECK FOR UNDERSTANDING

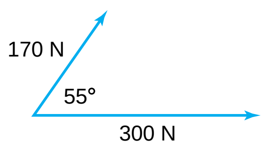
Communicating Mathematics

Read and study the lesson to answer each question.

1. **Determine** which would require more force—pushing an object up to the top of an incline, or lifting it straight up. Assume there is no friction.
2. **Describe** how the force exerted by the cables in Example 4 is affected if the angle between the cables is increased.
3. **Explain** what it means for forces to be in equilibrium.

Guided Practice

4. Make a sketch to show the forces acting on a ship traveling at 23 knots at an angle of 17° with the current.
5. Find the magnitude and direction of the resultant vector for the diagram.



6. A 100-newton force and a 50-newton force act on the same object. The angle between the forces measures 90°. Find the magnitude of the resultant force and the angle between the resultant force and the 50-pound force.

7. Denzel pulls a wagon along level ground with a force of 18 newtons on the handle. If the handle makes an angle of 40° with the horizontal, find the horizontal and vertical components of the force.

8. A 33-newton force at 90° and a 44-newton force at 60° are exerted on an object. What is the magnitude and direction of a third force that produces equilibrium on the object?

9. **Transportation** Two ferry landings are directly across a river from each other. A ferry that can travel at a speed of 12 miles per hour in still water is attempting to cross directly from one landing to the other. The current of the river is 4 miles per hour.
   a. Make a sketch of the situation.
   b. If a heading of 0° represents the line between the two landings, at what angle should the ferry's captain head?

EXERCISES

Practice

Make a sketch to show the given vectors.

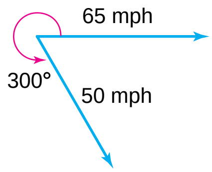
10. a force of 42 newtons acting on an object at an angle of 53° with the ground

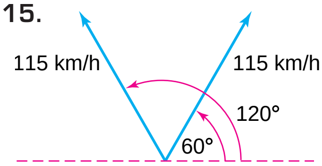
11. an airplane traveling at 256 miles per hour at an angle of 27° from the wind

12. a force of 342 pounds acting on an object while a force of 454 pounds acts on the same object an angle of 94° with the first force

Find the magnitude and direction of the resultant vector for each diagram.

13.


14.


15.


16. What would be the force required to push a 100-pound object along a ramp that is inclined 10° with the horizontal?

17. What is the magnitude and direction of the resultant of a 105-newton force along the *x*-axis and a 110-newton force at an angle of 50° to one another?

18. To keep a 75-pound block from sliding down an incline, a 52.1-pound force is exerted on the block along the incline. Find the angle that the incline makes with the horizontal.

19. Find the magnitude and direction of the resultant of two forces of 250 pounds and 45 pounds at angles of 25° and 250° with the *x*-axis, respectively.

20. Three forces with magnitudes of 70 pounds, 40 pounds, and 60 pounds act on an object at angles 330°, 45°, and 135°, respectively, with the positive *x*-axis. Find the direction and magnitude of the resultant of these forces.

21. A 23-newton force acting at 60° above the horizontal and a second 23-newton force acting at 120° above the horizontal act concurrently on a point. What is the magnitude and direction of a third force that produces equilibrium?

22. An object is placed on a ramp and slides to the ground. If the ramp makes an angle of 40° with the ground and the object weighs 25 pounds, find the acceleration of the object. Assume that there is no friction.

23. A force of 36 newtons pulls an object at an angle of 20° north of due east. A second force pulls on the object with a force of 48 newtons at an angle of 42° south of due west. Find the magnitude and direction of the resultant force.

24. Three forces in a plane act on an object. The forces are 70 pounds, 115 pounds and 135 pounds. The 70 pound force is exerted along the positive *x*-axis. The 115 pound force is applied below the *x*-axis at a 120° angle with the 70 pound force. The angle between the 115-pound and 135-pound forces is 75°, and between the 135-pound and 70-pound forces is 165°.

   a. Make a diagram showing the forces.

   b. Are the vectors in equilibrium? If not, find the magnitude and the direction of the resultant force.

Applications and Problem Solving

Real World Application

25. **Physics** While pulling a stalled car, a tow truck's cable makes an angle of 50° above the road. If the tension on the tow truck's cable is 1600 newtons, how much work is done by the truck on the car pulling it 1.5 kilometers down the road?

26. **Critical Thinking** The handle of a lawnmower you are pushing makes an angle of 60° with the ground.

   a. How could you increase the horizontal forward force you are applying without increasing the total force?

   b. What are some of the disadvantages of doing this?

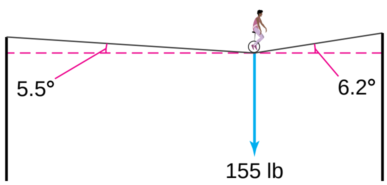
27. **Boating** A sailboat is headed east at 18 mph relative to the water. A current is moving the water south at 3 mph.

   a. What is the angle of the path of the sailboat?

   b. What is the sailboat's speed with respect to the ocean floor?

28. **Physics** Suzanne is pulling a wagon loaded with gardening bricks totaling 100 kilograms. She is applying a force of 100 newtons on the handle at 25° with the ground. What is the horizontal force on the wagon?

29. **Entertainment** A unicyclist is performing on a tightrope at a circus. The total weight of the performer and her unicycle is 155 pounds. How much tension is being exerted on each part of the cable?



30. **Critical Thinking** Chaz is using a rope tied to his tractor to remove an old tree stump from a field. Which method given below—a or b—will result in the greatest force applied to the stump? Assume that the tractor will exert the same amount of force using either method. Explain your answer.

   a. Tie the rope to the stump and pull.

   b. Tie one end to the stump and the other end to a nearby pole. Then pull on the rope perpendicular to it at a point about halfway between the two.

31. **Travel** A cruise ship is arriving at the port of Miami from the Bahamas. Two tugboats are towing the ship to the dock. They exert a force of 6000 tons along the axis of the ship. Find the tension in the towlines if each tugboat makes a 20° angle with the axis of the ship.

32. **Physics** A painting weighing 25 pounds is supported at the top corners by a taut wire that passes around a nail embedded in the wall. The wire forms a 120° angle at the nail. Assuming that the wire rests on the nail at its midpoint, find the pull on the wires.

Mixed Review

33. Find the inner product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ if $\vec{\mathbf{u}} = \langle 9, 5, 3 \rangle$ and $\vec{\mathbf{v}} = \langle -3, 2, 5 \rangle$. Are the vectors perpendicular? *(Lesson 8-4)*

34. If $A = (12, -5, 18)$ and $B = (0, -11, 21)$, write the ordered triple that represents $\overline{AB}$. *(Lesson 8-3)*

35. **Sports** Sybrina Floyd hit a golf ball on an approach shot with an initial velocity of 100 feet per second. The distance a golf ball travels is found by the formula $d = \frac{2v_0^2}{g} \sin \theta \cos \theta$, where v_0 is the initial velocity, g is the acceleration due to gravity, and θ is the measure of the angle that the initial path of the ball makes with the ground. Find the distance Ms. Floyd's ball traveled if the measure of the angle between the initial path of the ball and the ground is 65° and the acceleration due to gravity is 32 ft/s^2. *(Lesson 7-4)*

36. Write a polynomial function to model the set of data. *(Lesson 4-8)*

x	−1.5	−1	−0.5	0	0.5	1	1.5	2	2.5
f(x)	9	8	6.5	4.3	2	2.7	4.1	6.1	7.8

37. **Food Processing** A meat packer makes a kind of sausage using beef, pork, cereal, fat, water, and spices. The minimum cereal content is 12%, the minimum fat content is 15%, the minimum water content is 6.5%, and the spices are 0.5%. The remaining ingredients are beef and pork. There must be at least 30% beef content and at least 20% pork content for texture. The beef content must equal or exceed the pork content. The cost of all of the ingredients except beef and pork is $32 per 100 pounds. Beef can be purchased for $140 per 100 pounds and pork for $90 per 100 pounds. Find the combination of beef and pork for the minimum cost. What is the minimum cost per 100 pounds? *(Lesson 2-6)*

38. **SAT/ACT Practice** Let $*x$ be defined as $*x = x^3 - x$. What is the value of $*4 - *(-3)$?

   A 84 **B** 55 **C** −10
   D 22 **E** 4

8-6

Vectors and Parametric Equations

OBJECTIVES
- Write vector and parametric equations of lines.
- Graph parametric equations.

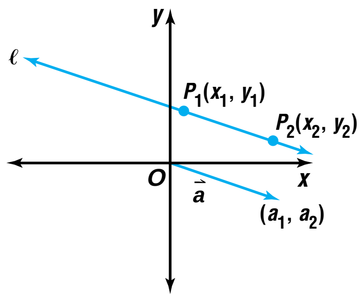
Real World Application

AIR RACING Air racing's origins date back to the early 1900s when aviation began. One of the most important races is the National Championship Air Race held every year in Reno, Nevada. In the competition, suppose pilot Bob Hannah passes the starting gate at a speed of 410.7 mph. Fifteen seconds later, Dennis Sanders flies by at 412.9 mph. They are racing for the next 500 miles. *This information will be used in Example 4.*

The relative positions of the airplanes are dependent upon their velocities at a given moment in time and their starting positions. **Vector equations** and equations known as **parametric equations** allow us to model that movement.

In Lesson 1-4, you learned how to write the equation of a line in the coordinate plane. For objects that are moving in a straight line, there is a more useful way of writing equations for the line describing the object's path using vectors.

If a line passes through the points P_1 and P_2 and is parallel to the vector $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$, the vector $\overrightarrow{P_1P_2}$ is also parallel to $\vec{\mathbf{a}}$. Thus, $\overrightarrow{P_1P_2}$ must be a scalar multiple of $\vec{\mathbf{a}}$. Using the scalar t, we can write the equation $\overrightarrow{P_1P_2} = t\vec{\mathbf{a}}$. Notice that both sides of the equation are vectors. This is called the vector equation of the line. Since $\vec{\mathbf{a}}$ is parallel to the line, it is called a **direction vector.** The scalar t is called a **parameter.**



Vector Equation of a Line

A line through $P_1(x_1, y_1)$ parallel to the vector $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ is defined by the set of points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ such that $\overrightarrow{P_1P_2} = t\vec{\mathbf{a}}$ for some real number t. Therefore, $\langle x_2 - x_1, y_2 - y_1 \rangle = t\langle a_1, a_2 \rangle$.

In Lesson 8-2, you learned that the vector from (x_1, y_1) to (x, y) is $\langle x - x_1, y - y_1 \rangle$.

Example 1 **Write a vector equation describing a line passing through $P_1(1, 4)$ and parallel to $\vec{\mathbf{a}} = \langle 3, -2 \rangle$.**

Let the line ℓ through $P_1(1, 4)$ be parallel to $\vec{\mathbf{a}}$. For any point $P(x, y)$ on ℓ, $\overrightarrow{P_1P} = \langle x - 1, y - 4 \rangle$. Since P_1P is on ℓ and is parallel to $\vec{\mathbf{a}}$, $\overrightarrow{P_1P} = t\vec{\mathbf{a}}$, for some value t. By substitution, we have $\langle x - 1, y - 4 \rangle = t\langle 3, -2 \rangle$.

Therefore, the equation $\langle x - 1, y - 4 \rangle = t\langle 3, -2 \rangle$ is a vector equation describing all of the points (x, y) on ℓ parallel to $\vec{\mathbf{a}}$ through $P_1(1, 4)$.

A vector equation like the one in Example 1 can be used to describe the coordinates for a point on the line for any value of the parameter t. For example, when $t = 4$ we can write the equation $\langle x - 1, y - 4 \rangle = 4\langle 3, -2 \rangle$ or $(12, -8)$. Then write the equation $x - 1 = 12$ and $y - 4 = -8$ to find the ordered pair $(13, -4)$. Likewise, when $t = 0$, the ordered pair $(1, 4)$ results. The parameter t often represents time. (In fact, that is the reason for the choice of the letter t.) An object moving along the line described by the vector equation $\langle x - 1, y - 4 \rangle = t\langle 3, 2 \rangle$ will be at the point $(1, 4)$ at time $t = 0$ and will be at the point $(13, -4)$ at time $t = 4$.

As you have seen, the vector equation $\langle x - x_1, y - y_1 \rangle = t\langle a_1, a_2 \rangle$ can be written as two equations relating the horizontal and vertical components of these two vectors separately.

$$x - x_1 = ta_1 \qquad y - y_1 = ta_2$$
$$x = x_1 + ta_1 \qquad y = y_1 + ta_2$$

The resulting equations, $x = x_1 + ta_1$ and $y = y_1 + ta_2$, are known as parametric equations of the line through $P_1(x_1, y_1)$ parallel to $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$.

Parametric Equations of a Line

A line through $P_1(x_1, y_1)$ that is parallel to the vector $\vec{\mathbf{a}} = \langle a_1, a_2 \rangle$ has the following parametric equations, where t is any real number.

$$x = x_1 + ta_1$$
$$y = y_1 + ta_2$$

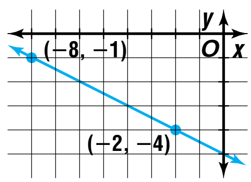
If we know the coordinates of a point on a line and its direction vector, we can write its parametric equations.

Example 2 **Find the parametric equations for a line parallel to $\vec{\mathbf{q}} = \langle 6, -3 \rangle$ and passing through the point at $(-2, -4)$. Then make a table of values and graph the line.**

Use the general form of the parametric equations of a line with $\langle a_1, a_2 \rangle = \langle 6, -3 \rangle$ and $\langle x_1, y_1 \rangle = \langle -2, -4 \rangle$.

$$x = x_1 + ta_1 \qquad y = y_1 + ta_2$$
$$x = -2 + t(6) \qquad y = -4 + t(-3)$$
$$x = -2 + 6t \qquad y = -4 - 3t$$

Now make a table of values for t. Evaluate each expression to find values for x and y. Then graph the line.



t	x	y
−1	−8	−1
0	−2	−4
1	4	−7
2	10	−10

Notice in Example 2 that each value of t establishes an ordered pair (x, y) whose graph is a point. As you have seen, these points can be considered the position of an object at various times t. Evaluating the parametric equations for a value of t gives us the coordinates of the position of the object after t units of time.

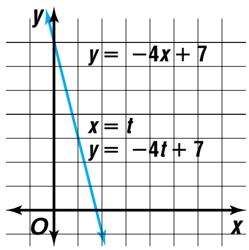
If the slope-intercept form of the equation of a line is given, we can write parametric equations for that line.

Example 3 **Write parametric equations of $y = -4x + 7$.**

In the equation $y = -4x + 7$, x is the independent variable, and y is the dependent variable. In parametric equations, t is the independent variable, and x and y are dependent variables. If we set the independent variables x and t equal, we can write two parametric equations in terms of t.

$x = t$
$y = -4t + 7$

Parametric equations for the line are $x = t$ and $y = -4t + 7$.



By making a table of values for t and evaluating each expression to find values for x and y and graphing the line, the parametric equations $x = t$ and $y = -4t + 7$ describe the same line as $y = -4x + 7$.

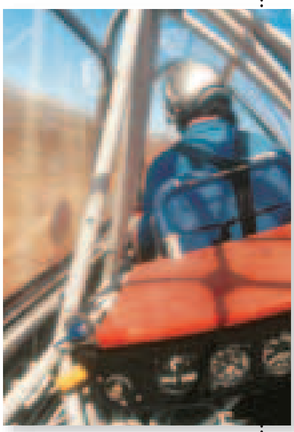
We can use vector equations and parametric equations to model physical situations, such as the National Championship Air Race, where t represents time in hours.

Example 4 Real World Application

AIR RACING **Refer to the application at the beginning of the lesson. Use parametric equations to model the situation. Assume that both planes maintain a constant speed.**

a. How long is it until the second plane overtakes the first?

b. How far have both planes traveled when the second plane overtakes the first?



a. First, write a set of parametric equations to represent each airplane's position at t hours.

Airplane 1: $x = 410.7t$ — $x = vt$
Airplane 2: $x = 412.9(t - 0.0042)$ — $15 \text{ s} \approx 0.0042 \text{ h}$

Since the time at which the second plane overtakes the first is when they have traveled the same distance, set the two expressions for x equal to each other.

$$410.7t \approx 412.9(t - 0.0042)$$
$$410.7t \approx 412.9t - 1.73418$$
$$1.73418 \approx 2.2t$$
$$0.788 \approx t$$

In about 0.788 hour or 47 minutes, the second plane overtakes the first.

b. Use the time to find the distance traveled when the planes pass.

$x = 410.7t$
$x = 410.7(0.788)$
$x = 323.6316$ or about 323.6 miles

Since the speeds are given in tenths of mph, we round the final answer to the nearest tenth.

The planes have traveled about 323.6 miles when the second plane overtakes the first plane.

We can also write the equation of a line in slope-intercept or standard form if we are given the parametric equations of the line.

Example 5 **Write an equation in slope-intercept form of the line whose parametric equations are $x = -5 + 4t$ and $y = 2 - 3t$.**

Solve each parametric equation for t.

$$x = -5 + 4t \qquad y = 2 - 3t$$
$$x + 5 = 4t \qquad y - 2 = -3t$$
$$\frac{x+5}{4} = t \qquad \frac{y-2}{3} = t$$

Use substitution to write an equation for the line without the variable t.

$$\frac{x+5}{4} = \frac{(y-2)}{-3} \quad \textit{Substitution}$$
$$(x + 5)(-3) = 4(y - 2) \quad \textit{Cross multiply.}$$
$$-3x - 15 = 4y - 8 \quad \textit{Simplify.}$$
$$y = -\frac{3}{4}x - \frac{7}{4} \quad \textit{Solve for y.}$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the graph of the parametric equations $x = 3 + 4t$ and $y = -1 + 2t$.
2. **Explain** how to find the parametric equations for the line through the point at (3, 6), parallel to the vector $\vec{\mathbf{v}} = \vec{\mathbf{i}} + 2\vec{\mathbf{j}}$.
3. **Describe** the line having parametric equations $x = 1 + t$ and $y = -t$. Include the slope of the line and the y-intercept in your description.

Guided Practice

Write a vector equation of the line that passes through point P and is parallel to $\vec{\mathbf{a}}$. Then write parametric equations of the line.

4. $P(-4, 11)$, $\vec{\mathbf{a}} = \langle -3, 8 \rangle$
5. $P(1, 5)$, $\vec{\mathbf{a}} = \langle -7, 2 \rangle$

Write parametric equations of each line with the given equation.

6. $3x + 2y = 5$
7. $4x - 6y = -12$

Write an equation in slope-intercept form of the line with the given parametric equations.

8. $x = -4t + 3$
$y = 5t - 3$

9. $x = 9t$
$y = 4t + 2$

10. Set up a table of values and then graph the line whose parametric equations are $x = 2 + 4t$ and $y = -1 + t$.

11. Sports A wide receiver catches a ball and begins to run for the endzone following a path defined by $\langle x - 5, y - 50\rangle = t\langle 0, -10\rangle$. A defensive player chases the receiver as soon as he starts running following a path defined by $\langle x - 10, y - 54\rangle = t\langle -0.9, -10.72\rangle$.

a. Write parametric equations for the path of each player.

b. If the receiver catches the ball on the 50-yard line ($y = 50$), will he reach the goal line ($y = 0$) before the defensive player catches him?

EXERCISES

Practice

Write a vector equation of the line that passes through point *P* and is parallel to $\vec{\mathbf{a}}$. Then write parametric equations of the line.

12. $P(5, 7)$, $\vec{\mathbf{a}} = \langle 2, 0\rangle$

13. $P(-1, 4)$, $\vec{\mathbf{a}} = \langle 6, -10\rangle$

14. $P(-6, 10)$, $\vec{\mathbf{a}} = \langle 3, 2\rangle$

15. $P(1, 5)$, $\vec{\mathbf{a}} = \langle -7, 2\rangle$

16. $P(1, 0)$, $\vec{\mathbf{a}} = \langle -2, -4\rangle$

17. $P(3, -5)$, $\vec{\mathbf{a}} = \langle -2, 5\rangle$

Write parametric equations of each line with the given equation.

18. $y = 4x - 5$

19. $-3x + 4y = 7$

20. $2x - y = 3$

21. $9x + y = -1$

22. $2x + 3y = 11$

23. $-4x + y = -2$

24. Write parametric equations for the line passing through the point at $(-2, 5)$ and parallel to the line $3x - 6y = -8$.

Write an equation in slope-intercept form of the line with the given parametric equations.

25. $x = 2t$
$y = 1 - t$

26. $x = -7 + \frac{1}{2}t$
$y = 3t$

27. $x = 4t - 11$
$y = t + 3$

28. $x = 4t - 8$
$y = 3 + t$

29. $x = 3 + 2t$
$y = -1 + 5t$

30. $x = 8$
$y = 2t + 1$

31. A line passes through the point at $(11, -4)$ and is parallel to $\vec{\mathbf{a}} = \langle 3, 7\rangle$.

a. Write a vector equation of the line.

b. Write parametric equations for the line.

c. Use the parametric equations to write the equation of the line in slope-intercept form.

Graphing Calculator

Use a graphing calculator to set up a table of values and then graph each line from its parametric form.

32. $x = 5t$
$y = -4 - t$

33. $x = 3t + 5$
$y = 1 + t$

34. $x = 1 + t$
$y = 1 - t$

Applications and Problem Solving

Real World Application

35. Geometry A line in a plane is defined by the parametric equations $x = 2 + 3t$ and $y = 4 + 7t$.

a. What part of this line is obtained by assigning only non-negative values to t?

b. How should t be defined to give the part of the line to the left of the y-axis?

36. Critical Thinking Graph the parametric equations $x = \cos^2 t$ and $y = \sin^2 t$.

37. Navigation During a military training exercise, an unmanned target drone has been detected on a radar screen following a path represented by the vector equation $\langle x, y\rangle = (3, 4) + t\langle -1, 0\rangle$. A surface to air missile is launched to intercept it and destroy it. This missile is following a trajectory represented by $\langle x, y\rangle = (2, 2) + t\langle 1, 2\rangle$.

a. Write the parametric equations for the path of the target drone and the missile.

b. Will the missile intercept the target drone?

38. Astronomy Astronomers have traced the path of two asteroids traveling through space. At a particular time t, the position of asteroid Ceres can be represented by $(-1 + t, 4 - t, -1 + 2t)$. Asteroid Pallas' path at any time t can be expressed by $(-7 + 2t, -6 + 2t, -1 + t)$.

a. Write the parametric equations for the path of each asteroid.

b. Do the paths of the asteroids cross? If so, where?

c. Following these paths, will the asteroids collide? If so, where?

39. Critical Thinking Find the parametric equations for the line passing through points at $\left(-\frac{1}{3}, 1, 1\right)$ and $(0, 5, -8)$. (*Hint:* Equations are needed for x, y, and z)

Mixed Review

40. An airplane flies at 150 km/h and heads 30° south of east. A 50-km/h wind blows in the direction 25° west of south. Find the ground speed and direction of the plane. *(Lesson 8-5)*

41. Find the inner product $\langle 1, 3\rangle \cdot \langle 3, -2\rangle$ and state whether the vectors are perpendicular. Write *yes* or *no*. *(Lesson 8-4)*

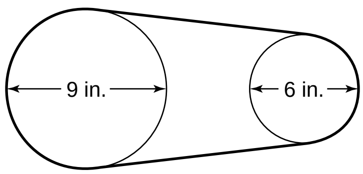
42. Solve $\triangle ABC$ if $A = 40°$, $b = 16$, and $a = 9$. *(Lesson 5-7)*

43. Approximate the real zero(s) of $f(x) = x^5 + 3x^3 - 4$ to the nearest tenth. *(Lesson 4-5)*

44. Find the inverse of the function $y = \frac{3}{2}x - 2$. *(Lesson 3-4)*

45. Write the standard form of the equation of the line that is parallel to the graph of $y = x - 8$ and passes through the point at $(-3, 1)$. *(Lesson 1-5)*

46. SAT/ACT Practice A pulley having a 9-inch diameter is belted to a pulley having a 6-inch diameter, as shown in the figure. If the larger pulley runs at 120 rpm (revolutions per minute), how fast does the smaller pulley run?



A 80 rpm **B** 100 rpm **C** 160 rpm
D 180 rpm **E** 240 rpm

Extra Practice See p. A41.

GRAPHING CALCULATOR EXPLORATION

8-6B Modeling With Parametric Equations

An Extension of Lesson 8-6

OBJECTIVE

- Investigate the use of parametric equations.

You can use a graphing calculator and parametric equations to investigate real-world situations, such as simulating a 500-mile, two airplane race. Suppose the first plane averages a speed of 408.7 miles per hour for the entire race. The second plane starts 30 seconds after the first and averages 418.3 miles per hour.

TRY THIS

Write a set of parametric equations to represent each plane's position at t hours. This will simulate the race on two parallel race courses so that it is easy to make a visual comparison. Use the formula $x = vt$.

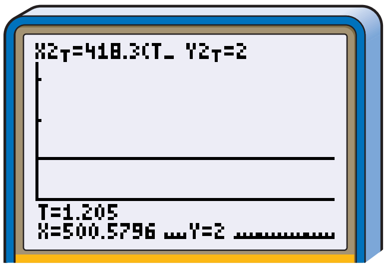
Plane 1: $x = 408.7t$ — *$x = vt$*

$y = 1$ — *y represents the position of plane 1 on course 1.*

Plane 2: $x = 418.3(t - 0.0083)$ — *Remember that plane 2 started 30 seconds or 0.0083 hour later.*

$y = 2$ — *Plane 2 is on course 2.*

Set the graphing calculator to parametric and simultaneous modes. Next, set the viewing window to the following values: **Tmin** = 0, **Tmax** = 5, **Tstep** = .005, **Xmin** = 0, **Xmax** = 500, **Xscl** = 10, **Ymin** = 0, **Ymax** = 5, and **Yscl** = 1. Then enter the parametric equations on the **Y=** menu. Then press [GRAPH] to "see the race."



Which plane finished first? Notice that the line on top reached the edge of the screen first. That line represented the position of the second plane, so plane 2 finished first. You can confirm the conclusion using the [TRACE] function. The plane with the smaller t-value when x is about 500 is the winner. Note that when x_1 is about 500, $t = 1.225$ and when x_2 is about 500, $t = 1.205$.

WHAT DO YOU THINK?

1. How long does it take the second plane to overtake the first?
2. How far have the two planes traveled when the second plane overtakes the first?
3. How much would the pilot of the first plane have to increase the plane's speed to win the race?

8-7

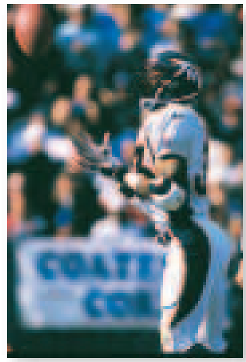
Modeling Motion Using Parametric Equations

OBJECTIVES

- Model the motion of a projectile using parametric equations.
- Solve problems related to the motion of a projectile, its trajectory, and range.

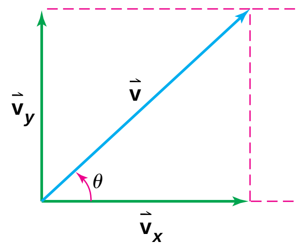
Real World Application

SPORTS Suppose a professional football player kicks a football with an initial velocity of 29 yards per second at an angle of 68° to the horizontal. Suppose a kick returner catches the ball 5 seconds later. How far has the ball traveled horizontally and what is its vertical height at that time? *This problem will be solved in Example 2.*

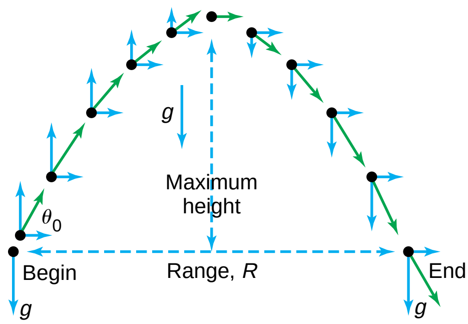


Objects that are launched, like a football, are called *projectiles*. The path of a projectile is called its *trajectory*. The horizontal distance that a projectile travels is its *range*. Physicists describe the motion of a projectile in terms of its position, velocity, and acceleration. All these quantities can be represented by vectors.

The figure at the right describes the initial trajectory of a punted football as it leaves the kicker's foot. The magnitude of the initial velocity $|\vec{\mathbf{v}}|$ and the direction θ of the ball can be described by a vector that can be expressed as the sum of its horizontal and vertical components $\vec{\mathbf{v}}_x$ and $\vec{\mathbf{v}}_y$.



As the ball moves, gravity will act on it in the vertical direction. The horizontal component is unaffected by gravity. So, discounting air resistance, the horizontal speed is constant throughout the flight of the ball.



The vertical component of the velocity of the ball is large and positive at the beginning, decreasing to zero at the top of its trajectory, then increasing in the negative direction as it falls. When the ball returns to the ground, its vertical speed is the same as when it left the kicker's foot, but in the opposite direction.

Parametric equations can represent the position of the ball relative to the starting point in terms of the parameter of time.

In order to find parametric equations that represent the path of a projectile like a football, we must write the horizontal $\vec{\mathbf{v}}_x$ and vertical $\vec{\mathbf{v}}_y$ components of the initial velocity.

$$\cos\theta = \frac{|\vec{\mathbf{v}}_x|}{|\vec{\mathbf{v}}|} \qquad \sin\theta = \frac{|\vec{\mathbf{v}}_y|}{|\vec{\mathbf{v}}|}$$

$$|\vec{\mathbf{v}}_x| = |\vec{\mathbf{v}}|\cos\theta \qquad |\vec{\mathbf{v}}_y| = |\vec{\mathbf{v}}|\sin\theta$$

Example 1 **Find the initial horizontal velocity and vertical velocity of a stone kicked with an initial velocity of 16 feet per second at an angle of 38° with the ground.**

$$|\vec{\mathbf{v}}_x| = |\vec{\mathbf{v}}|\cos\theta \qquad |\vec{\mathbf{v}}_y| = |\vec{\mathbf{v}}|\sin\theta$$

$$|\vec{\mathbf{v}}_x| = 16\cos 38^\circ \qquad |\vec{\mathbf{v}}_y| = 16\sin 38^\circ$$

$$|\vec{\mathbf{v}}_x| \approx 13 \qquad |\vec{\mathbf{v}}_y| \approx 10$$

The initial horizontal velocity is about 13 feet per second and the initial vertical velocity is about 10 feet per second.

Because horizontal velocity is unaffected by gravity, it is the magnitude of the horizontal component of the initial velocity. Therefore, the horizontal position of a projectile, after t seconds is given by the following equation.

$$\underbrace{\textit{horizontal distance}}_{x} = \underbrace{\textit{horizontal velocity}}_{|\vec{\mathbf{v}}|\cos\theta} \cdot \underbrace{\textit{time}}_{t}$$

$$x = t|\vec{\mathbf{v}}|\cos\theta$$

Since vertical velocity is affected by gravity, we must adjust the vertical component of initial velocity. By subtracting the vertical displacement due to gravity from the vertical displacement caused by the initial velocity, we can determine the height of the projectile after T seconds. The height, in feet or meters, of a free-falling object affected by gravity is given by the equation $h = \frac{1}{2}gt^2$, where $g \approx 9.8$ m/s^2 or 32 ft/s^2 and t is the time in seconds.

$$\underbrace{\textit{vertical displacement}}_{y} = \underbrace{\textit{displacement due to initial velocity}}_{(|\vec{\mathbf{v}}|\sin\theta)t} - \underbrace{\textit{displacement due to gravity}}_{\frac{1}{2}gt^2}$$

$$y = t|\vec{\mathbf{v}}|\sin\theta - \frac{1}{2}gt^2$$

Therefore, the path of a projectile can be expressed in terms of parametric equations.

Parametric Equations for the Path of a Projectile

If a projectile is launched at an angle of θ with the horizontal with an initial velocity of magnitude $|\vec{\mathbf{v}}|$, the path of the projectile may be described by these equations, where t is time and g is acceleration due to gravity.

$$x = t|\vec{\mathbf{v}}| \cos \theta$$
$$y = t|\vec{\mathbf{v}}| \sin \theta - \frac{1}{2}gt^2$$

Example 2 Real World Application

SPORTS **Refer to the application at the beginning of the lesson.**

a. How far has the ball traveled horizontally and what is its vertical height at that time?

b. Suppose the kick returner lets the ball hit the ground instead of catching it. What is the hang time, the elapsed time between the moment the ball is kicked and the time it hits the ground?

a. Write the position of the ball as a pair of parametric equations defining the path of the ball for any time t in seconds.

$x = t|\vec{\mathbf{v}}| \cos \theta$ $\qquad$ $y = t|\vec{\mathbf{v}}| \sin \theta - \frac{1}{2}gt^2$

The initial velocity of 29 yards per second must be expressed as 87 feet per second as gravity is expressed in terms of feet per second squared.

$x = t(87) \cos 68°$ $\qquad$ $y = t(87) \sin 68° - \frac{1}{2}(32)t^2$

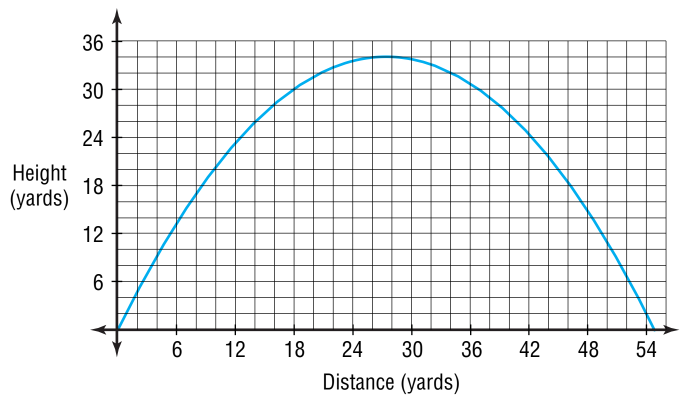
$x = 87t \cos 68°$ $\qquad$ $y = 87t \sin 68° - 16t^2$

Find x and y when $t = 5$.

$x = 87(5) \cos 68°$ $\qquad$ $y = 87(5) \sin 68° - 16(5)^2$

$x \approx 163$ $\qquad$ $y \approx 3$

After 5 seconds, the football has traveled about 163 feet or $54\frac{1}{3}$ yards horizontally and is about 3 feet or 1 yard above the ground.



b. Determine when the vertical height is 0 using the equation for y.

$y = 87t \sin 68° - 16t^2$

$y = 80.66t - 16t^2$

$y = t(80.66 - 16t)$ $\qquad$ *Factor.*

(continued on the next page)

Now let $y = 0$ and solve for t.

$$0 = 80.66 - 16t$$
$$16t = 80.66$$
$$t = 5.04$$

The hang time is about 5 seconds.

The parametric equations describe the path of an object that is launched from ground level. Some objects are launched from above ground level. For example, a baseball may be hit at a height of 3 feet. So, you must add the initial vertical height to the expression for y. This accounts for the fact that at time 0, the object will be above the ground.

Example 3 Real World Application

SOFTBALL **Kaci Clark led the Women's Pro Softball League in strikeouts in 1998. Suppose she throws the ball at an angle of 5.2° with the horizontal at a speed of 67 mph. The distance from the pitcher's mound to home plate is 43 feet. If Kaci releases the ball 2.7 feet above the ground, how far above the ground is the ball when it crosses home plate?**

First, write parametric equations that model the path of the softball. Remember to convert 67 mph to about 98.3 feet per second.

$$x = t|\vec{\mathbf{v}}| \cos \theta \qquad y = t|\vec{\mathbf{v}}| \sin \theta - \frac{1}{2}gt^2 + h$$
$$x = t(98.3)\cos 5.2^\circ \qquad y = t(98.3)\sin 5.2^\circ - \frac{1}{2}(32)t^2 + 2.7$$
$$x = 98.3t \cos 5.2^\circ \qquad y = 98.3t \sin 5.2^\circ - 16t^2 + 2.7$$

Kaci Clark

Then, find the amount of time that it will take the baseball to travel 43 feet horizontally. This will be the moment when it crosses home plate.

$$43 = 98.3t \cos 5.2^\circ$$
$$t = \frac{43}{98.3 \cos 5.2^\circ}$$
$$t \approx 0.439$$

The softball will cross home plate in about 0.44 second.

To find the vertical position of the ball at that time, find y when $t = 0.44$.

$$y = 98.3t \sin 5.2^\circ - 16t^2 + 2.7$$
$$y = 98.3(0.44) \sin 5.2^\circ - 16(0.44)^2 + 2.7$$
$$y \approx 3.522$$

The softball will be about 3.5 feet above home plate.

CONTENTS

Check for Understanding

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** situations in which a projectile travels vertically. At what angle with the horizontal must the projectile be launched?
2. **Describe** the vertical velocities of a projectile at its launch and its landing.
3. **Explain** the effect the angle of a golf club's head has on the angle of initial velocity of a golf ball.

Guided Practice

4. Find the initial vertical velocity of a stone thrown with an initial velocity of 50 feet per second at an angle of 40° with the horizontal.
5. Find the initial horizontal velocity of a discus thrown with an initial velocity of 20 meters per second at an angle of 50° with the horizontal.

Find the initial horizontal velocity and vertical velocity for each situation.

6. A soccer ball is kicked with an initial velocity of 45 feet per second at an angle of 32° with the horizontal.
7. A stream of water is shot from a sprinkler head with an initial velocity of 7.5 meters per second at an angle of 20° with the ground.
8. **Meteorology** An airplane flying at an altitude of 3500 feet is dropping research probes into the eye of a hurricane. The path of the plane is parallel to the ground at the time the probes are released with an initial velocity of 300 mph.
   a. Write the parametric equations that represent the path of the probes.
   b. Sketch the graph describing the path of the probes.
   c. How long will it take the probes to reach the ground?
   d. How far will the probes travel horizontally before they hit the ground?

Exercises

Practice

Find the initial horizontal velocity and vertical velocity for each situation.

9. a javelin thrown at 65 feet per second at an angle of 60° with the horizontal
10. an arrow released at 47 meters per second at an angle of 10.7° with the horizontal
11. a cannon shell fired at 1200 feet per second at a 42° angle with the ground
12. a can is kicked with an initial velocity of 17 feet per second at an angle of 28° with the horizontal
13. a golf ball hit with an initial velocity of 69 yards per second at 37° with the horizontal
14. a kangaroo leaves the ground at an angle of 19° at 46 kilometers per hour

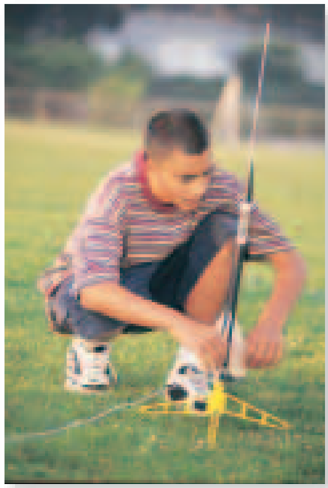
Applications and Problem Solving

Real World Application

15. **Golf** Professional golfer Nancy Lopez hits a golf ball with a force to produce an initial velocity of 175 feet per second at an angle of 35° above the horizontal. She estimates the distance to the hole to be 225 yards.
    a. Write the position of the ball as a pair of parametric equations.
    b. Find the range of the ball.

16. **Projectile Motion** What is the relationship between the angle at which a projectile is launched, the time it stays in the air, and the distance it covers?

17. **Physics** Alfredo and Kenishia are performing a physics experiment in which they will launch a model rocket. The rocket is supposed to release a parachute 300 feet in the air, 7 seconds after liftoff. They are firing the rocket at a 78° angle from the horizontal.
    a. Find the initial velocity of the rocket.
    b. To protect other students from the falling rockets, the teacher needs to place warning signs 50 yards from where the parachute is released. How far should the signs be from the point where the rockets are launched?



18. **Critical Thinking** Is it possible for a projectile to travel in a circular arc rather than a parabolic arc? Explain your answer.

19. **Critical Thinking** Janelle fired a projectile and measured its range. She hypothesizes that if the magnitude of the initial velocity is doubled and the angle of the velocity remains the same, the projectile will travel twice as far as it did before. Do you agree with her hypothesis? Explain.

20. **Aviation** Commander Patrick Driscoll flies with the U.S. Navy's Blue Angels. Suppose he places his aircraft in a 45° dive at an initial speed of 800 km/h.
    a. Write parametric equations to represent the descent path of the aircraft.
    b. How far has the aircraft descended after 2.5 seconds?
    c. What is the average rate at which the plane is losing altitude during the first 2.5 seconds?

21. **Entertainment** The "Human Cannonball" is shot out of a cannon with an initial velocity of 70 mph 10 feet above the ground at an angle of 35°.
    a. What is the maximum range of the cannon?
    b. How far from the launch point should a safety net be placed if the "Human Cannonball" is to land on it at a point 8 feet above the ground?
    c. How long is the flight of the "Human Cannonball" from the time he is launched to the time he lands in the safety net?

22. **Critical Thinking** If a circle of radius 1 unit is rolled along the x-axis at a rate of 1 unit per second, then the path of a point P on the circle is called a *cycloid*.
    a. Sketch what you think a cycloid must look like. (*Hint:* Use a coin or some other circular object to simulate the situation.)

    b. The parametric equations of a cycloid are $x = t - \sin t$ and $y = 1 - \cos t$ where t is measured in radians. Use a graphing calculator to graph the cycloid. Set your calculator in radian mode. An appropriate range is **Tmin** = 0, **Tmax** = 18.8, **Tstep** = 0.2, **Xmin** = −6.5, **Xmax** = 25.5, **Xscl** = 2, **Ymin** = −8.4, **Ymax** = 12.9, and **Yscl** = 1. Compare the results with your sketch.

23. Entertainment The Independence Day fireworks at Memorial Park are fired at an angle of 82° with the horizontal. The technician firing the shells expects them to explode about 300 feet in the air 4.8 seconds after they are fired.

a. Find the initial velocity of a shell fired from ground level.

b. Safety barriers will be placed around the launch area to protect spectators. If the barriers are placed 100 yards from the point directly below the explosion of the shells, how far should the barriers be from the point where the fireworks are launched?

24. Baseball Derek Jeter, shortstop for the New York Yankees, comes to bat with runners on first and third bases. Greg Maddux, pitcher for the Atlanta Braves, throws a slider across the plate about waist high, 3 feet above the ground. Derek Jeter hits the ball with an initial velocity of 155 feet per second at an angle of 22° above the horizontal. The ball travels straight at the 420 foot mark on the center field wall which is 15 feet high.

a. Write parametric equations that describe the path of the ball.

b. Find the height of the ball after it has traveled 420 feet horizontally. Will the ball clear the fence for a home run, or will the center fielder be able to catch it?

c. If there were no outfield seats, how far would the ball travel before it hits the ground?

Mixed Review

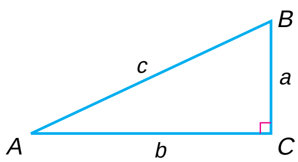
25. Write an equation of the line with parametric equations $x = 11 - t$ and $y = 8 - 6t$ in slope-intercept form. *(Lesson 8-6)*

26. Food Industry Fishmongers will often place ice over freshly-caught fish that are to be shipped to preserve the freshness. Suppose a block of ice with a mass of 300 kilograms is held on an ice slide by a rope parallel to the slide. The slide is inclined at an angle of 22°. *(Lesson 8-5)*

a. What is the pull on the rope?

b. What is the force on the slide?

27. If $b = 17.4$ and $c = 21.9$, find the measure of angle A to the nearest degree. *(Lesson 5-5)*



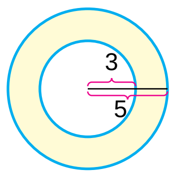
28. Solve the system of equations. *(Lesson 2-2)*

$2x - y + z = 2$

$x + 3y - 2z = -3.25$

$-4x - 5y + z = 2.5$

29. SAT/ACT Practice What is the area of the shaded region?



A 19π **B** 16π

C 9π **D** 4π

E 2π

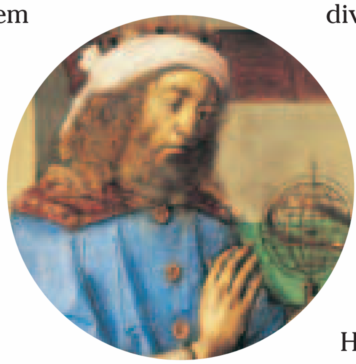
Extra Practice See p. A41.

CONTENTS

HISTORY of MATHEMATICS

ANGLE MEASURE

You know that the direction of a vector is the directed angle, usually measured in degrees, between the positive x-axis and the vector. How did that system originate? How did other angle measurement systems develop?


Ptolemy

Early Evidence The division of a circle into 360° is based upon a unit of distance used by the **Babylonians,** which was equal to about 7 miles. They related time and miles by observing the time that it took for a person to travel one of their units of distance. An entire day was 12 "time-miles," so one complete revolution of the sky by the sun was divided into 12 units. They also divided each time-mile into 30 units. So 12×30 or 360 units represented a complete revolution of the sun. Other historians feel that the Babylonians used 360° in a circle because they had a base 60 number system, and thought that there were 360 days in a year.

The Greeks adopted the 360° circle. **Ptolemy** (85-165) used the degree system in his work in astronomy, introducing symbols for degrees, minutes and seconds in his mathematical work, *Almagest,* though they differed from our modern symbols. When his work was translated into Arabic, sixtieths were called "first small parts," sixtieths of sixtieths were called "second small parts," and so on. The later Latin translations were "partes minutae primae," now our "minutes," and "partes minutae secundae," now our "seconds."

The Renaissance The degree symbol (°) became more widely used after the publication of a book by Dutch mathematician, **Gemma Frisius** in 1569.

Modern Era Radian angle measure was introduced by **James Thomson** in 1873. The radian was developed to simplify formulas used in trigonometry and calculus. Radian measure can be given as an exact quantity not just an approximation.

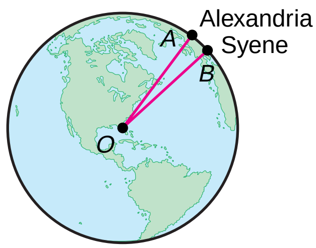
Other measurement units for angles have been developed to meet a specific need. These include mils, decimally- or centesimally-divided degrees, and millicycles.

Today mechanical engineers, like **Mark Korich,** use angle measure in designing electric and hybrid vehicles. Mr. Korich, a senior staff engineer for an auto manufacturer, calculates the draft angle that allows molded parts to pull away from the mold more easily. He also uses vectors to calculate the magnitude and direction of forces acting upon the vehicle under normal conditions.

ACTIVITIES

1. Greek mathematician, **Eratosthenes,** calculated the circumference of Earth using proportion and angle measure. By using the angle of a shadow cast by a rod in Alexandria, he determined that $\angle AOB$ was equal to $7°12'$. He knew that the distance from Alexandria to Syene was 5000 stadia (singular stadium). One stadium equals 500 feet. Use the diagram to find the circumference of Earth. Compare your result to the actual circumference of Earth.

2. Research one of the famous construction problems of the Greeks—trisecting an angle using a straightedge and compass. Try to locate a solution to this problem.



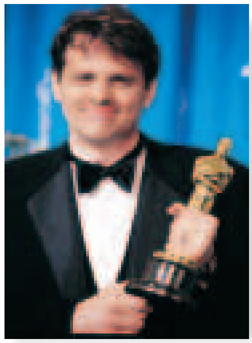
3. interNET CONNECTION Find out more about those who contributed to the history of angle measure. Visit **www.amc.glencoe.com**

8-8 Transformation Matrices in Three-Dimensional Space

OBJECTIVE
- Transform three-dimensional figures using matrix operations to describe the transformation.

Real World Application

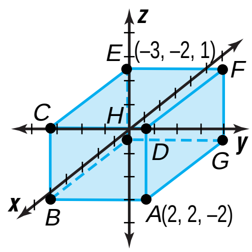
COMPUTER ANIMATION Chris Wedge of Blue Sky Studios, Inc. used software to create the film that won the 1999 Academy Award for Animated Short Film. The computer software allows Mr. Wedge to draw three-dimensional objects and manipulate or transform them to create motion, color, and light direction. The mathematical processes used by the computer are very complex. *A problem related to animation will be solved in Example 2.*



Look Back
Refer to Lesson 2-3 to review ordered pairs and matrices.

Basic movements in three-dimensional space can be described using vectors and transformation matrices. Recall that a point at (x, y) in a two-dimensional coordinate system can be represented by the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. This idea can be extended to three-dimensional space. A point at (x, y, z) can be expressed as the matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Example 1 **Find the coordinates of the vertices of the rectangular prism and represent them as a vertex matrix.**



$\overrightarrow{AE} = \langle -3 - 2, -2 - 2, 1 - (-2) \rangle$ or $\langle -5, -4, 3 \rangle$

You can use the coordinates of $\overrightarrow{AE}$ to find the coordinates of the other vertices.

$B(2, 2 + (-4), -2) = B(2, -2, -2)$
$C(2, 2 + (-4), -2 + 3) = C(2, -2, 1)$
$D(2, 2, -2 + 3) = D(2, 2, 1)$
$F(2 + (-5), 2, -2 + 3) = F(-3, 2, 1)$
$G(2 + (-5), 2, -2) = G(-3, 2, -2)$
$H(2 + (-5), 2 + (-4), -2) = H(-3, -2, -2)$

You could also find the coordinates of B, G, and H first, then add 3 to the z-coordinates of A, B, and G to find the coordinates of D, C, and F.

The vertex matrix for the prism is

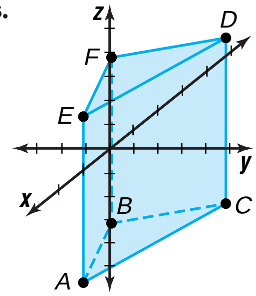
$$\begin{matrix} \\ x \\ y \\ z \end{matrix} \begin{matrix} A & B & C & D & E & F & G & H \\ \left[\begin{matrix} 2 \\ 2 \\ -2 \end{matrix}\right. & \begin{matrix} 2 \\ -2 \\ -2 \end{matrix} & \begin{matrix} 2 \\ -2 \\ 1 \end{matrix} & \begin{matrix} 2 \\ 2 \\ 1 \end{matrix} & \begin{matrix} -3 \\ -2 \\ 1 \end{matrix} & \begin{matrix} -3 \\ 2 \\ 1 \end{matrix} & \begin{matrix} -3 \\ 2 \\ -2 \end{matrix} & \left.\begin{matrix} -3 \\ -2 \\ -2 \end{matrix}\right] \end{matrix}.$$

In Lesson 2-4, you learned that certain matrices could transform a polygon on a coordinate plane. Likewise, transformations of three-dimensional figures, called **polyhedra,** can also be represented by certain matrices. A polyhedron (singular of *polyhedra*) is a closed three-dimensional figure made up of flat polygonal regions.

Example 2 **COMPUTER ANIMATION** **Maria is working on a computer animation project.**

Real World Application

She needs to translate a prism using the vector $\vec{\mathbf{a}} = \langle 3, 3, 0\rangle$. The vertices of the prism have the following coordinates.



$A(2, 1, -4)$ $\quad$ $B(-1, -1, -4)$ $\quad$ $C(-2, 3, -4)$

$D(-2, 3, 3)$ $\quad$ $E(2, 1, 3)$ $\quad$ $F(-1, -1, 3)$

a. Write a matrix that will have such an effect on the figure.

b. Find the coordinates of the vertices of the translated image.

c. Graph the translated image.

a. To translate the prism by the vector $\vec{\mathbf{a}} = \langle 3, 3, 0\rangle$, we must first add 3 to each of the x- and y-coordinates. The z-coordinates remain the same. The translation matrix can be written as

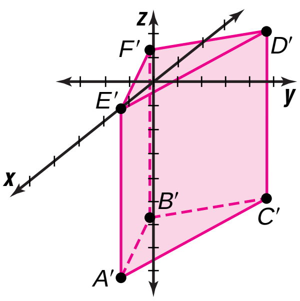
$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

b. Write the vertices of the prism in a 6×3 matrix. Then add it to the translation matrix to find the vertices of the translated image.

$$\begin{bmatrix} 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{matrix} A & B & C & D & E & F \\ \end{matrix} \begin{bmatrix} 2 & -1 & -2 & -2 & 2 & -1 \\ 1 & -1 & 3 & 3 & 1 & -1 \\ -4 & -4 & -4 & 3 & 3 & 3 \end{bmatrix}$$

$$= \begin{matrix} A' & B' & C' & D' & E' & F' \end{matrix} \begin{bmatrix} 5 & 2 & 1 & 1 & 5 & 2 \\ 4 & 2 & 6 & 6 & 4 & 2 \\ -4 & -4 & -4 & 3 & 3 & 3 \end{bmatrix}$$

c. Draw the graph of the image.



Recall that certain 2×2 matrices could be used to reflect a plane figure across an axis. Likewise, certain 3×3 matrices can be used to reflect three-dimensional figures in space.

Example 3 **Let M represent the vertex matrix of the rectangular prism in Example 1.**

a. Find TM if $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

b. Graph the resulting image.

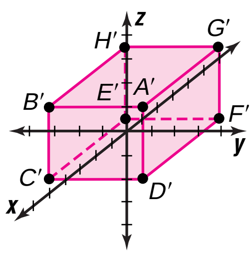
c. Describe the transformation represented by matrix T.

a. First find TM.

$$TM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 & 2 & 2 & -3 & -3 & -3 & -3 \\ 2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 \\ -2 & -2 & 1 & 1 & 1 & 1 & -2 & -2 \end{bmatrix}$$

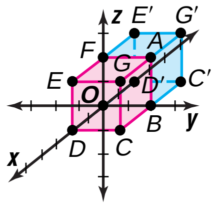
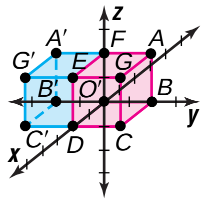
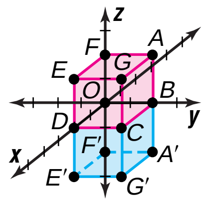
$$TM = \begin{matrix} A' & B' & C' & D' & E' & F' & G' & H' \end{matrix}$$
$$TM = \begin{bmatrix} 2 & 2 & 2 & 2 & -3 & -3 & -3 & -3 \\ 2 & -2 & -2 & 2 & -2 & 2 & 2 & -2 \\ 2 & 2 & -1 & -1 & -1 & -1 & 2 & 2 \end{bmatrix}$$

b. Then graph the points represented by the resulting matrix.



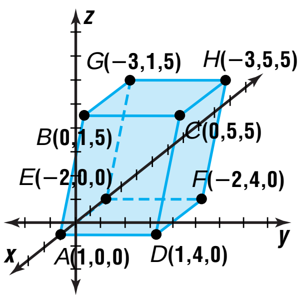
c. The transformation matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ reflects the image of each vertex over the xy-plane. This results in a reflection of the prism when the new vertices are connected by segments.

The transformation matrix in Example 3 resulted in a reflection over the xy-plane. Similar transformations will result in reflections over the xz- and yz-planes. These transformations are summarized in the chart on the next page.

Reflection Matrices		
For a reflection over the:	**Multiply the vertex matrix by:**	**Resulting image**
yz-plane	$R_{yz\text{-}plane} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
xz-plane	$R_{xz\text{-}plane} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
xy-plane	$R_{xy\text{-}plane} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

One other transformation of two-dimensional figures that you have studied is the *dilation*. A dilation with scale factor k, for $k \neq 0$, can be represented by the matrix $D = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$.

Example 4 **A parallelepiped is a prism whose faces are all parallelograms as shown in the graph.**



a. Find the vertex matrix for the transformation D where $k = 2$.

b. Draw a graph of the resulting figure.

c. What effect does transformation D have on the original figure?

a. If $k = 2$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Find the coordinates of the vertices of the parallelepiped. Write them as vertex matrix P.

$$P = \begin{matrix} A & B & C & D & E & F & G & H \end{matrix}$$
$$P = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & -2 & -3 & -3 \\ 0 & 1 & 5 & 4 & 0 & 4 & 1 & 5 \\ 0 & 5 & 5 & 0 & 0 & 0 & 5 & 5 \end{bmatrix}$$

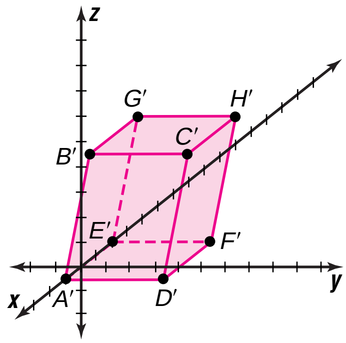
Then, find the product of D and P.

$$DP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 & -2 & -2 & -3 & -3 \\ 0 & 1 & 5 & 4 & 0 & 4 & 1 & 5 \\ 0 & 5 & 5 & 0 & 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\begin{matrix} A' & B' & C' & D' & E' & F' & G' & H' \end{matrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 2 & -4 & -4 & -6 & -6 \\ 0 & 2 & 10 & 8 & 0 & 8 & 2 & 10 \\ 0 & 10 & 10 & 0 & 0 & 0 & 10 & 10 \end{bmatrix}$$

b. Graph the transformation.



c. The transformation matrix D is a dilation. The dimensions of the prism have increased by a factor of 2.

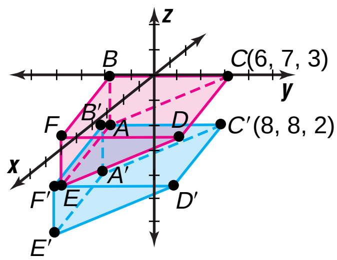
CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** the transformation that matrix $T = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ produces on a three-dimensional figure.

2. **Write** a transformation matrix that represents the translation shown at the right.



3. **Determine** if multiplying a vertex matrix by the transformation matrix

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ produces the same result as multiplying by

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ and then by $$V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. **Math Journal** Using a chart, describe the effects reflection, translation, and dilation have on

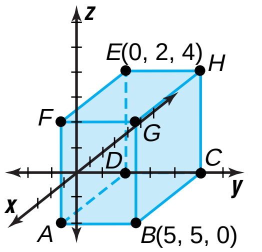
a. the figure's orientation on the coordinate system.

b. the figure's size.

c. the figure's shape.

Guided Practice

5. Refer to the rectangular prism at the right.



a. Write the matrix for the figure.

b. Write the resulting matrix if you translate the figure using the vector $\langle 4, -1, 2\rangle$.

c. Transform the figure using the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Graph the image and describe the result.

d. Describe the transformation on the figure resulting from its product with the matrix $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$.

6. **Architecture** Trevor is revising a design for a playground that contains a piece of equipment shaped like a rectangular prism. He needs to enlarge the prism 4 times its size and move it along the x-axis 2 units.

a. What are the transformation matrices?

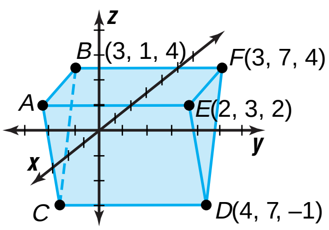
b. Graph the rectangular prism and its image after the transformation.

EXERCISES

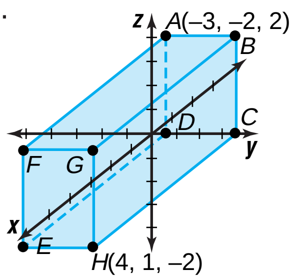
Practice

Write the matrix for each prism.

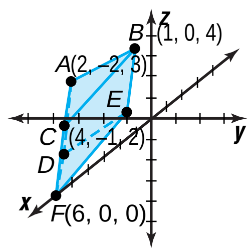
7.



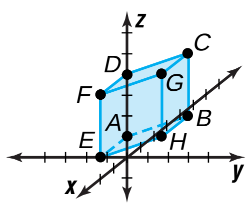
8.



9.



Use the prism at the right for Exercises 10–15. The vertices are $A(0, 0, 1)$, $B(0, 3, 2)$, $C(0, 5, 5)$, $D(0, 0, 4)$, $E(2, 0, 1)$, $F(2, 0, 4)$, $G(2, 3, 5)$, and $H(2, 3, 2)$. Translate the prism using the given vectors. Graph each image and describe the result.



10. $\vec{\mathbf{a}}\ \langle 0, -2, 4\rangle$ 11. $\vec{\mathbf{b}}\ \langle 1, -2, -2\rangle$ 12. $\vec{\mathbf{c}}\ \langle 1, 5, -3\rangle$

Refer to the figure for Exercises 10-12. Transform the figure using each matrix. Graph each image and describe the result.

13. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **14.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ **15.** $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Describe the result of the product of a vertex matrix and each matrix.

16. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **17.** $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ **18.** $\begin{bmatrix} -0.75 & 0 & 0 \\ 0 & -0.75 & 0 \\ 0 & 0 & -0.75 \end{bmatrix}$

19. A transformation in three-dimensional space can also be represented by $T(x, y, z) \rightarrow (2x, 2y, 5z)$.

a. Determine the transformation matrix.

b. Describe the transformation that such a matrix will have on a figure.

Applications and Problem Solving

20. Marine Biology A researcher studying a group of dolphins uses the matrix

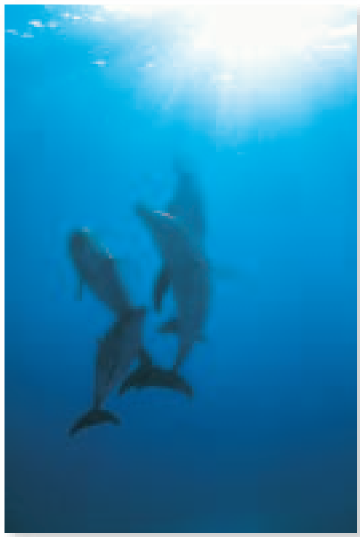
$$\begin{bmatrix} 20 & 136 & 247 & 302 & 351 \\ -58 & -71 & -74 & -83 & -62 \\ 27 & 53 & 59 & 37 & 52 \end{bmatrix},$$

with the ship at the origin, to track the dolphins' movement. Later, the researcher will translate the matrix to a fixed reference point using the vector $\langle 23.6, 72, 0 \rangle$.

a. Write the translation matrix used for the transformation.

b. What is the resulting matrix?

c. Describe the result of the translation.



21. Critical Thinking Write a transformational matrix that would first reflect a rectangular prism over the *yz*-plane and then reduce its dimensions by half.

22. Meteorology At the National Weather Service Center in Miami, Florida, meteorologists test models to forecast weather phenomena such as hurricanes and tornadoes. One weather disturbance, wind shear, can be modeled using a transformation on a cube. (*Hint:* You can use any size cube for the model.)

a. Write a matrix to transform a cube into a slanted parallelepiped.

b. Graph a cube and its image after the transformation.

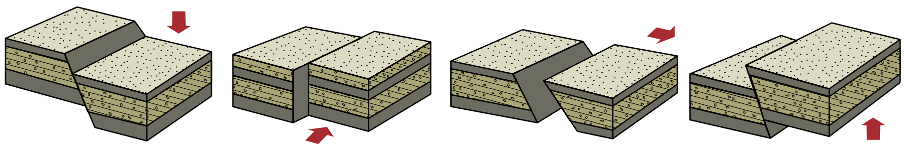
23. Geometry Suppose a cube is transformed by two matrices,

first by $U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, and then by $T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Describe the resulting image.

24. Critical Thinking Matrix T maps a point $P(x, y, z)$ to the point $P'(3x, 2y, x - 4z)$. Write a 3×3 matrix for T.

25. Seismology Seismologists classify movements in the earth's crust by determining the direction and amount of movement which has taken place on a fault. Some of the classifications are shown using *block diagrams.*



dip-slip　strike-slip　oblique slip　reverse fault

a. A seismologist used the matrix below to describe a particular feature along a fault.

$$\begin{bmatrix} 123.9 & -41.3 & 201.7 & 73.8 & -129.4 & 36.4 \\ 86.4 & 144.2 & -29.9 & -84.2 & 95.5 & -125.5 \\ 206.5 & 247.8 & 262.7 & 213.2 & -165.2 & -84.1 \end{bmatrix}$$

After a series of earthquakes he determined the matrix describing the feature had changed to the following matrix.

$$\begin{bmatrix} 123.9 & -41.3 & 201.7 & 73.8 & -129.4 & 36.4 \\ 88.0 & 145.8 & -28.3 & -82.6 & 97.1 & -123.9 \\ 205.3 & 246.6 & 261.5 & 212.0 & -166.4 & -85.3 \end{bmatrix}$$

What classification of movement bests describes the transformation that has occurred?

b. Determine the matrix that describes the movement that occurred.

Mixed Review

26. Physics Jaimie and LaShawna are standing at the edge of a cliff that is 150 feet high. At the same time, LaShawna drops a stone and Jaimie throws a stone horizontally at a velocity of 35 ft/s. *(Lesson 8-7)*

a. About how far apart will the stones be when they land?

b. Will the stones land at the same time? Explain.

27. Write an equation in slope-intercept form of the line with parametric equations $x = -5t - 1$ and $y = 2t + 10$. *(Lesson 8-6)*

28. Evaluate $\sec\left(\cos^{-1}\frac{2}{5}\right)$ if the angle is in Quadrant I. *(Lesson 6-8)*

29. Medicine Maria was told to take 80 milligrams of medication each morning for three days. The amount of medicine in her body on the fourth day is modeled by $M(x) = 80x^3 + 80x^2 + 80x$, where x represents the absorption rate per day. Suppose Maria has 24.2 milligrams of the medication present in her body on the fourth day. Find the absorption rate of the medication. *(Lesson 4-5)*

30. SAT/ACT Practice Which of the following equations are equivalent?

I. $2x + 4y = 8$　　II. $3x + 6y = 12$

III. $4x + 8y = 8$　　IV. $6x + 12y = 16$

A I and II only　**B** I and IV only　**C** II and IV only

D III and IV only　**E** I, II, and III

Extra Practice See p. A41.

STUDY GUIDE AND ASSESSMENT

VOCABULARY

component (p. 488)
cross product (p. 507)
direction (p. 485)
direction vector (p. 520)
dot product (p. 506)
equal vectors (p. 485)
inner product (p. 505)
magnitude (p. 485)
opposite vectors (p. 487)
parallel vectors (p. 488)
parameter (p. 520)
parametric equation (p. 520)
polyhedron (p. 535)
resultant (p. 486)
scalar (p. 488)
scalar quantity (p. 488)
standard position (p. 485)
unit vector (p. 495)
vector (p. 485)
vector equation (p. 520)
zero vector (p. 485)

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term from the list to complete each sentence.

1. The __?__ of two or more vectors is the sum of the vectors.
2. A(n) __?__ vector has a magnitude of one unit.
3. Two vectors are equal if and only if they have the same direction and __?__.
4. The __?__ product of two vectors is a vector.
5. Two vectors in space are perpendicular if and only if their __?__ product is zero.
6. Velocity can be represented mathematically by a(n) __?__.
7. Vectors with the same direction and different magnitudes are __?__.
8. A vector with its initial point at the origin is in __?__ position.
9. A __?__ vector is used to describe the slope of a line.
10. Two or more vectors whose sum is a given vector are called __?__ of the given vector.

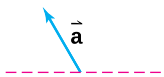
components
cross
direction
equal
inner
magnitude
parallel
parameter
resultant
scalar
standard
unit
vector

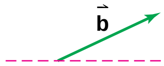
SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

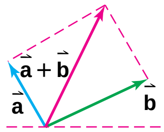
Lesson 8-1 Add and subtract vectors geometrically.

- Find the sum of $\vec{a}$ and $\vec{b}$ using the parallelogram method.





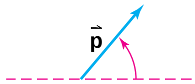
Place the initial points of $\vec{a}$ and $\vec{b}$ together. Draw dashed lines to form a complete parallelogram. The diagonal from the initial points to the opposite vertex of the parallelogram is the resultant.



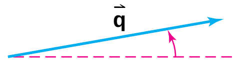
REVIEW EXERCISES

Use a ruler and a protractor to determine the magnitude (in centimeters) and direction of each vector.

11.



12.



Use $\vec{p}$ and $\vec{q}$ above to find the magnitude and direction of each resultant.

13. $\vec{p} + \vec{q}$ **14.** $2\vec{p} + \vec{q}$

15. $3\vec{p} - \vec{q}$ **16.** $4\vec{p} - \vec{q}$

Find the magnitude of the horizontal and vertical components of each vector shown for Exercises 11 and 12.

17. $\vec{p}$ **18.** $\vec{q}$

Lesson 8-2 Find ordered pairs that represent vectors, and sums and products of vectors.

- Write the ordered pair that represents the vector from $M(3, 1)$ to $N(-7, 4)$. Then find the magnitude of $\overline{MN}$.

$$\overline{MN} = \langle -7 - 3, 4 - 1 \rangle \text{ or } \langle -10, 3 \rangle$$

$$|\overline{MN}| = \sqrt{(-7-3)^2 + (4-1)^2} \text{ or } \sqrt{109}$$

Find $\vec{a} + \vec{b}$ if $\vec{a} = \langle 1, -5 \rangle$ and $\vec{b} = \langle -2, 4 \rangle$.

$$\vec{a} + \vec{b} = \langle 1, -5 \rangle + \langle -2, 4 \rangle$$
$$= \langle 1 + (-2), -5 + 4 \rangle \text{ or } \langle -1, -1 \rangle$$

Write the ordered pair that represents $\overline{CD}$. Then find the magnitude of $\overline{CD}$.

19. $C(2, 3)$, $D(7, 15)$

20. $C(-2, 8)$, $D(4, 12)$

21. $C(2, -3)$, $D(0, 9)$

22. $C(-6, 4)$, $D(-5, -4)$

Find an ordered pair to represent $\vec{u}$ in each equation if $\vec{v} = \langle 2, -5 \rangle$ and $\vec{w} = \langle 3, -1 \rangle$.

23. $\vec{u} = \vec{v} + \vec{w}$

24. $\vec{u} = \vec{v} - \vec{w}$

25. $\vec{u} = 3\vec{v} + 2\vec{w}$

26. $\vec{u} = 3\vec{v} - 2\vec{w}$

Lesson 8-3 Find the magnitude of vectors in three-dimensional space.

- Write the ordered triple that represents the vector from $R(-2, 0, 8)$ to $S(5, -4, -1)$. Then find the magnitude of $\overline{RS}$.

$$\overline{RS} = \langle 5 - (-2), -4 - 0, -1 - 8 \rangle$$
$$= \langle 7, -4, -9 \rangle$$

$$|\overline{RS}| = \sqrt{(5 - (-2))^2 + (-4 - 0)^2 + (-1 - 8)^2}$$
$$= \sqrt{49 + 16 + 81} \text{ or about } 12.1$$

Write the ordered triple that represents $\overline{EF}$. Then find the magnitude of $\overline{EF}$.

27. $E(2, -1, 4)$, $F(6, -2, 1)$

28. $E(9, 8, 5)$, $F(-1, 5, 11)$

29. $E(-4, -3, 0)$, $F(2, -1, 7)$

30. $E(3, 7, -8)$, $F(-4, 0, 5)$

Find an ordered triple to represent $\vec{u}$ in each equation if $\vec{v} = \langle -1, 7, -4 \rangle$ and $\vec{w} = \langle 4, -1, 5 \rangle$.

31. $\vec{u} = 2\vec{w} - 5\vec{v}$ **32.** $\vec{u} = 0.25\vec{v} + 0.4\vec{w}$

CONTENTS

OBJECTIVES AND EXAMPLES

Lesson 8-4 Find the inner and cross products of vectors.

- Find the inner product of $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ if $\vec{\mathbf{a}} = \langle 3, -1, 7 \rangle$ and $\vec{\mathbf{b}} = \langle 0, -2, -4 \rangle$. Are $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ perpendicular?

  $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 3(0) + (-1)(-2) + 7(-4)$ or -26

  $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are not perpendicular since their inner product is not zero.

  Find $\vec{\mathbf{c}} \times \vec{\mathbf{d}}$ if $\vec{\mathbf{c}} = \langle -2, 1, 1 \rangle$ and $\vec{\mathbf{d}} = \langle 1, -3, 0 \rangle$.

  $$\vec{\mathbf{c}} \times \vec{\mathbf{d}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ -2 & 1 & 1 \\ 1 & -3 & 0 \end{vmatrix}$$

  $$= \begin{vmatrix} 1 & 1 \\ -3 & 0 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} \vec{\mathbf{k}}$$

  $$= 3\vec{\mathbf{i}} + 1\vec{\mathbf{j}} + 5\vec{\mathbf{k}} \text{ or } \langle 3, 1, 5 \rangle$$

  Determine if $\vec{\mathbf{a}}$ is perpendicular to $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ if $\vec{\mathbf{a}} = \langle 3, 1, 5 \rangle$, $\vec{\mathbf{b}} = \langle -2, 1, 1 \rangle$, and $\vec{\mathbf{c}} = \langle 1, -3, 0 \rangle$.

  $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 3(-2) + 1(1) + 5(1)$ or 0

  $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} = 3(1) + 1(-3) + 5(0)$ or 0

  Since the inner products are zero, $\vec{\mathbf{a}}$ is perpendicular to both $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.

REVIEW EXERCISES

Find each inner product and state whether the vectors are perpendicular. Write *yes* or *no*.

33. $\langle 5, -1 \rangle \cdot \langle -2, 6 \rangle$

34. $\langle 2, 6 \rangle \cdot \langle 3, -4 \rangle$

35. $\langle 4, 1, -2 \rangle \cdot \langle 3, -4, 4 \rangle$

36. $\langle 2, -1, 4 \rangle \cdot \langle 6, -2, 1 \rangle$

37. $\langle 5, 2, -10 \rangle \cdot \langle 2, -4, -4 \rangle$

Find each cross product. Then verify if the resulting vector is perpendicular to the given vectors.

38. $\langle 5, -2, 5 \rangle \times \langle -1, 0, -3 \rangle$

39. $\langle -2, -3, 1 \rangle \times \langle 2, 3, -4 \rangle$

40. $\langle -1, 0, 4 \rangle \times \langle 5, 2, -1 \rangle$

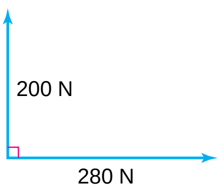
41. $\langle 7, 2, 1 \rangle \times \langle 2, 5, 3 \rangle$

42. Find a vector perpendicular to the plane containing the points (1, 2, 3), (−4, 2, −1), and (5, −3, 0).

OBJECTIVES AND EXAMPLES

Lesson 8-5 Solve problems using vectors and right triangle trigonometry.

- Find the magnitude and direction of the resultant vector for the diagram.

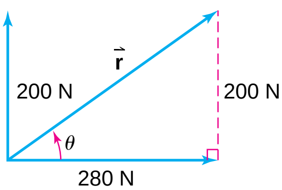


  $|\vec{\mathbf{r}}|^2 = 200^2 + 280^2$

  $|\vec{\mathbf{r}}|^2 = 118{,}400$

  $|\vec{\mathbf{r}}| = \sqrt{118{,}400}$

  ≈ 344 N



  $\tan \theta = \frac{200}{280}$

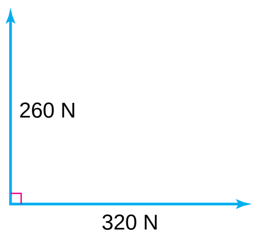
  $\theta = \tan^{-1} \frac{200}{280}$

  $\approx 35.6°$

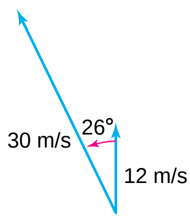
REVIEW EXERCISES

Find the magnitude and direction of the resultant vector for each diagram.

43.



44.



OBJECTIVES AND EXAMPLES

Lesson 8-6 Write vector and parametric equations of lines.

Write a vector equation of the line that passes through $P(-6, 3)$ and is parallel to $\vec{\mathbf{v}} = \langle 1, 4 \rangle$. Then write parametric equations of the line.

The vector equation is $\langle x + 6, y - 3 \rangle = t\langle 1, 4 \rangle$.

Write the parametric equations of a line with $(a_1, a_2) = (-6, 3)$ and $(x_1, y_1) = (1, 4)$.

$x = x_1 + ta_1 \qquad y = y_1 + ta_2$
$x = -6 + t(1) \qquad y = 3 + t(4)$
$x = -6 + t \qquad y = 3 + 4t$

REVIEW EXERCISES

Write a vector equation of the line that passes through point P and is parallel to $\vec{\mathbf{v}}$. Then write parametric equations of the line.

45. $P(3, -5)$, $\vec{\mathbf{v}} = \langle 4, 2 \rangle$

46. $P(-1, 9)$, $\vec{\mathbf{v}} = \langle -7, -5 \rangle$

47. $P(4, 0)$, $\vec{\mathbf{v}} = \langle 3, -6 \rangle$

Write parametric equations of each line with the given equation.

48. $y = -8x - 7$

49. $y = -\frac{1}{2}x + \frac{5}{2}$

Lesson 8-7 Model the motion of a projectile using parametric equations.

Find the initial horizontal and vertical velocity for an arrow released at 52 m/s at an angle of 12° with the horizontal.

$|\vec{\mathbf{v}}_x| = |\vec{\mathbf{v}}| \cos \theta \qquad |\vec{\mathbf{v}}_y| = |\vec{\mathbf{v}}| \sin \theta$
$|\vec{\mathbf{v}}_x| = 52 \cos 12° \qquad |\vec{\mathbf{v}}_y| = 52 \sin 12°$
$|\vec{\mathbf{v}}_x| \approx 50.86 \qquad |\vec{\mathbf{v}}_y| \approx 10.81$

The initial horizontal velocity is about 50.86 m/s and the initial vertical velocity is about 10.81 m/s.

Find the initial horizontal and vertical velocity for each situation.

50. a stone thrown with an initial velocity of 15 feet per second at an angle of 55° with the horizontal

51. a baseball thrown with an initial velocity of 13.2 feet per second at an angle of 66° with the horizontal

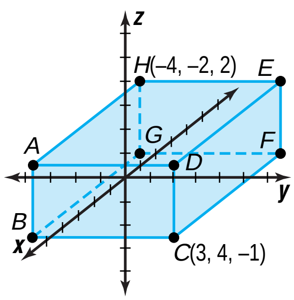
52. a soccer ball kicked with an initial velocity of 18 meters per second at an angle of 28° with the horizontal

Lesson 8-8 Transform three-dimensional figures using matrix operations to describe the transformation.

A prism needs to be translated using the vector $\langle 0, 1, 2 \rangle$. Write a matrix that will have such an effect on a figure.

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Use the prism for Exercises 53 and 54.



53. Translate the figure using the vector $\vec{\mathbf{n}} = \langle 2, 0, 3 \rangle$. Graph the image and describe the result.

54. Transform the figure using the matrix $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Graph the image and describe the result.

APPLICATIONS AND PROBLEM SOLVING

55. **Physics** Karen uses a large wrench to change the tire on her car. She applies a downward force of 50 pounds at an angle of 60° one foot from the center of the lug nut. Find the torque. *(Lesson 8-4)*

56. **Sports** Bryan punts a football with an initial velocity of 38 feet per second at an angle of 40° from the horizontal. If the ball is 2 feet above the ground when it is kicked, how high is it after 0.5 second? *(Lesson 8-7)*

57. **Navigation** A boat that travels at 16 km/h in calm water is sailing across a current of 3 km/h on a river 250 meters wide. The boat makes an angle of 35° with the current heading into the current. *(Lesson 8-5)*

   a. Find the resultant velocity of the boat.

   b. How far upstream is the boat when it reaches the other shore?

58. **Physics** Mario and Maria are moving a stove. They are applying forces of 70 N and 90 N at an angle of 30° to each other. If the 90 N force is applied along the x-axis find the magnitude and direction of the resultant of these forces. *(Lesson 8-5)*

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. The ordered pair $\langle -3, 2\rangle$ represents $\overrightarrow{XY}$.

   a. Give possible coordinates for X and Y. Show that your coordinates are correct.

   b. Find the magnitude of $\overrightarrow{XY}$. Did you need to know the coordinates for X and Y to find this magnitude? Explain.

2. a. $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are parallel. Give ordered pairs for P, Q, R, and S for which this is true. Explain how you know $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are parallel.

   b. $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are perpendicular. Give ordered pairs to represent $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Explain how you know that $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are perpendicular.

PORTFOLIO

Devise a real-world problem that can be solved using vectors. Explain why vectors are needed to solve your problem. Solve your problem. Be sure to show and explain all of your work.

Unit 2 interNET Project
THE CYBERCLASSROOM

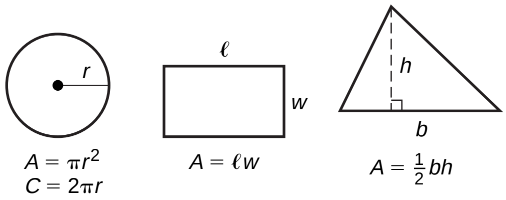
Vivid Vectors!

- Search the Internet to find websites that have lessons about vectors and their applications. Find at least three different sources of information.
- Make diagrams of examples that use vectors to solve problems by combining the ideas you found in Internet lessons and your textbook.
- Prepare a presentation to summarize the Internet Project for Unit 2. Design the presentation as a webpage. Use appropriate software to create the webpage.

Additional Assessment See p. A63 for Chapter 8 practice test.

Geometry Problems—Perimeter, Area, and Volume

Many SAT and ACT problems use perimeter, circumference, and area. A few problems use volume. Even though the formulas are often given on the test, it is more efficient if you memorize them.



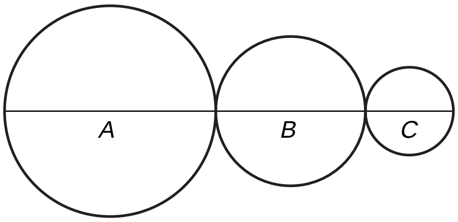
THE PRINCETON REVIEW

TEST-TAKING TIP

Recall that the value of π is about 3.14. When you need to estimate a quantity that involves π, use 3 as a rough approximation.

SAT EXAMPLE

1. In the figure below, the radius of circle A is twice the radius of the circle B and four times the radius of C. If the sum of the areas of the circles is 84π, what is the measure of $\overline{AC}$?



HINT Use a variable when necessary. Choose it carefully.

Solution This is a grid-in problem. You must find the length of $\overline{AC}$. This segment contains the radii of the circles. Let r be the radius of circle C. Then the radius of circle B is $2r$, and the radius of circle A is $4r$.

You know the total area. Write an equation for the sum of the three areas.

$$\text{Area } A + \text{Area } B + \text{Area } C = 84\pi$$
$$\pi(4r)^2 + \pi(2r)^2 + \pi(r)^2 = 84\pi$$
$$16r^2\pi + 4r^2\pi + r^2\pi = 84\pi$$
$$21r^2\pi = 84\pi$$
$$r^2 = 4$$
$$r = 2$$

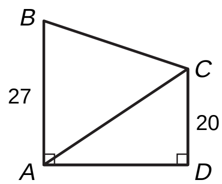
The radius of C is 2; the radius of B is 4; the radius of A is 8. Recall that the problem asked for the length of $\overline{AC}$. Use the diagram to see which lengths to add—one radius of A, two radii of B, and one radius of C.

$$AC = 8 + 4 + 4 + 2 \text{ or } 18$$

The answer is 18.

ACT EXAMPLE

2. In the figure below, if $AB = 27$, $CD = 20$, and the area of $\triangle ADC = 240$, what is the area of polygon $ABCD$ in square units?



A 420 **B** 480 **C** 540 **D** 564 **E** 1128

HINT You may write in the test booklets. Mark all information you discover on the figure.

Solution The polygon is made up of two triangles, so find the area of each triangle. You know that the area of $\triangle ADC$ is 240 and its height is 20.

$$A = \frac{1}{2}bh$$
$$240 = \frac{1}{2}(b)(20)$$
$$240 = 10b$$
$$24 = b \qquad \text{So, } AD = 24.$$

Now find the area of $\triangle ABC$. The height of this triangle can be measured along $\overline{AD}$. So, $h = 24$, and the height is the measure of $\overline{AB}$, or 27. The area of $\triangle ABC = \frac{1}{2}(27)(24)$ or 324. You need to add the areas of the two triangles.

$$240 + 324 = 564$$

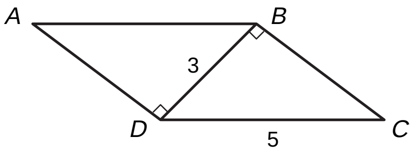
The answer is choice **D.**

SAT AND ACT PRACTICE

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

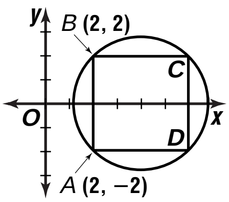
Multiple Choice

1. In parallelogram $ABCD$ below, $BD = 3$ and $CD = 5$. What is the area of $ABCD$?



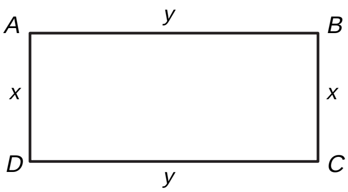
A 12 **B** 15 **C** 18 **D** 20
E It cannot be determined from the information given.

2. In square $ABCD$ below, what is the equation of circle Q that is circumscribed around the square?



A $(x - 4)^2 + y^2 = 4$ **B** $(x - 4)^2 + y^2 = 8$
C $(x + 4)^2 + y^2 = 8$ **D** $(x - 4)^2 + y^2 = 32$
E $(x + 4)^2 + y^2 = 32$

3. If the perimeter of the rectangle $ABCD$ is equal to p, and $x = \frac{2}{3}y$, what is the value of y in terms of p?



A $\frac{p}{10}$ **B** $\frac{3p}{10}$ **C** $\frac{p}{3}$ **D** $\frac{2p}{5}$ **E** $\frac{3p}{5}$

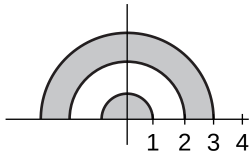
4. If two sides of a triangle have lengths of 40 and 80, which of the following cannot be the length of the third side?

A 40 **B** 41 **C** 50 **D** 80 **E** 81

5. $\sqrt[3]{x^2} \cdot \sqrt[9]{x^3} =$

A $x^{\frac{2}{9}}$ **B** $x^{\frac{1}{3}}$ **C** $x^{\frac{1}{2}}$ **D** $x^{\frac{2}{3}}$ **E** x

6. The figure below is made of three concentric semi-circles. What is the area of the shaded region in square units?



A 3π **B** $\frac{9}{2}\pi$ **C** 6π **D** 7π **E** 9π

7. $\frac{1}{5} + \frac{2}{25} + \frac{3}{50} =$

A 0.170 **B** 0.240 **C** 0.320
D 0.340 **E** 0.463

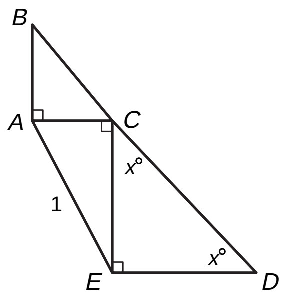
8. A 30-inch by 40-inch rectangular surface is to be completely covered with 1-inch square tiles, which cannot overlap one another and cannot overhang. If white tiles are to cover the interior and red tiles are to form a 1-inch wide border along the edge of the surface, how many red tiles will be needed?

A 70 **B** 136 **C** 140 **D** 142 **E** 144

9. Which of the following is the closest approximation of the area of a circle with radius x?

A $4x^2$
B $3x^2$
C $2x^2$
D x^2
E $0.75x^2$

10. Grid-In
In the figure, if $AE = 1$, what is the sum of the area of $\triangle ABC$ and the area of $\triangle CDE$?



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