UNIT 3

Advanced Functions and Graphing

You are now ready to apply what you have learned in earlier units to more complex functions. The three chapters in this unit contain very different topics; however, there are some similarities among them. The most striking similarity is that all three chapters require that you have had some experience with graphing. As you work through this unit, try to make connections between what you have already studied and what you are currently studying. This will help you use the skills you have already mastered more effectively.

Chapter 9 **Polar Coordinates and Complex Numbers**
Chapter 10 **Conics**
Chapter 11 **Exponential and Logarithmic Functions**

Unit 3 interNET Projects

SPACE—THE FINAL FRONTIER

People have been fascinated by space since the beginning of time. Until 1961, however, human beings have been bound to Earth, unable to feel and experience life in space. Current space programs undertaken by NASA are exploring our solar system and beyond using sophisticated unmanned satellites such as the Hubble Space Telescope in orbit around Earth and the Mars Global Surveyor in orbit around Mars. In these projects, you will look at some interesting mathematics related to space—the final frontier. At the end of each chapter in Unit 3, you will be given specific tasks to explore space using the Internet.

CHAPTER 9
(page 611)

From Point A to Point B In Chapter 9, you will learn about the polar coordinate system, which is quite different from the rectangular coordinate system. Do scientists use the rectangular coordinate system or the polar coordinate system as they record the position of objects in space? Or, do they use some other system?
Math Connection: Research coordinate systems by using the Internet. Write a summary that describes each coordinate system that you find. Include diagrams and any information about converting between systems.

CHAPTER 10
(page 691)

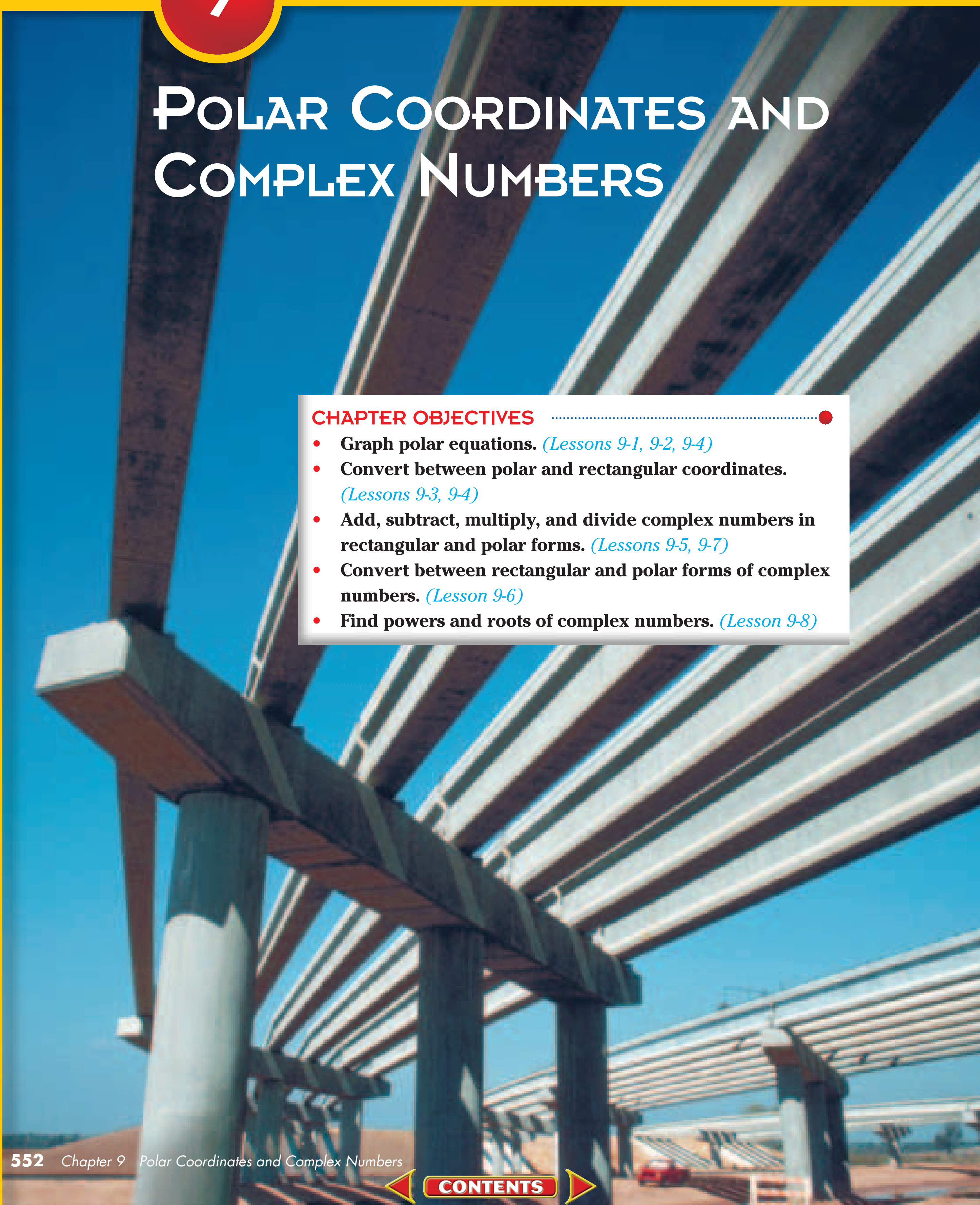
Out in Orbit! What types of orbits do planets, artificial satellites, or space exploration vehicles have? Can orbits be modeled by the conic sections?
Math Connection: Use the Internet to find data about the orbit of a space vehicle, satellite, or planet. Make a scale drawing of the object's orbit labeling important features and dimensions. Then, write a summary describing the orbit of the object, being sure to discuss which conic section best models the orbit.

CHAPTER 11
(page 753)

Kepler is Still King! Johannes Kepler (1571–1630) was an important mathematician and scientist of his time. He observed the planets and stars and developed laws for the motion of those bodies. His laws are still used today. It is truly amazing how accurate his laws are considering the primitive observation tools that he used.
Math Connection: Research Kepler's Laws by using the Internet. Kepler's Third Law relates the distance of planets from the sun and the period of each planet. Use the Internet to find the distance each planet is from the sun and to find each planet's period. Verify Kepler's Third Law.

interNET CONNECTION For more information on the Unit Project, visit: www.amc.glencoe.com

Polar Coordinates and Complex Numbers



CHAPTER OBJECTIVES

- **Graph polar equations.** *(Lessons 9-1, 9-2, 9-4)*
- **Convert between polar and rectangular coordinates.** *(Lessons 9-3, 9-4)*
- **Add, subtract, multiply, and divide complex numbers in rectangular and polar forms.** *(Lessons 9-5, 9-7)*
- **Convert between rectangular and polar forms of complex numbers.** *(Lesson 9-6)*
- **Find powers and roots of complex numbers.** *(Lesson 9-8)*

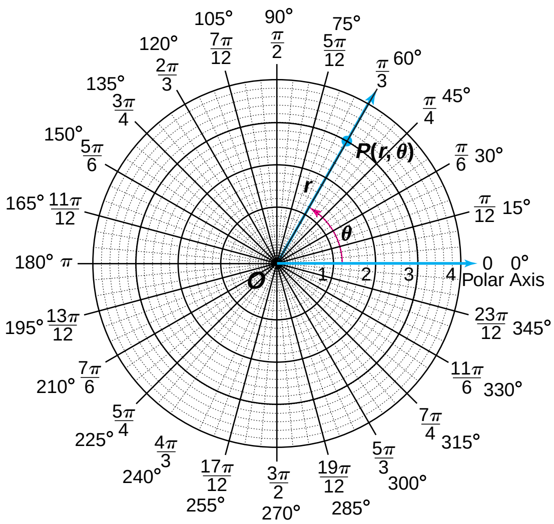
9-1 Polar Coordinates

OBJECTIVES
- Graph points in polar coordinates.
- Graph simple polar equations.
- Determine the distance between two points with polar coordinates.

Real World Application

SURVEYING Before large road construction projects, or even the construction of a new home, take place, a surveyor maps out characteristics of the land. A surveyor uses a device called a *theodolite* to measure angles. The precise locations of various land features are determined using distances and the angles measured with the theodolite. While mapping out a level site, a surveyor identifies a landmark 450 feet away and 30° to the left and another landmark 600 feet away and 50° to the right. What is the distance between the two landmarks? *This problem will be solved in Example 5.*

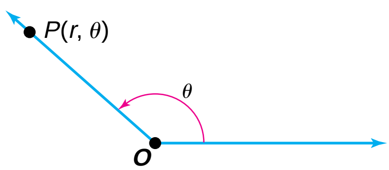
Recording the position of an object using the distance from a fixed point and an angle made with a fixed ray from that point uses a **polar coordinate system.** When surveyors record the locations of objects using distances and angles, they are using **polar coordinates.**



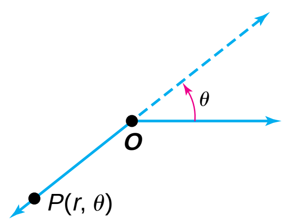
In a polar coordinate system, a fixed point O is called the **pole** or origin. The **polar axis** is usually a horizontal ray directed toward the right from the pole. The location of a point P in the polar coordinate system can be identified by polar coordinates in the form (r, θ). If a ray is drawn from the pole through point P, the distance from the pole to point P is $|r|$. The measure of the angle formed by $\overrightarrow{OP}$ and the polar axis is θ. The angle can be measured in degrees or radians. *This grid is sometimes called the polar plane.*

Consider positive and negative values of r.

Suppose $r > 0$. Then θ is the measure of any angle in standard position that has $\overrightarrow{OP}$ as its terminal side.

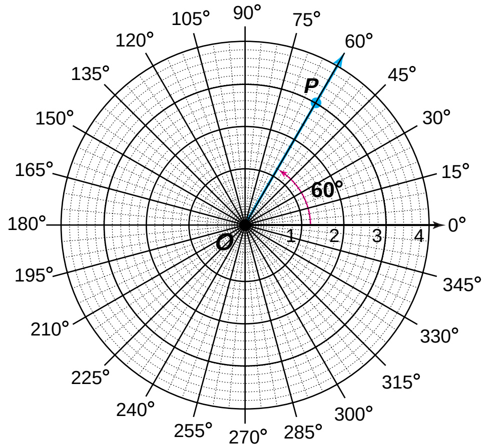


Suppose $r < 0$. Then θ is the measure of any angle that has the ray opposite $\overrightarrow{OP}$ as its terminal side.



Example 1 **Graph each point.**

a. $P(3, 60°)$

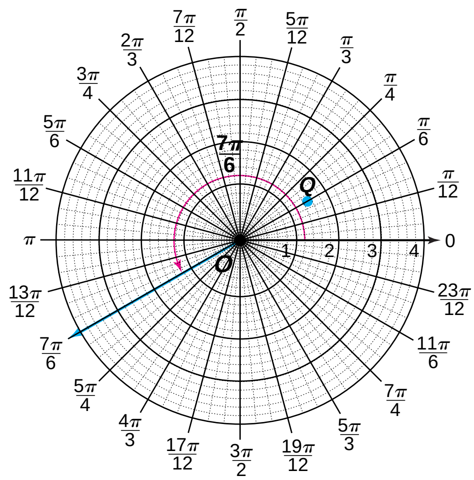


On a polar plane, sketch the terminal side of a 60° angle in standard position.

Since r is positive ($r = 3$), find the point on the terminal side of the angle that is 3 units from the pole.

Notice that point P is on the third circle from the pole.

b. $Q\left(-1.5, \frac{7\pi}{6}\right)$



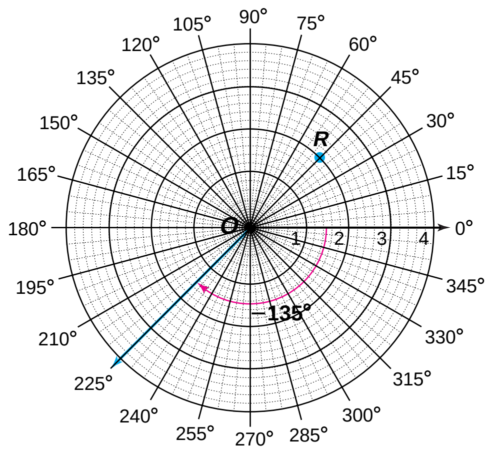
Sketch the terminal side of an angle measuring $\frac{7\pi}{6}$ radians in standard position.

Since r is negative, extend the terminal side of the angle in the opposite direction. Find the point Q that is 1.5 units from the pole along this extended ray.

Notice that point Q is halfway between the first and second circles from the pole.

As you have seen, the r-coordinate can be any real value. The angle θ can also be negative. If $\theta > 0$, then θ is measured counterclockwise from the polar axis. If $\theta < 0$, then θ is measured clockwise from the polar axis.

Example 2 **Graph $R(-2, -135°)$.**

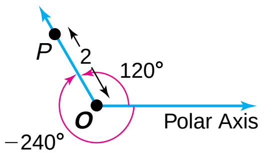


Negative angles are measured clockwise. Sketch the terminal side of an angle of $-135°$ in standard position.

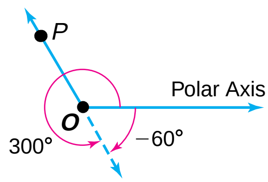
Since r is negative, the point $R(-2, -135°)$ is 2 units from the pole along the ray opposite the terminal side that is already drawn.

In Example 2, the point $R(-2, -135°)$ lies in the polar plane 2 units from the pole on the terminal side of a 45° angle in standard position. This means that point R could also be represented by the coordinates $(2, 45°)$. In general, the polar coordinates of a point are not unique. Every point can be represented by infinitely many pairs of polar coordinates. This happens because any angle in standard position is coterminal with infinitely many other angles.

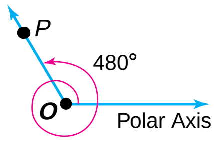
If a point has polar coordinates (r, θ), then it also has polar coordinates $(r, \theta + 2\pi)$ in radians, or $(r, \theta + 360°)$ in degrees. In fact, you can add any integer multiple of 2π to θ and find another pair of polar coordinates for the same point. If you use the opposite r-value, the angle will change by π, giving $(-r, \theta + \pi)$ as another ordered pair for the same point. You can then find even more polar coordinates for the same point by adding multiples of 2π to $\theta + \pi$. The following graphs illustrate six of the different ways to name the polar coordinates of the same point.



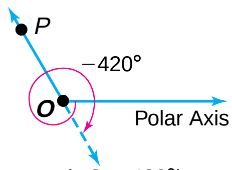
(2, 120°) or (2, −240°)



(−2, −60°) or (−2, 300°)



(2, 480°)



(−2, −420°)

Here is a summary of all the ways to represent a point in polar coordinates.

Multiple Representations of (r, θ)	If a point P has polar coordinates (r, θ), then P can also be represented by polar coordinates $(r, \theta + 2\pi k)$ or $(-r, \theta + (2k + 1)\pi)$, where k is any integer.

In degrees, the representations are $(r, \theta + 360k°)$ and $(-r, \theta + (2k + 1)180°)$. For every angle, there are infinitely many representations.

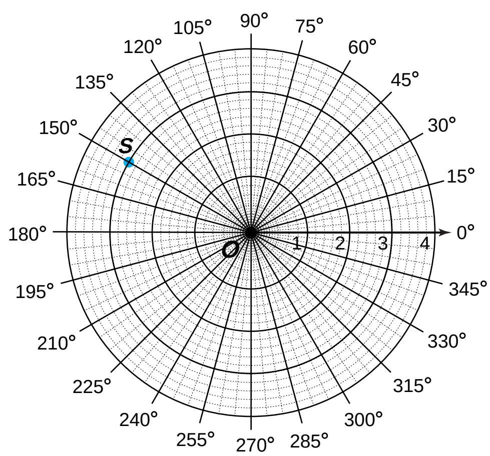
Example 3 **Name four different pairs of polar coordinates that represent point S on the graph with the restriction that $-360° \leq \theta \leq 360°$.**

One pair of polar coordinates for point S is $(3, 150°)$.

To find another representation, use $(r, \theta + 360k°)$ with $k = -1$.
$(3, 150° + 360(-1)°) = (3, -210°)$

To find additional polar coordinates, use $(-r, \theta + (2k + 1)180°)$.
$(-3, 150° + 180°) = (-3, 330°)$ $k = 0$

To find a fourth pair, use $(-r, \theta + (2k + 1)180°)$ with $k = -1$.
$(-3, 150° + (-1)180°) = (-3, -30°)$



Therefore, $(3, 150°)$, $(3, -210°)$, $(-3, 330°)$, and $(-3, -30°)$ all represent the same point in the polar plane.

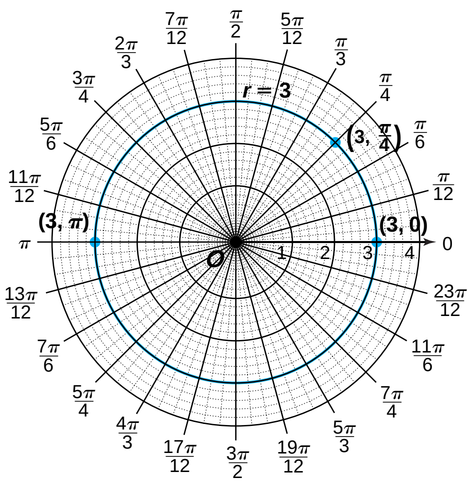
An equation expressed in terms of polar coordinates is called a **polar equation.** For example, $r = 2 \sin \theta$ is a polar equation. A **polar graph** is the set of all points whose coordinates (r, θ) satisfy a given polar equation.

You already know how to graph equations in the *Cartesian,* or *rectangular, coordinate system.* Graphs of equations involving constants like $x = 2$ and $y = -3$ are considered basic in the Cartesian coordinate system. Similarly, the polar coordinate system has some basic graphs. Graphs of the polar equations $r = k$ and $\theta = k$, where k is a constant, are considered basic.

Example 4 **Graph each polar equation.**

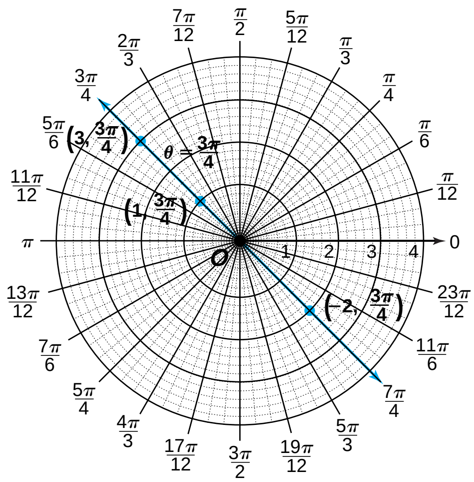
a. $r = 3$

The solutions to $r = 3$ are the ordered pairs (r, θ) where $r = 3$ and θ is any real number. Some examples are $(3, 0)$, $\left(3, \frac{\pi}{4}\right)$, and $(3, \pi)$. In other words, the graph of this equation is the set of all ordered pairs of the form $(3, \theta)$. Any point that is 3 units from the pole is included in this graph. The graph is the circle centered at the origin with radius 3.

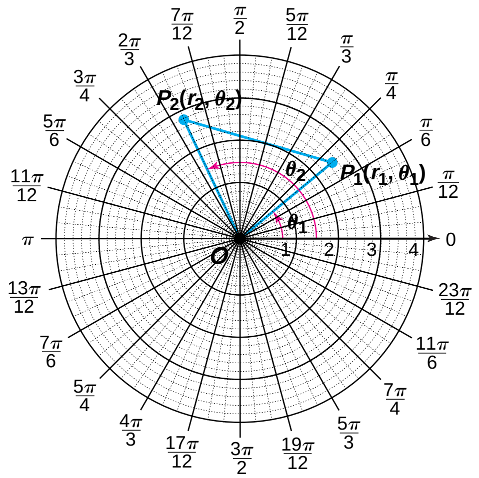


b. $\theta = \frac{3\pi}{4}$

The solutions to this equation are the ordered pairs (r, θ) where $\theta = \frac{3\pi}{4}$ and r is any real number. Some examples are $\left(1, \frac{3\pi}{4}\right)$, $\left(-2, \frac{3\pi}{4}\right)$, and $\left(3, \frac{3\pi}{4}\right)$. In other words, the graph of this equation is the set of all ordered pairs of the form $\left(r, \frac{3\pi}{4}\right)$. The graph is the line that includes the terminal side of the angle $\frac{3\pi}{4}$ in standard position.



Just as you can find the distance between two points in a rectangular coordinate plane, you can derive a formula for the distance between two points in a polar plane.



Given two points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ in the polar plane, draw $\triangle P_1OP_2$. $\angle P_1OP_2$ has measure $\theta_2 - \theta_1$. Apply the Law of Cosines to $\triangle P_1OP_2$.

$$(P_1P_2)^2 = (OP_1)^2 + (OP_2)^2 - 2(OP_1)(OP_2)\cos(\theta_2 - \theta_1)$$

Now substitute r_1 for OP_1 and r_2 for OP_2.

$$(P_1P_2)^2 = r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_2 - \theta_1)$$
$$P_1P_2 = \sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_2 - \theta_1)}$$

Distance Formula in Polar Plane

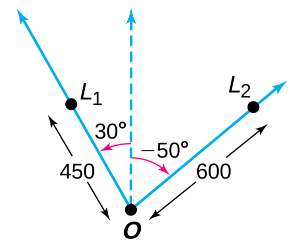
If $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ are two points in the polar plane, then

$$P_1P_2 = \sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_2 - \theta_1)}.$$

Example 5 Real World Application

SURVEYING **Refer to the application at the beginning of the lesson. What is the distance between the two landmarks?**

Set up a coordinate system so that turning to the left is a positive angle and turning to the right is a negative angle. With this convention, the landmarks are at $L_1(450, 30°)$ and $L_2(600, -50°)$.



Now use the Polar Distance Formula.

$$P_1P_2 = \sqrt{r_1^{\,2} + r_2^{\,2} - 2r_1r_2\cos(\theta_2 - \theta_1)} \qquad (r_1, \theta_1) = (450, 30°)$$
$$L_1L_2 = \sqrt{450^2 + 600^2 - 2(450)(600)\cos(-50° - 30°)} \qquad (r_2, \theta_2) = (600, -50°)$$
$$\approx 684.6 \text{ feet}$$

The landmarks are about 685 feet apart.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** why a point in the polar plane cannot be named by a unique ordered pair (r, θ).
2. **Explain** how to graph (r, θ) if $r < 0$ and $\theta > 0$.
3. **Name** two values of θ such that $(-4, \theta)$ represents the same point as $(4, 120°)$.
4. **Explain** why the graph in Example 4a is also the graph of $r = -3$.
5. **Describe** the polar coordinates of the pole.

Guided Practice

Graph each point.

6. $A(1, 135°)$ **7.** $B\left(2.5, -\frac{\pi}{6}\right)$ **8.** $C(-3, -120°)$ **9.** $D\left(-2, \frac{13\pi}{6}\right)$

10. Name four different pairs of polar coordinates that represent the point at $\left(-2, \frac{\pi}{6}\right)$.

Graph each polar equation.

11. $r = 1$ **12.** $\theta = -\frac{\pi}{3}$ **13.** $r = 3.5$

14. Find the distance between $P_1\left(2.5, \frac{\pi}{6}\right)$ and $P_2\left(-3, -\frac{\pi}{4}\right)$ on the polar plane.

15. Gardening A lawn sprinkler can cover the part of a circular region determined by the polar inequalities $-30° \leq \theta \leq 210°$ and $0 \leq r \leq 20$, where r is measured in feet.

a. Sketch a graph of the region that the sprinkler can cover.

b. Find the area of the region.

EXERCISES

Practice

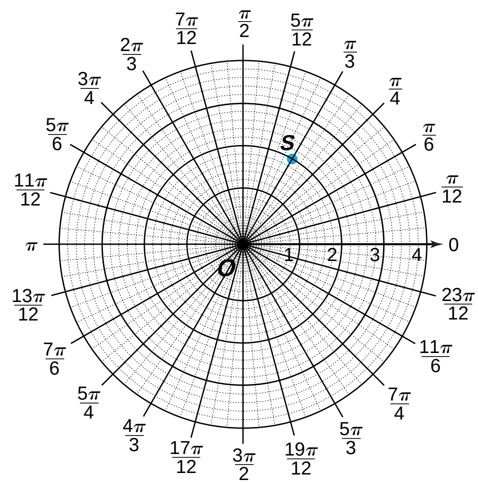
Graph each point.

16. $E(2, 30°)$ **17.** $F\left(1, \frac{\pi}{2}\right)$ **18.** $G(5, 240°)$ **19.** $H\left(\frac{1}{2}, \frac{3\pi}{4}\right)$

20. $J\left(1.5, -\frac{\pi}{4}\right)$ **21.** $K\left(\frac{5}{2}, -210°\right)$ **22.** $L\left(3, -\frac{\pi}{6}\right)$ **23.** $M(2, -90°)$

24. $N(-1, 120°)$ **25.** $P\left(-0.5, -\frac{11\pi}{6}\right)$ **26.** $Q\left(-2, \frac{25\pi}{3}\right)$ **27.** $R\left(-\frac{7}{2}, 1050°\right)$

28. List four pairs of polar coordinates that represent the point S in the graph. Use both radians and degrees.



Name four other pairs of polar coordinates for each point.

29. $T(1.5, 180°)$ **30.** $U\left(-1, \frac{\pi}{3}\right)$ **31.** $V(4, 315°)$

Graph each polar equation.

32. $r = 1.5$ **33.** $\theta = \frac{5\pi}{4}$ **34.** $r = 2$

35. $\theta = 30°$ **36.** $\theta = -150°$ **37.** $\theta = -\frac{\pi}{4}$

38. $\theta = 840°$ **39.** $r = 0$ **40.** $r = -1$

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41. Write a polar equation for the circle centered at the origin with radius $\sqrt{2}$.

Find the distance between the points with the given polar coordinates.

42. $P_1(4, 170°)$ and $P_2(6, 105°)$

43. $P_1\left(1, \frac{\pi}{6}\right)$ and $P_2\left(5, \frac{3\pi}{4}\right)$

44. $P_1\left(-2.5, \frac{\pi}{8}\right)$ and $P_2\left(-1.75, -\frac{2\pi}{5}\right)$

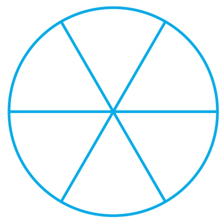
45. $P_1(1.3, -47°)$ and $P_2(-3.6, -62°)$

46. Find an ordered pair of polar coordinates to represent the point whose rectangular coordinates are $(-3, 4)$.

Applications and Problem Solving

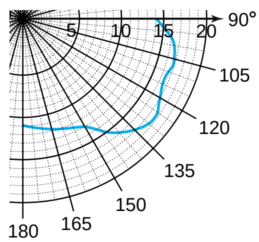
Real World Application

47. Web Page Design When designing websites with circular graphics, it is often convenient to use polar coordinates, sometimes called "pizza coordinates" in this context. If the origin is at the center of the screen, what are the polar equations of the lines that cut the region into the six congruent slices shown?



48. Critical Thinking Prove that if $P_1(r_1, \theta)$ and $P_2(r_2, \theta)$ are two points in the polar plane, then the distance formula reduces to $P_1P_2 = |r_1 - r_2|$.

49. Sailing A graph of the maximum speed of a sailboat versus the angle of the wind is called a "performance curve" or just a "polar." The graph at the right is the polar for a particular boat when the wind speed is 20 knots. θ represents the angle of the wind in degrees, and r is the maximum speed of the boat in knots.



a. What is the maximum speed when $\theta = 120°$?

b. What is the maximum speed when $\theta = 150°$?

50. Acoustics Polar coordinates can be used to model a concert amphitheater. Suppose the performer is placed at the pole and faces the direction of the polar axis. The seats have been built to occupy the region with $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$ and $0.25 \leq r \leq 3$, where r is measured in hundreds of feet.

a. Sketch this region in the polar plane.

b. How many seats are there if each person has 6 square feet of space?

51. Critical Thinking Explain why the order of the points used in the distance formula is irrelevant. That is, why can you choose either point to be P_1 and the other to be P_2?

U.S. Navy Blue Angels

52. Aviation Two jets at the same altitude are being tracked on an air traffic controller's radar screen. The coordinates of the planes are (5, 310°) and (6, 345°), with r measured in miles.

a. Sketch a graph of this situation.

b. If regulations prohibit jets from passing within three miles of each other, are these planes in violation? Explain.

Mixed Review

53. Transportation Two docks are directly across a river from each other. A boat that can travel at a speed of 8 miles per hour in still water is attempting to cross directly from one dock to the other. The current of the river is 3 miles per hour. At what angle should the captain head? *(Lesson 8-5)*

54. Find the inner product $\langle 3, -2, 4 \rangle \cdot \langle 1, -4, 0 \rangle$. Then state whether the vectors are perpendicular. Write *yes* or *no*. *(Lesson 8-4)*

55. Find the distance from the line with equation $y = 9x - 3$ to the point at $(-3, 2)$. *(Lesson 7-7)*

56. Simplify the expression $\frac{1 - \sin^2 \alpha}{\sin^2 \alpha}$. *(Lesson 7-1)*

57. Find Arccos $\frac{\sqrt{3}}{2}$. *(Lesson 6-8)*

58. State the amplitude and period for the function $y = 5 \cos 4\theta$. *(Lesson 6-4)*

59. Determine the number of possible solutions and, if a solution exists, solve $\triangle ABC$ if $A = 30°$, $b = 18.6$, and $a = 9.3$. Round to the nearest tenth. *(Lesson 5-7)*

60. Find the number of possible positive real zeros and the number of possible negative real zeros of the function $f(x) = x^3 - 4x^2 + 4x - 1$. Then determine the rational zeros. *(Lesson 4-4)*

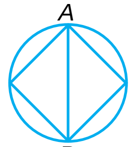
61. Determine the slant asymptote of $f(x) = \frac{x^2 + 2x - 3}{x + 5}$. *(Lesson 3-7)*

62. Determine whether the graph of $f(x) = x^4 + 3x^2 + 2$ is symmetric with respect to the x-axis, the y-axis, both, or neither. *(Lesson 3-1)*

63. Find the value of the determinant $\begin{vmatrix} -2 & 4 & -1 \\ 1 & -1 & 0 \\ -3 & 4 & 5 \end{vmatrix}$. *(Lesson 2-5)*

64. Given that x is an integer, state the relation representing $y = 11 - x$ and $-3 \leq x \leq 0$ by listing a set of ordered pairs. Then state whether this relation is a function. *(Lesson 1-1)*

65. SAT/ACT Practice The circumference of the circle is 50π. What is the length of the diagonal $\overline{AB}$ of the square inscribed in the circle?



A $12\frac{1}{2}$ **B** $10\sqrt{2}$ **C** 25

D $25\sqrt{2}$ **E** 50

Extra Practice See p. A42.

9-2 Graphs of Polar Equations

OBJECTIVE

- Graph polar equations.

Real World Application

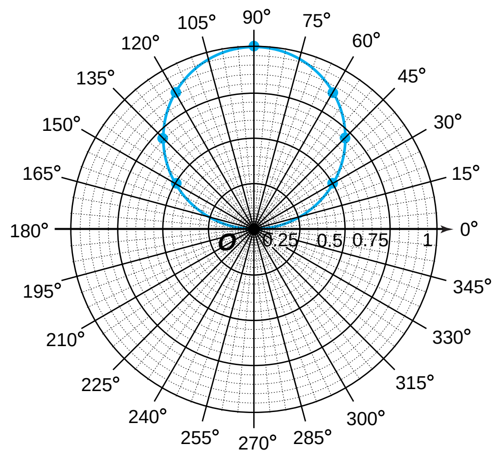
AUDIO TECHNOLOGY One way to describe the ability of a microphone to pick up sounds from different directions is to examine its polar pattern. A polar coordinate system is set up with the microphone at the origin. θ is used to locate a source of sound that moves in a horizontal circle around the microphone, and r measures the amplitude of the signal that the microphone detects. The polar graph of r as a function of θ is called the *polar pattern* of the microphone. A cardioid microphone is a microphone whose polar pattern is shaped like the graph of the equation $r = 2.5 + 2.5 \cos \theta$. A graph of this polar pattern provides information about the microphone. *A problem related to this will be solved in Example 3.*

In Lesson 9-1, you learned to graph polar equations of the form $r = k$ and $\theta = k$, where k is a constant. In this lesson, you will learn how to graph more complicated types of polar equations. When you graph any type of polar equation for the first time, you can create a table of values for r and θ. Then you can use the table to plot points in the polar plane.

Example 1 **Graph $r = \sin \theta$.**

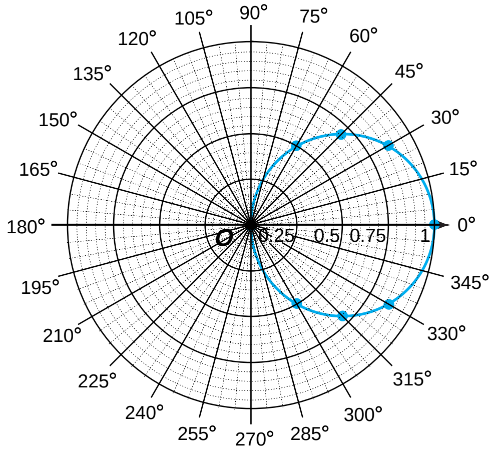
Make a table of values. Round the values of r to the nearest tenth. Graph the ordered pairs and connect them with a smooth curve.

θ	$\sin \theta$	(r, θ)
0°	0	(0, 0°)
30°	0.5	(0.5, 30°)
45°	0.7	(0.7, 45°)
60°	0.9	(0.9, 60°)
90°	1	(1, 90°)
120°	0.9	(0.9, 120°)
135°	0.7	(0.7, 135°)
150°	0.5	(0.5, 150°)
180°	0	(0, 180°)
210°	−0.5	(−0.5, 210°)
225°	−0.7	(−0.7, 225°)
240°	−0.9	(−0.9, 240°)
270°	−1	(−1, 270°)
300°	−0.9	(−0.9, 300°)
315°	−0.7	(−0.7, 315°)
330°	−0.5	(−0.5, 330°)
360°	0	(0, 360°)



Notice that the ordered pairs obtained when $180° \leq \theta \leq 360°$ represent the same points as the ordered pairs obtained when $0° \leq \theta \leq 180°$.

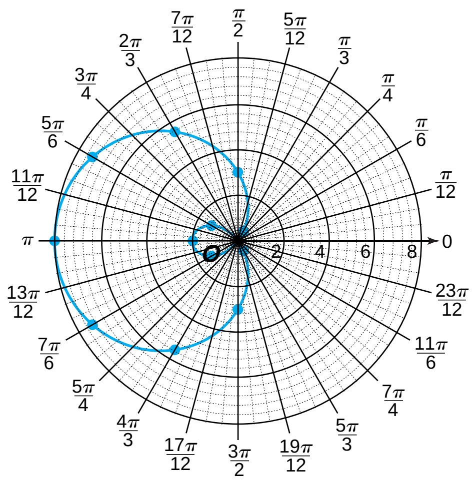
The graph of $r = \cos \theta$ is shown at the right. Notice that this graph and the graph in Example 1 are circles with a diameter of 1 unit. Both pass through the origin. As with the families of graphs you studied in Chapter 3, you can alter the position and shape of a polar graph by multiplying the function by a number or by adding to it. You can also multiply θ by a number or add a number to it in order to alter the graph. However, the changes in the graphs of polar equations can be quite different from those you studied in Chapter 3.



Examples 2 **Graph each polar equation.**

a. $r = 3 - 5 \cos \theta$

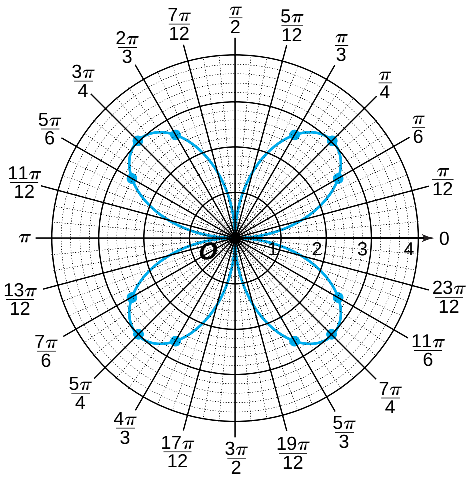
θ	$3 - 5 \cos \theta$	(r, θ)
0	−2	(−2, 0)
$\frac{\pi}{6}$	−1.3	$\left(-1.3, \frac{\pi}{6}\right)$
$\frac{\pi}{3}$	0.5	$\left(0.5, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	3	$\left(3, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	5.5	$\left(5.5, \frac{2\pi}{3}\right)$
$\frac{5\pi}{6}$	7.3	$\left(7.3, \frac{5\pi}{6}\right)$
π	8	$(8, \pi)$
$\frac{7\pi}{6}$	7.3	$\left(7.3, \frac{7\pi}{6}\right)$
$\frac{4\pi}{3}$	5.5	$\left(5.5, \frac{4\pi}{3}\right)$
$\frac{3\pi}{2}$	3	$\left(3, \frac{3\pi}{2}\right)$
$\frac{5\pi}{3}$	0.5	$\left(0.5, \frac{5\pi}{3}\right)$
$\frac{11\pi}{6}$	−1.3	$\left(-1.3, \frac{11\pi}{6}\right)$
2π	−2	$(-2, 2\pi)$



This type of curve is called a **limaçon.** Some limaçons have inner loops like this one. Other limaçons come to a point, have a dimple, or just curve outward.

b. $r = 3 \sin 2\theta$

θ	$3 \sin 2\theta$	(r, θ)
0	0	$(0, 0)$
$\frac{\pi}{6}$	2.6	$\left(2.6, \frac{\pi}{6}\right)$
$\frac{\pi}{4}$	3	$\left(3, \frac{\pi}{4}\right)$
$\frac{\pi}{3}$	2.6	$\left(2.6, \frac{\pi}{3}\right)$
$\frac{\pi}{2}$	0	$\left(0, \frac{\pi}{2}\right)$
$\frac{2\pi}{3}$	-2.6	$\left(-2.6, \frac{2\pi}{3}\right)$
$\frac{3\pi}{4}$	-3	$\left(-3, \frac{3\pi}{4}\right)$
$\frac{5\pi}{6}$	-2.6	$\left(-2.6, \frac{5\pi}{6}\right)$
π	0	$(0. \pi)$
$\frac{7\pi}{6}$	2.6	$\left(2.6, \frac{7\pi}{6}\right)$
$\frac{5\pi}{4}$	3	$\left(3, \frac{5\pi}{4}\right)$
$\frac{4\pi}{3}$	2.6	$\left(2.6, \frac{4\pi}{3}\right)$
$\frac{3\pi}{2}$	0	$\left(0, \frac{3\pi}{2}\right)$
$\frac{5\pi}{3}$	-2.6	$\left(-2.6, \frac{5\pi}{3}\right)$
$\frac{7\pi}{4}$	-3	$\left(-3, \frac{7\pi}{4}\right)$
$\frac{11\pi}{6}$	-2.6	$\left(-2.6, \frac{11\pi}{6}\right)$



This type of curve is called a **rose**.

3 **AUDIO TECHNOLOGY** **Refer to the application at the beginning of the lesson. The polar pattern of a microphone can be modeled by the polar equation $r = 2.5 + 2.5 \cos \theta$.**

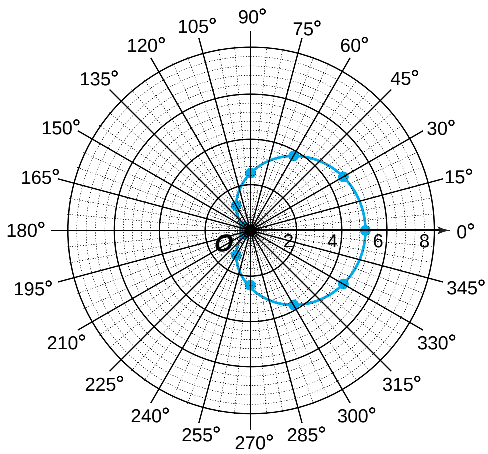
Real World Application

a. Sketch a graph of the polar pattern.

b. Describe what the polar pattern tells you about the microphone.

a.

θ	$2.5 + 2.5 \cos \theta$	(r, θ)
0°	5	(5, 0°)
30°	4.7	(4.7, 30°)
60°	3.8	(3.8, 60°)
90°	2.5	(2.5, 90°)
120°	1.3	(1.3, 120°)
150°	0.3	(0.3, 150°)
180°	0	(0, 180°)
210°	0.3	(0.3, 210°)
240°	1.3	(1.3, 240°)
270°	2.5	(2.5, 270°)
300°	3.8	(3.8, 300°)
330°	4.7	(4.7, 330°)

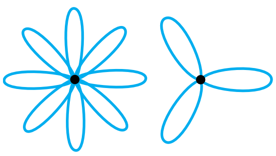
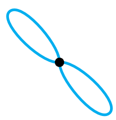
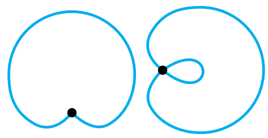
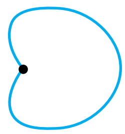
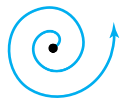


(continued on the next page)

This type of curve is called a **cardioid.** A cardioid is a special type of limaçon.

b. The polar pattern indicates that the microphone will pick up only very loud sounds from behind it. It will detect much softer sounds from in front.

Limaçons, roses, and cardioids are examples of **classical curves.** The classical curves are summarized in the chart below.

Classical Curves					
Curve	rose	lemniscate *(pronounced lehm NIHS kuht)*	limaçon *(pronounced lee muh SOHN)*	cardioid *(pronounced KARD ee oyd)*	spiral of Archimedes *(pronounced ar kih MEED eez)*
Polar Equation	$r = a \cos n\theta$ $r = a \sin n\theta$ *n is a positive integer.*	$r^2 = a^2 \cos 2\theta$ $r^2 = a^2 \sin 2\theta$	$r = a + b \cos \theta$ $r = a + b \sin \theta$	$r = a + a \cos \theta$ $r = a + a \sin \theta$	$r = a\theta$ (θ in radians)
General Graph					

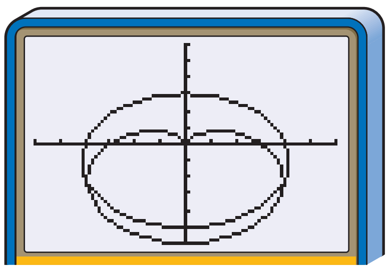
It is possible to graph more than one polar equation at a time on a polar plane. However, the points where the graphs intersect do not always represent common solutions to the equations, since every point can be represented by infinitely many polar coordinates.

Example 4 **Graph the system of polar equations. Solve the system using algebra and trigonometry and compare the solutions to those on your graph.**

$$\boldsymbol{r = 3 - 3 \sin \theta}$$
$$\boldsymbol{r = 4 - \sin \theta}$$

To solve the system of equations, substitute $3 - 3 \sin \theta$ for r in the second equation.

$$3 - 3 \sin \theta = 4 - \sin \theta$$
$$-2 \sin \theta = 1$$
$$\sin \theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6} \text{ or } \theta = \frac{11\pi}{6}$$



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Substituting each angle into either of the original equations gives $r = 4.5$, so the solutions of the system are $\left(4.5, \frac{7\pi}{6}\right)$ and $\left(4.5, \frac{11\pi}{6}\right)$. Tracing the curves shows that these solutions correspond with the intersection points of the graphs.

Graphing Calculator Tip

A graphing calculator can graph polar equations in its polar mode. You can use the simultaneous graphing mode to help determine if the graphs pass through the same point at the same time.

CHECK FOR UNDERSTANDING

Communicating Mathematics

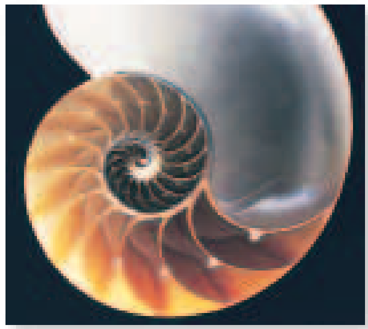
Read and study the lesson to answer each question.

1. **Write** a polar equation whose graph is a rose.
2. **Determine** the maximum value of r in the equation $r = 3 + 5 \sin \theta$. What is the minimum value of r?
3. **State** the reason that algebra and trigonometry do not always find all the points of intersection of the graphs of polar equations.
4. **You Decide** Linh and Barbara were working on their homework together. Linh said that she thought that when graphing polar equations, you only need to generate points for which $0 \leq \theta \leq \pi$ because other values of θ would just generate the same points. Barbara said she thought she remembered an example from class where values of θ ranging from 0 to 4π had to be considered. Who is correct? Explain.

Guided Practice

Graph each polar equation. Identify the type of curve each represents.

5. $r = 1 + \sin \theta$
6. $r = 2 - 3 \sin \theta$
7. $r = \cos 2\theta$
8. $r = 1.5\theta$
9. Graph the system of polar equations $r = 2 \sin \theta$ and $r = 2 \cos 2\theta$. Solve the system using algebra and trigonometry. Assume $0 \leq \theta < 2\pi$.
10. **Biology** The chambered nautilus is a mollusk whose shell can be modeled by the polar equation $r = 2\theta$.
    a. Graph this equation for $0 \leq \theta \leq 2\pi$.
    b. Determine an approximate interval for θ that would result in a graph that models the chambered nautilus shown in the photo.



EXERCISES

Practice

Graph each polar equation. Identify the type of curve each represents.

11. $r = -3 \sin \theta$
12. $r = 3 + 3 \cos \theta$
13. $r = 3\theta$
14. $r^2 = 4 \cos 2\theta$
15. $r = 2 \sin 3\theta$
16. $r = -2 \sin 3\theta$
17. $r = \frac{5}{2}\theta$
18. $r = -5 + 3 \cos \theta$
19. $r = -2 - 2 \sin \theta$
20. $r^2 = 9 \sin 2\theta$
21. $r = \sin 4\theta$
22. $r = 2 + 2 \cos \theta$
23. Write an equation for a rose with 3 petals.
24. What is an equation for a spiral of Archimedes that passes through $A\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$?

Graph each system of polar equations. Solve the system using algebra and trigonometry. Assume $0 \leq \theta < 2\pi$.

25. $r = 3$
    $r = 2 + \cos \theta$
26. $r = 1 + \cos \theta$
    $r = 1 - \cos \theta$
27. $r = 2 \sin \theta$
    $r = 2 \sin 2\theta$

Graphing Calculator

Graph each system using a graphing calculator. Find the points of intersection. Round coordinates to the nearest tenth. Assume $0 \leq \theta < 2\pi$.

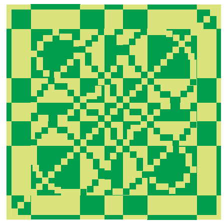
28. $r = 1$
$r = 2 \cos 2\theta$

29. $r = 3 + 3 \sin \theta$
$r = 2$

30. $r = 2 + 2 \cos \theta$
$r = 3 + \sin \theta$

Applications and Problem Solving

31. Textiles Patterns in fabric can often be created by modifying a mathematical graph. The pattern at the right can be modeled by a lemniscate.



a. Suppose the designer wanted to begin with a lemniscate that was 6 units from end to end. What polar equation could have been used?

b. What polar equation could have been used to generate a lemniscate that was 8 units from end to end?

32. Audio Technology Refer to the application at the beginning of the lesson and Example 3. Another microphone has a polar pattern that can be modeled by the polar equation $r = 3 + 2 \cos \theta$. Graph this polar pattern and compare it to the pattern of the cardioid microphone.

33. Music The curled part at the end of a violin is called the *scroll.* The scroll in the picture appears to curl around twice. For what interval of θ-values will the graph of $r = \theta$ model this violin scroll?

34. Critical Thinking

a. Graph $r = \cos \frac{\theta}{n}$ for $n = 2, 4, 6,$ and 8. Predict the shape of the graph of $r = \cos \frac{\theta}{10}$.

b. Graph $r = \cos \frac{\theta}{n}$ for $n = 3, 5, 7,$ and 9. Predict the shape of the graph of $r = \cos \frac{\theta}{11}$.

35. Communication Suppose you want to use your computer to make a heart-shaped card for a friend. Write a polar equation that you could use to generate the shape of a heart. Make sure the graph of your equation points in the right direction.

36. Critical Thinking The general form for a limaçon is $r = a + b \cos \theta$ or $r = a + b \sin \theta$.

a. When will a limaçon have an inner loop?

b. When will it have a dimple?

c. When will it have neither an inner loop nor a dimple? (This is called a *convex limaçon.*)

37. **Critical Thinking** Describe the transformation necessary to obtain the graph of each equation from the graph of the polar function $r = f(\theta)$.

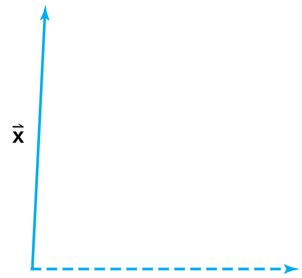
a. $r = f(\theta - \alpha)$ (α is a constant.)
b. $r = f(-\theta)$
c. $r = -f(\theta)$
d. $r = cf(\theta)$ (c is a constant, $c > 0$.)

Mixed Review

38. Find four other pairs of polar coordinates that represent the same point as $(4, 45°)$. *(Lesson 9-1)*

39. Find the cross product of $\vec{\mathbf{v}}\langle 2, 3, 0\rangle$ and $\vec{\mathbf{w}}\langle -1, 2, 4\rangle$. Verify that the resulting vector is perpendicular to $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. *(Lesson 8-4)*

40. Use a ruler and protractor to determine the magnitude (in centimeters) and direction of the vector shown at the right. *(Lesson 8-1)*



41. Verify that $\dfrac{\sin^2 x}{\cos^4 x + \cos^2 x \sin^2 x} = \tan^2 x$ is an identity. *(Lesson 7-2)*

42. Solve $\triangle ABC$ if $A = 21°15'$, $B = 49°40'$, and $c = 28.9$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-6)*

43. **Travel** Adita is trying to decide where to go on vacation. He prefers not to fly, so he wants to take a bus or a train. The table below shows the round-trip fares for trips from his home in Kansas City, Missouri to various cities. Represent this data with a matrix. *(Lesson 2-3)*

Round–Trip Fares to Kansas City

	New York	Los Angeles	Miami
Bus	$240	$199	$260
Train	$254	$322	$426

44. **SAT Practice** Which fraction is the simplified form of $\dfrac{\frac{1}{8} + \frac{6}{4}}{\frac{3}{16}}$?

A $\frac{26}{3}$

B $\frac{14}{3}$

C $\frac{28}{9}$

D $\frac{7}{3}$

E $\frac{4}{3}$

Extra Practice See p. A42.

9-3

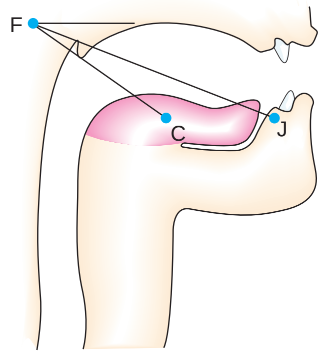
Polar and Rectangular Coordinates

OBJECTIVE

- Convert between polar and rectangular coordinates.

Real World Application

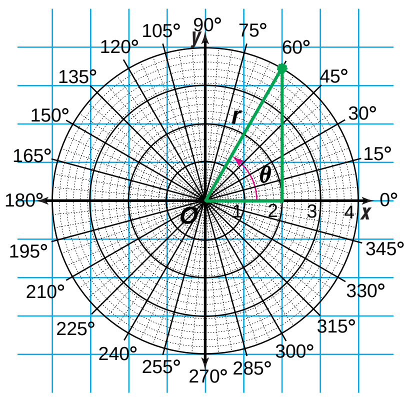
PHYSIOLOGY A laboratory has designed a voice articulation program that synthesizes speech by controlling the speech articulators: the jaw, tongue, lips, and so on. To create a mathematical model that is manageable, each of these speech articulators is identified by a point.



J = the edge of the jaw
C = the center of the tongue
F = a fixed point about which the jaw rotates

The positions of these points are given in polar coordinates with F as the pole and the horizontal as the polar axis. Changing these positions alters the sounds that are synthesized. *A problem related to this will be solved in Example 2.*

For some real-world phenomena, it is useful to be able to convert between polar coordinates and rectangular coordinates.



Suppose a rectangular coordinate system is superimposed on a polar coordinate system so that the origins coincide and the x-axis aligns with the polar axis. Let P be any point in the plane.

Polar coordinates: $P(r, \theta)$
Rectangular coordinates: $P(x, y)$

Trigonometric functions can be used to convert polar coordinates to rectangular coordinates.

Converting Polar Coordinates to Rectangular Coordinates

The rectangular coordinates (x, y) of a point named by the polar coordinates (r, θ) can be found by using the following formulas.

$$x = r \cos \theta$$
$$y = r \sin \theta$$

You will derive these formulas in Exercise 42.

Examples **1** **Find the rectangular coordinates of each point.**

a. $\boldsymbol{P\left(5, \frac{\pi}{3}\right)}$

For $P\left(5, \frac{\pi}{3}\right)$, $r = 5$ and $\theta = \frac{\pi}{3}$.

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 5\cos\frac{\pi}{3} & &= 5\sin\frac{\pi}{3} \\ &= 5\left(\frac{1}{2}\right) \text{ or } \frac{5}{2} & &= 5\left(\frac{\sqrt{3}}{2}\right) \text{ or } \frac{5\sqrt{3}}{2} \end{aligned}$$

The rectangular coordinates of P are $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$ or (2.5, 4.33) to the nearest hundredth.

b. $\boldsymbol{Q(-13, -70°)}$

For $Q(-13, -70°)$, $r = -13$ and $\theta = -70°$.

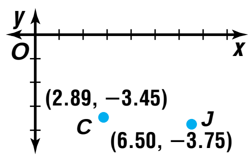
$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= -13\cos(-70°) & &= -13\sin(-70°) \\ &\approx -13(0.34202) & &\approx -13(-0.93969) \\ &\approx -4.45 & &\approx 12.22 \end{aligned}$$

The rectangular coordinates of Q are approximately $(-4.45, 12.22)$.

2 **PHYSIOLOGY** **Refer to the application at the beginning of the lesson. Suppose the computer model assigns polar coordinates (7.5, 330°) to point *J* and (4.5, 310°) to point *C* in order to create a particular sound. Each unit represents a centimeter. Is the center of the tongue above or below the edge of the jaw? by how far?**

Real World Application

Find the rectangular coordinates of each point.



For $J(7.5, 330°)$, $r = 7.5$ and $\theta = 330°$.

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 7.5\cos 330° & &= 7.5\sin 330° \\ &\approx 6.50 & &= -3.75 \end{aligned}$$

$J(7.5, 330°) \rightarrow J(6.50, -3.75)$

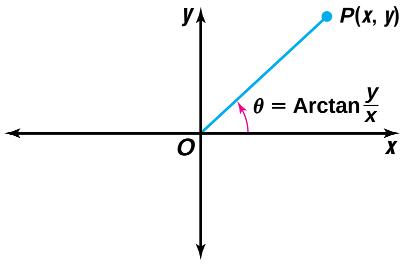
For $C(4.5, 310°)$, $r = 4.5$ and $\theta = 310°$.

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= 4.5\cos 310° & &= 4.5\sin 310° \\ &\approx 2.89 & &\approx -3.45 \end{aligned}$$

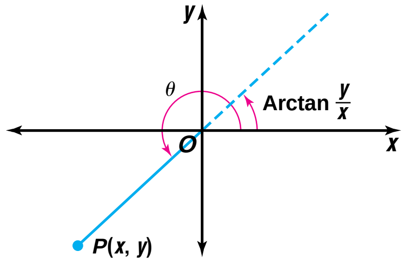
$C(4.5, 310°) \rightarrow C(2.89, -3.45)$

Since $-3.45 > -3.75$, the center of the tongue is above the edge of the jaw when this sound is made. Subtracting the y-coordinates, we see that the center of the tongue is about 0.3 centimeter, or 3 millimeters, higher than the edge of the jaw.

If a point is named by the rectangular coordinates (x, y), you can find the corresponding polar coordinates by using the Pythagorean Theorem and the Arctangent function. Since the Arctangent function only determines angles in the first and fourth quadrants, you must add π radians to the value of θ for points with coordinates (x, y) that lie in the second or third quadrants.



When $x > 0$, $\theta = \text{Arctan } \frac{y}{x}$.



When $x < 0$, $\theta = \text{Arctan } \frac{y}{x} + \pi$.

When x is zero, $\theta = \pm\frac{\pi}{2}$. Why?

Converting Rectangular Coordinates to Polar Coordinates

The polar coordinates (r, θ) of a point named by the rectangular coordinates (x, y) can be found by the following formulas.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \text{Arctan } \frac{y}{x}, \text{ when } x > 0$$

$$\theta = \text{Arctan } \frac{y}{x} + \pi, \text{ when } x < 0$$

Example 3 **Find the polar coordinates of $R(-8, -12)$.**

For $R(-8, -12)$, $x = -8$ and $y = -12$.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-8)^2 + (-12)^2} \\ &= \sqrt{208} \\ &\approx 14.42 \end{aligned}$$

$$\begin{aligned} \theta &= \text{Arctan } \frac{y}{x} + \pi && x < 0 \\ &= \text{Arctan } \frac{-12}{-8} + \pi \\ &= \text{Arctan } \frac{3}{2} + \pi \\ &\approx 4.12 \end{aligned}$$

The polar coordinates of R are approximately (14.42, 4.12).

Other polar coordinates can also represent this point.

The conversion equations can also be used to convert equations from one coordinate system to the other.

Examples 4 **Write the polar equation $r = 6 \cos \theta$ in rectangular form.**

$$\begin{aligned} r &= 6 \cos \theta \\ r^2 &= 6r \cos \theta && \text{Multiply each side by } r. \\ x^2 + y^2 &= 6x && r^2 = x^2 + y^2,\ r\cos\theta = x \end{aligned}$$

5 **Write the rectangular equation $(x - 3)^2 + y^2 = 9$ in polar form.**

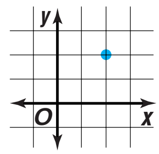
$$\begin{aligned}
(x-3)^2 + y^2 &= 9 & & \\
(r\cos\theta - 3)^2 + (r\sin\theta)^2 &= 9 & & \textit{x = r cos θ, y = r sin θ} \\
r^2\cos^2\theta - 6r\cos\theta + 9 + r^2\sin^2\theta &= 9 & & \textit{Multiply.} \\
r^2\cos^2\theta - 6r\cos\theta + r^2\sin^2\theta &= 0 & & \textit{Subtract 9 from each side.} \\
r^2\cos^2\theta + r^2\sin^2\theta &= 6r\cos\theta & & \textit{Isolate squared terms.} \\
r^2(\cos^2\theta + \sin^2\theta) &= 6r\cos\theta & & \textit{Factor.} \\
r^2(1) &= 6r\cos\theta & & \textit{Pythagorean Identity} \\
r^2 &= 6r\cos\theta & & \\
r &= 6\cos\theta & & \textit{Simplify.}
\end{aligned}$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Write** the polar coordinates of the point in the graph at the right.



2. **Explain** why you have to consider what quadrant a point lies in when converting from rectangular coordinates to polar coordinates.
3. **Determine** the polar equation for $x = 2$.
4. *Math Journal* **Write** a paragraph explaining how to convert from polar coordinates to rectangular coordinates and vice versa. Include a diagram with your explanation.

Guided Practice

Find the polar coordinates of each point with the given rectangular coordinates. Use $0 \leq \theta < 2\pi$ and $r \geq 0$.

5. $\left(-\sqrt{2}, \sqrt{2}\right)$
6. $(-2, -5)$

Find the rectangular coordinates of each point with the given polar coordinates.

7. $\left(-2, \frac{4\pi}{3}\right)$
8. $(2.5, 250°)$

Write each rectangular equation in polar form.

9. $y = 2$
10. $x^2 + y^2 = 16$

Write each polar equation in rectangular form.

11. $r = 6$
12. $r = -\sec\theta$

13. **Acoustics** The polar pattern of a cardioid lavalier microphone is given by $r = 2 + 2\cos\theta$.
    a. Graph the polar pattern.
    b. Will the microphone detect a sound that originates from the point with rectangular coordinates $(-2, 0)$? Explain.

EXERCISES

Practice

Find the polar coordinates of each point with the given rectangular coordinates. Use $0 \le \theta < 2\pi$ and $r \ge 0$.

14. $(2, -2)$ **15.** $(0, 1)$ **16.** $(1, \sqrt{3})$

17. $\left(-\frac{1}{4}, -\frac{\sqrt{3}}{4}\right)$ **18.** $(3, 8)$ **19.** $(4, -7)$

Find the rectangular coordinates of each point with the given polar coordinates.

20. $\left(3, \frac{\pi}{2}\right)$ **21.** $\left(\frac{1}{2}, \frac{3\pi}{4}\right)$ **22.** $\left(-1, -\frac{\pi}{6}\right)$

23. $(-2, 270°)$ **24.** $(4, 210°)$ **25.** $(14, 130°)$

Write each rectangular equation in polar form.

26. $x = -7$ **27.** $y = 5$

28. $x^2 + y^2 = 25$ **29.** $x^2 + y^2 = 2y$

30. $x^2 - y^2 = 1$ **31.** $x^2 + (y - 2)^2 = 4$

Write each polar equation in rectangular form.

32. $r = 2$ **33.** $r = -3$

34. $\theta = \frac{\pi}{3}$ **35.** $r = 2 \csc \theta$

36. $r = 3 \cos \theta$ **37.** $r^2 \sin 2\theta = 8$

38. Write the equation $y = x$ in polar form.

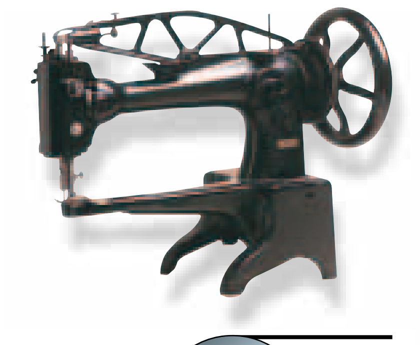
39. What is the rectangular form of $r = \sin \theta$?

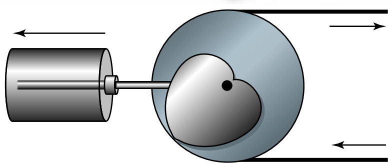
Applications and Problem Solving

Real World Application

40. Surveying A surveyor identifies a landmark at the point with polar coordinates $(325, 70°)$. What are the rectangular coordinates of this point?

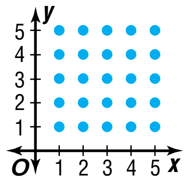
41. Machinery An arc of a spiral of Archimedes is used to create a disc that drives the spindle on a sewing machine. Note from the figure that the spinning disc drives the rod that moves the piston. The outline of the disc can be modeled by the graph of $r = \frac{\theta}{6}$ for $\frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$ and its reflection in the line $\theta = \frac{\pi}{4}$. How far does the piston move from right to left?

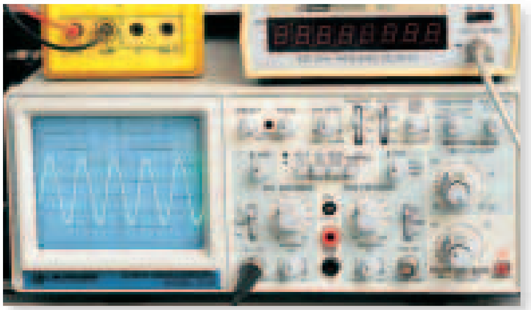




42. Critical Thinking Write a convincing argument to prove that, when converting polar coordinates to rectangular coordinates, the formulas $x = r \cos \theta$ and $y = r \sin \theta$ are true. Include a labeled drawing in your answer.

43. **Irrigation** A sod farm can use a combined Cartesian and polar coordinate system to identify points in the field. Points have coordinates of the form (x, y, r, θ). The sprinkler heads are spaced 25 meters apart. The Cartesian coordinates x and y, which are positive integers, indicate how many sprinkler heads to go across and up in the grid. The polar coordinates give the location of the point relative to the sprinkler head. If a point has coordinates (4, 3, 2, 120°), then how far to the east and north of the origin is that point?





44. **Electrical Engineering** When adding sinusoidal expressions like $4 \sin (3.14t + 20°)$ and $5 \sin (3.14t + 70°)$, electrical engineers sometimes use *phasors,* which are vectors in polar form.

   a. The phasors for these two expressions are written as $4\angle 20°$ and $5\angle 70°$. Find the rectangular forms of these vectors.

   b. Add the two vectors you found in part a.

   c. Write the sum of the two vectors in phasor notation.

   d. Write the sinusoidal expression corresponding to the phasor in part c.

45. **Critical Thinking** Identify the type of graph generated by the polar equation $r = 2a \sin \theta + 2a \cos \theta$. Write the equivalent rectangular equation.

Mixed Review

46. Graph the polar equation $r = 2 + \sin \theta$. *(Lesson 9-2)*

47. Name four different pairs of polar coordinates that represent the point at $(-2, 45°)$. *(Lesson 9-1)*

48. **Aviation** An airplane flies at an air speed of 425 mph on a heading due south. It flies against a headwind of 50 mph from a direction 30° east of south. Find the airplane's ground speed and direction. *(Lesson 8-5)*

49. Solve $\sin^2 A = \cos A - 1$ for principal values of A. *(Lesson 7-5)*

50. Graph $y = 2 \cos \theta$. *(Lesson 6-4)*

51. Use the unit circle to find the exact value of $\cos 210°$. *(Lesson 5-3)*

52. Write a polynomial function to model the set of data. *(Lesson 4-8)*

x	−10	−7	−4	−1	2	5	8	11	14
f(x)	−15	−9.2	−6.9	−3	−0.1	2	1.1	−2.3	−4.5

53. Use synthetic division to divide $x^5 - 3x^2 - 20$ by $x - 2$. *(Lesson 4-3)*

54. Write the equation of best fit for a set of data using the ordered pairs (17, 145) and (25, 625). *(Lesson 1-6)*

55. **SAT/ACT Practice** x, y, and z are different positive integers. $\frac{x}{y}$ and $\frac{y}{z}$ are also positive integers. Which of the following cannot be a positive integer?

A $\frac{x}{z}$ **B** $(x)(y)$ **C** $\frac{z}{x}$ **D** $(x + y)z$ **E** $(x - z)y$

Extra Practice See p. A42.

CONTENTS

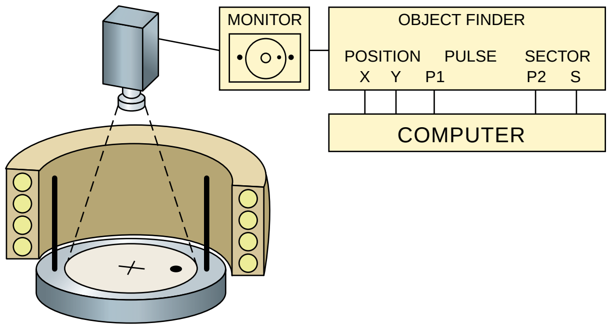
9-4

Polar Form of a Linear Equation

OBJECTIVES
- Write the polar form of a linear equation.
- Graph the polar form of a linear equation.

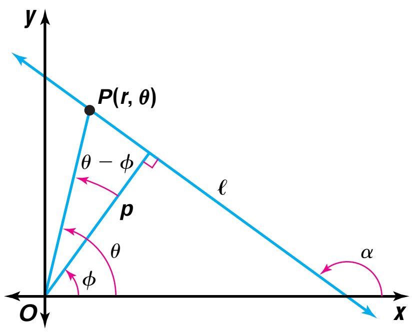
Real World Application

BIOLOGY Scientists hope that by studying the behavior of flies, they will gain insight into the genetics and brain functions of flies. Karl Götz of the Max Planck Institute in Tübingen, Germany, designed the Buridan Paradigm, which is a device that tracks the path of a fruit fly walking between two visual landmarks. The position of the fly is recorded in both rectangular and polar coordinates. If the fly walks in a straight line from the point with polar coordinates (6, 2) to the point with polar coordinates (2, −0.5), what is the equation of the path of the fly? How close did the fly come to the origin? *This problem will be solved in Example 4.*



Look Back
You can refer to Lesson 7-6 to review normal form.

The polar form of the equation for a line ℓ is closely related to the normal form, which is $x \cos \phi + y \sin \phi - p = 0$. We have learned that $x = r \cos \theta$ and $y = r \sin \theta$. The polar form of the equation of line ℓ can be obtained by substituting these values into the normal form.



$$x \cos \phi + y \sin \phi - p = 0$$
$$(r \cos \theta) \cos \phi + (r \sin \theta) \sin \phi - p = 0$$
$$r(\cos \theta \cos \phi + \sin \theta \sin \phi) = p$$
$$r \cos (\theta - \phi) = p \quad \cos \theta \cos \phi + \sin \theta \sin \phi = \cos (\theta - \phi)$$

Polar Form of a Linear Equation

The polar form of a linear equation, where p is the length of the normal and ϕ is the positive angle between the positive x-axis and the normal, is

$$p = r \cos (\theta - \phi).$$

In the polar form of a linear equation, θ and r are variables, and p and ϕ are constants. Values for p and ϕ can be obtained from the normal form of the standard equation of a line. Remember to choose the value for ϕ according to the quadrant in which the normal lies.

Example 1 **Write each equation in polar form.**

a. $5x + 12y = 26$

The standard form of the equation is $5x + 12y - 26 = 0$. First, write the equation in normal form to find the values of p and ϕ. To convert to normal form, we need to find the value of $\pm\sqrt{A^2 + B^2}$.

$$\pm\sqrt{A^2 + B^2} = \pm\sqrt{5^2 + 12^2}, \text{ or } \pm 13$$

Since C is negative, use $+13$. The normal form of the equation is

$\frac{5}{13}x + \frac{12}{13}y - 2 = 0.$ *The normal form is $x \cos \phi + y \sin \phi - p = 0$.*

We can see from the normal form that $p = 2$, $\cos \phi = \frac{5}{13}$, and $\sin \phi = \frac{12}{13}$.

Since $\cos \phi$ and $\sin \phi$ are both positive, the normal lies in the first quadrant.

$$\tan \phi = \frac{\sin \phi}{\cos \phi}$$

$$\tan \phi = \frac{\frac{12}{13}}{\frac{5}{13}}$$

$$\tan \phi = \frac{12}{5}$$

$\phi \approx 1.18$ *Use the Arctangent function.*

Substitute the values for p and ϕ into the polar form.

$p = r \cos (\theta - \phi) \rightarrow 2 = r \cos (\theta - 1.18)$

The polar form of $5x + 12y = 26$ is $2 = r \cos (\theta - 1.18)$.

b. $2x - 7y = -5$

The standard form of this equation is $2x - 7y + 5 = 0$.

So, $\pm\sqrt{A^2 + B^2} = \sqrt{\pm 2^2 + (-7)^2}$, or $\pm\sqrt{53}$.

Since C is positive, use $-\sqrt{53}$. Then the normal form of the equation is

$$-\frac{2}{\sqrt{53}}x + \frac{7}{\sqrt{53}}y - \frac{5}{\sqrt{53}} = 0.$$

We see that $p = \frac{5}{\sqrt{53}}$ or $\frac{5\sqrt{53}}{53}$, $\cos \phi = -\frac{2}{\sqrt{53}}$, and $\sin \phi = \frac{7}{\sqrt{53}}$.

Since $\cos \phi < 0$ but $\sin \phi > 0$, the normal lies in the second quadrant.

$$\tan \phi = \frac{\sin \phi}{\cos \phi}$$

$\tan \phi = -\frac{7}{2}$ $\quad \frac{7}{\sqrt{53}} \div \left(-\frac{2}{\sqrt{53}}\right) = \frac{7}{\sqrt{53}} \cdot \left(-\frac{\sqrt{53}}{2}\right)$

$\phi \approx 106°$ *Add 180° to the Arctangent value.*

Substitute the values for p and ϕ into the polar form.

$$p = r \cos (\theta - \phi) \rightarrow \frac{5\sqrt{53}}{53} = r \cos (\theta - 106°)$$

The polar form of the equation is $\frac{5\sqrt{53}}{53} = r \cos (\theta - 106°)$.

The polar form of a linear equation can be converted to rectangular form by using the angle sum and difference identities for cosine and the polar coordinate conversion equations.

Example 2 **Write $2 = r \cos(\theta - 60°)$ in rectangular form.**

$2 = r\cos(\theta - 60°)$

$2 = r(\cos\theta\cos 60° + \sin\theta\sin 60°)$ *Difference identity for cosine*

$2 = r\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$ *$\cos 60° = \frac{1}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$*

$2 = \frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta$ *Distributive Property*

$2 = \frac{1}{2}x + \frac{\sqrt{3}}{2}y$ *Polar to rectangular conversion equations*

$4 = x + \sqrt{3}y$ *Multiply each side by 2.*

$0 = x + \sqrt{3}y - 4$ *Subtract 4 from each side.*

The rectangular form of $2 = r\cos(\theta - 60°)$ is $x + \sqrt{3}y - 4 = 0$.

The polar form of a linear equation can be graphed by preparing a table of coordinates and then graphing the ordered pairs in the polar system.

Examples 3 **Graph the equation $r = 4\sec\left(\theta + \frac{2\pi}{3}\right)$.**

Graphing Calculator Tip

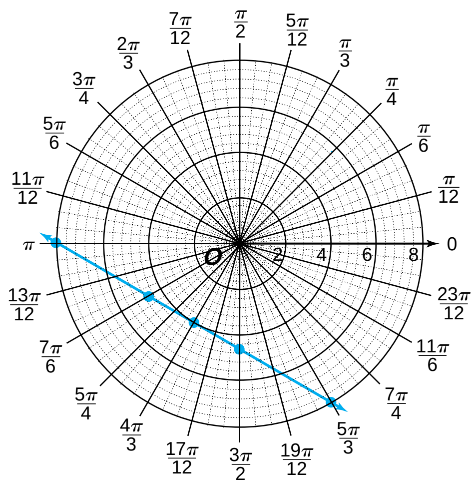
You can use the **TABLE** feature with Δ**Tbl** $= \frac{\pi}{6}$ to generate these values.

Write the equation in the form $r = \dfrac{4}{\cos\left(\theta + \frac{2\pi}{3}\right)}$.

Use your calculator to make a table of values.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	-8	-4.6	-4	-4.6	-8	undefined	8

Graph the ordered pairs on a polar plane.



4 **BIOLOGY** **Refer to the application at the beginning of the lesson.**

a. If the fly walks in a straight line from the point with polar coordinates (6, 2) to the point with polar coordinates (2, −0.5), what is the equation of the path of the fly?

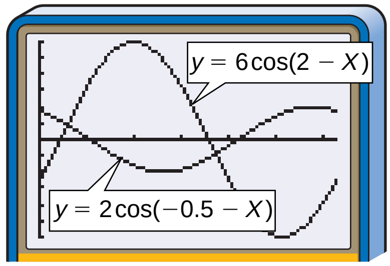
b. If r is measured in centimeters, how close did the fly come to the origin?

a. The two points must satisfy the equation $p = r\cos(\theta - \phi)$. Substitute both ordered pairs into this form to create a system of equations.

$p = 6\cos(2 - \phi)$ $\quad (r, \theta) = (6, 2)$
$p = 2\cos(-0.5 - \phi)$ $\quad (r, \theta) = (2, -0.5)$

Substituting either expression for p into the other equation results in the equation $6\cos(2 - \phi) = 2\cos(-0.5 - \phi)$.

A graphing calculator shows that there are two solutions to this equation between 0 and 2π: $\phi \approx 0.59$ and $\phi \approx 3.73$.



[0, 2π] scl:1 by [−6, 6] scl:1

Substituting these values into $p = 6\cos(2 - \phi)$ yields $p \approx 0.93$ and $p \approx -0.93$, respectively. Since p, the length of the normal, must be positive, we use $\phi \approx 0.59$ and $p \approx 0.93$.

Therefore, the polar form of the equation of the fly's path is $0.93 = r\cos(\theta - 0.59)$.

b. The closest that the fly came to the origin is the value of p, 0.93 centimeter or 9.3 millimeters.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **State** the general form for the polar equation of a line. Explain the significance of each part of the equation.
2. **Determine** what value of θ will result in $r = p$ in the polar equation of a line.
3. **Explain** how to find ϕ and the length of the normal for a rectangular equation of the form $x = k$. Write the polar form of the equation.
4. **Explain** why it is a good idea to find several ordered pairs when graphing the polar equation of a line, even though only two points are needed to determine a line.

Guided Practice

Write each equation in polar form. Round ϕ to the nearest degree.

5. $3x - 4y - 10 = 0$
6. $-2x + 4y = 9$

Write each equation in rectangular form.

7. $3 = r\cos(\theta - 60°)$
8. $r = 2\sec\left(\theta + \frac{\pi}{4}\right)$

Graph each polar equation.

9. $3 = r \cos\left(\theta - \frac{\pi}{3}\right)$

10. $r = 2 \sec(\theta + 45°)$

11. Aviation An air traffic controller is looking at a radar screen with the control tower at the origin.

a. If the path of an airplane can be modeled by the equation $5 = r \cos\left(\theta - \frac{5\pi}{6}\right)$, then what are the polar coordinates of the plane when it comes the closest to the tower?

b. Sketch a graph of the path of the plane.

EXERCISES

Practice

Write each equation in polar form. Round ϕ to the nearest degree.

12. $7x - 24y + 100 = 0$

13. $21x + 20y = 87$

14. $6x - 8y = 21$

15. $3x + 2y - 5 = 0$

16. $4x - 5y = 10$

17. $-x + 3y = 7$

Write each equation in rectangular form.

18. $6 = r \cos(\theta - 120°)$

19. $4 = r \cos\left(\theta + \frac{\pi}{4}\right)$

20. $2 = r \cos(\theta + \pi)$

21. $1 = r \cos(\theta - 330°)$

22. $r = 11 \sec\left(\theta + \frac{7\pi}{6}\right)$

23. $r = 5 \sec(\theta - 60°)$

Graph each polar equation.

24. $6 = r \cos(\theta - 45°)$

25. $1 = r \cos\left(\theta - \frac{\pi}{6}\right)$

26. $2 = r \cos(\theta + 60°)$

27. $3 = r \cos(\theta + 90°)$

28. $r = 3 \sec\left(\theta + \frac{\pi}{3}\right)$

29. $r = 4 \sec\left(\theta - \frac{\pi}{4}\right)$

30. Write the polar form of the equation of the line that passes through points with rectangular coordinates $(4, -1)$ and $(-2, 3)$.

31. What is the polar form of the equation of the line that passes through points with polar coordinates $\left(3, \frac{\pi}{4}\right)$ and $\left(2, \frac{7\pi}{6}\right)$?

Applications and Problem Solving

Real World Application

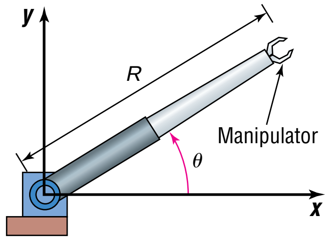
32. Biology Refer to the application at the beginning of the lesson. Suppose the Buridan Paradigm tracks a fruit fly whose path is modeled by the polar equation $r = 6 \sec(\theta - 15°)$, where r is measured in centimeters.

a. How close did the fly come to the origin?

b. What were the polar coordinates of the fly when it was closest to the origin?

33. Critical Thinking Write the polar forms of the equations of two lines, neither of which is vertical, such that the lines intersect at a 90° angle and have normal segments of length 2.

34. Robotics The diagram at the right shows a robot with a telescoping arm. The "hand" at the end of such an arm is called the manipulator. What polar equation should the arm be programmed to follow in order to move the manipulator in a straight line from the point with rectangular coordinates (5, 4) to the point with rectangular coordinates (15, 4)?



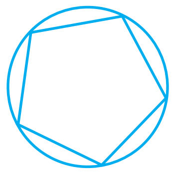
35. Surveying A surveyor records the locations of points in a plot of land by means of polar coordinates. In a circular plot of radius 500 feet, stakes are placed at (125, 130°) and (300, 70°), where r is measured in feet. The stakes are at the same elevation.

a. Draw a diagram of the plot of land. Include the locations of the two stakes.

b. Find the polar equation of the line determined by the stakes.

36. Critical Thinking Show that $k = r \sin (\theta - \alpha)$ is also the equation of a line in polar coordinates. Identify the significance of k and α in the graph.

37. Design A carnival ride designer wants to create a Ferris wheel with a 40-foot radius. The designer wants the interior of the circle to have lines of lights that form a regular pentagon. Find the polar equation of the line that contains (40, 0°) and (40, 72°).



Mixed Review

38. Write the polar equation $r = 6$ in rectangular form. *(Lesson 9-3)*

39. Identify the type of curve represented by the equation $r = \sin 6\theta$. *(Lesson 9-2)*

40. Write parametric equations of the line with equation $x - 3y = 6$. *(Lesson 8-6)*

41. Gardening A sprinkler is set to rotate 65° and spray a distance of 6 feet. What is the area of the gound being watered? *(Lesson 6-1)*

42. Use the unit circle to find the value of sin 360° *(Lesson 5-3)*

43. Solve $2x^3 + 5x^2 - 12x = 0$. *(Lesson 4-1)*

44. SAT Practice **Grid-In** If $c + d = 12$ and $c^2 - d^2 = 48$, then $c - d = ?$

MID-CHAPTER QUIZ

Graph each polar equation. (Lesson 9-1)

1. $r = 4$ **2.** $\theta = \frac{2\pi}{3}$

Graph each polar equation. (Lesson 9-2)

3. $r = 3 + 3 \sin \theta$ **4.** $r = \cos 2\theta$

Find the polar coordinates of each point with the given rectangular coordinates. (Lesson 9-3)

5. $P(-\sqrt{2}, -\sqrt{2})$ **6.** $Q(0, -4)$

7. Write the polar form of the equation $x^2 + y^2 = 36$. (Lesson 9-3)

8. Find the rectangular form of $r = 2 \csc \theta$. (Lesson 9-3)

Write each equation in polar form. Round ϕ to the nearest degree. (Lesson 9-4)

9. $5x - 12y = -3$

10. $-2x - 6y = 2$

Extra Practice See p. A42.

CONTENTS

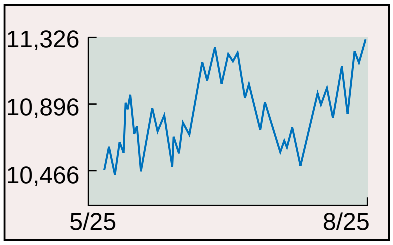
9-5 Simplifying Complex Numbers

OBJECTIVE

- Add, subtract, multiply, and divide complex numbers in rectangular form.

Real World Application

DYNAMICAL SYSTEMS Dynamical systems is a branch of mathematics that studies constantly changing systems like the stock market, the weather, and population. In many cases, one can catch a glimpse of the system at some point in time, but the forces that act on the system cause it to change quickly. By analyzing how a dynamical system changes over time, it may be possible to predict the behavior of the system in the future. One of the basic mathematical models of a dynamical system is iteration of a complex function. *A problem related to this will be solved in Example 4.*



A Typical Graph of the Stock Market

Recall that complex numbers are numbers of the form $a + bi$, where a and b are real numbers and i, the imaginary unit, is defined by $i^2 = -1$. The first few powers of i are shown below.

$i^1 = i$	$i^2 = -1$	$i^3 = i^2 \cdot i = -i$	$i^4 = (i^2)^2 = 1$
$i^5 = i^4 \cdot i = i$	$i^6 = i^4 \cdot i^2 = -1$	$i^7 = i^4 \cdot i^3 = -i$	$i^8 = (i^2)^4 = 1$

Notice the repeating pattern of the powers of i.

$$i, -1, -i, 1, i, -1, -i, 1$$

In general, the value of i^n, where n is a whole number, can be found by dividing n by 4 and examining the remainder as summarized in the table at the right.

You can also simplify any integral power of i by rewriting the exponent as a multiple of 4 plus a positive remainder.

To find the value of i^n, let R be the remainder when n is divided by 4.	
if R = 0	$i^n = 1$
if R = 1	$i^n = i$
if R = 2	$i^n = -1$
if R = 3	$i^n = -i$

Example 1 **Simplify each power of i.**

a. i^{53}

Method 1
$53 \div 4 = 13 \text{ R1}$
If R = 1, $i^n = i$.
$i^{53} = i$

Method 2
$i^{53} = (i^4)^{13} \cdot i$
$= (1)^{13} \cdot i$
$= i$

b. i^{-13}

Method 1
$-13 \div 4 = -4 \text{ R3}$
If R = 3, $i^n = -i$.
$i^{-13} = -i$

Method 2
$i^{-13} = (i^4)^{-4} \cdot i^3$
$= (1)^{-4} \cdot i^3$
$= -i$

The complex number $a + bi$, where a and b are real numbers, is said to be in **rectangular form.** a is called the **real part** and b is called the **imaginary part.** If $b = 0$, the complex number is a real number. If $b \neq 0$, the complex number is an **imaginary number.** If $a = 0$ and $b \neq 0$, as in $4i$, then the complex number is a **pure imaginary number.** Complex numbers can be added and subtracted by performing the chosen operation on both the real and imaginary parts.

Example 2 **Simplify each expression.**

Graphing Calculator Tip

Some calculators have a complex number mode. In this mode, they can perform complex number arithmetic.

a. $(5 - 3i) + (-2 + 4i)$

$$\begin{aligned}(5 - 3i) + (-2 + 4i) &= [5 + (-2)] + [-3i + 4i] \\ &= 3 + i\end{aligned}$$

b. $(10 - 2i) - (14 - 6i)$

$$\begin{aligned}(10 - 2i) - (14 - 6i) &= 10 - 2i - 14 + 6i \\ &= -4 + 4i\end{aligned}$$

The product of two or more complex numbers can be found using the same procedures you use when multiplying binomials.

Example 3 **Simplify $(2 - 3i)(7 - 4i)$.**

$$\begin{aligned}(2 - 3i)(7 - 4i) &= 7(2 - 3i) - 4i(2 - 3i) && \textit{Distributive property} \\ &= 14 - 21i - 8i + 12i^2 && \textit{Distributive property} \\ &= 14 - 21i - 8i + 12(-1) && i^2 = -1 \\ &= 2 - 29i\end{aligned}$$

Iteration is the process of repeatedly applying a function to the output produced by the previous input. When using complex numbers with functions, it is traditional to use z for the independent variable.

Example 4 **DYNAMICAL SYSTEMS** **If $f(z) = (0.5 + 0.5i)z$, find the first five iterates of f for the initial value $z_0 = 1 + i$. Describe any pattern that you see.**

Real World Application

$f(z) = (0.5 + 0.5i)z$

$$\begin{aligned}f(1 + i) &= (0.5 + 0.5i)(1 + i) && \textit{Replace } z \textit{ with } 1 + i. \\ &= 0.5 + 0.5i + 0.5i + 0.5i^2 \\ &= i && z_1 = i\end{aligned}$$

$$\begin{aligned}f(i) &= (0.5 + 0.5i)i \\ &= 0.5i + 0.5i^2 \\ &= \underbrace{-0.5 + 0.5i} && z_2 = -0.5 + 0.5i\end{aligned}$$

$$\begin{aligned}f(-0.5 + 0.5i) &= (0.5 + 0.5i)(-0.5 + 0.5i) \\ &= -0.25 + 0.25i - 0.25i + 0.25i^2 \\ &= -0.5 && z_3 = -0.5\end{aligned}$$

(continued on the next page)

interNET CONNECTION

Graphing Calculator Programs
To download a graphing calculator program that performs complex iteration, visit: **www.amc.glencoe.com**

$$f(-0.5) = (0.5 + 0.5\boldsymbol{i})(-0.5)$$
$$= -0.25 - 0.25\boldsymbol{i} \qquad z_4 = -0.25 - 0.25\boldsymbol{i}$$

$$f(-0.25 - 0.25\boldsymbol{i}) = (0.5 + 0.5\boldsymbol{i})(-0.25 - 0.25\boldsymbol{i})$$
$$= -0.125 - 0.125\boldsymbol{i} - 0.125\boldsymbol{i} - 0.125\boldsymbol{i}^2$$
$$= -0.25\boldsymbol{i} \qquad z_5 = -0.25\boldsymbol{i}$$

The first five iterates of $1 + \boldsymbol{i}$ are $\boldsymbol{i}$, $-0.5 + 0.5\boldsymbol{i}$, -0.5, $-0.25 - 0.25\boldsymbol{i}$, and $-0.25\boldsymbol{i}$. The absolute values of the nonzero real and imaginary parts (1, 0.5, 0.25) stay the same for two steps and then are halved.

Two complex numbers of the form $a + b\boldsymbol{i}$ and $a - b\boldsymbol{i}$ are called **complex conjugates.** Recall that if a quadratic equation with real coefficients has complex solutions, then those solutions are complex conjugates. Complex conjugates also play a useful role in the division of complex numbers. To simplify the quotient of two complex numbers, multiply the numerator and denominator by the conjugate of the denominator. The process is similar to rationalizing the denominator in an expression like $\frac{1}{3 + \sqrt{2}}$.

Example 5 **Simplify $(5 - 3\boldsymbol{i}) \div (1 - 2\boldsymbol{i})$.**

$$(5 - 3\boldsymbol{i}) \div (1 - 2\boldsymbol{i}) = \frac{5 - 3\boldsymbol{i}}{1 - 2\boldsymbol{i}}$$
$$= \frac{5 - 3\boldsymbol{i}}{1 - 2\boldsymbol{i}} \cdot \frac{1 + 2\boldsymbol{i}}{1 + 2\boldsymbol{i}} \qquad \textit{Multiply by 1; } 1 + 2\boldsymbol{i} \textit{ is the conjugate of } 1 - 2\boldsymbol{i}.$$
$$= \frac{5 + 10\boldsymbol{i} - 3\boldsymbol{i} - 6\boldsymbol{i}^2}{1 - 4\boldsymbol{i}^2}$$
$$= \frac{5 + 7\boldsymbol{i} - 6(-1)}{1 - (-4)} \qquad \boldsymbol{i}^2 = -1$$
$$= \frac{11 + 7\boldsymbol{i}}{5}$$
$$= \frac{11}{5} + \frac{7}{5}\boldsymbol{i} \qquad \textit{Write the answer in the form } a + b\boldsymbol{i}.$$

The list below summarizes the operations with complex numbers presented in this lesson.

Operations with Complex Numbers

For any complex numbers $a + bi$ and $c + di$, the following are true.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$
$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Describe** how to simplify any integral power of $\boldsymbol{i}$.
2. **Draw** a Venn diagram to show the relationship between real, pure imaginary, and complex numbers.
3. **Explain** why it is useful to multiply by the conjugate of the denominator over itself when simplifying a fraction containing complex numbers.
4. **Write** a quadratic equation that has two complex conjugate solutions.

Guided Practice

Simplify.

5. $\boldsymbol{i}^{-6}$

6. $\boldsymbol{i}^{10} + \boldsymbol{i}^{2}$

7. $(2 + 3\boldsymbol{i}) + (-6 + \boldsymbol{i})$

8. $(2.3 + 4.1\boldsymbol{i}) - (-1.2 - 6.3\boldsymbol{i})$

9. $(2 + 4\boldsymbol{i}) + (-1 + 5\boldsymbol{i})$

10. $(-2 - \boldsymbol{i})^2$

11. $\dfrac{\boldsymbol{i}}{1 + 2\boldsymbol{i}}$

Look Back
You can refer to Lessons 8-1 and 8-2 to review vectors, components, and resultants.

12. **Vectors** It is sometimes convenient to use complex numbers to represent vectors. A vector with a horizontal component of magnitude a and a vertical component of magnitude b can be represented by the complex number $a + b\boldsymbol{i}$. If an object experiences a force with a horizontal component of 2.5 N and a vertical component of 3.1 N as well as a second force with a horizontal component of -6.2 N and a vertical component of 4.3 N, find the resultant force on the object. Write your answer as a complex number.

EXERCISES

Practice

Simplify.

13. $\boldsymbol{i}^{6}$

14. $\boldsymbol{i}^{19}$

15. $\boldsymbol{i}^{1776}$

16. $\boldsymbol{i}^{9} + \boldsymbol{i}^{-5}$

17. $(3 + 2\boldsymbol{i}) + (-4 + 6\boldsymbol{i})$

18. $(7 - 4\boldsymbol{i}) + (2 - 3\boldsymbol{i})$

19. $\left(\frac{1}{2} + \boldsymbol{i}\right) - (2 - \boldsymbol{i})$

20. $(-3 - \boldsymbol{i}) - (4 - 5\boldsymbol{i})$

21. $(2 + \boldsymbol{i})(4 + 3\boldsymbol{i})$

22. $(1 + 4\boldsymbol{i})^2$

23. $\left(1 + \sqrt{7}\boldsymbol{i}\right)\left(-2 - \sqrt{5}\boldsymbol{i}\right)$

24. $\left(2 + \sqrt{-3}\right)\left(-1 + \sqrt{-12}\right)$

25. $\dfrac{2 + \boldsymbol{i}}{1 + 2\boldsymbol{i}}$

26. $\dfrac{3 - 2\boldsymbol{i}}{-4 - \boldsymbol{i}}$

27. $\dfrac{5 - \boldsymbol{i}}{5 + \boldsymbol{i}}$

28. Write a quadratic equation with solutions $\boldsymbol{i}$ and $-\boldsymbol{i}$.

29. Write a quadratic equation with solutions $2 + \boldsymbol{i}$ and $2 - \boldsymbol{i}$.

Simplify.

30. $(2 - \boldsymbol{i})(3 + 2\boldsymbol{i})(1 - 4\boldsymbol{i})$

31. $(-1 - 3\boldsymbol{i})(2 + 2\boldsymbol{i})(1 - 2\boldsymbol{i})$

32. $\dfrac{\frac{1}{2} + \sqrt{3}\boldsymbol{i}}{1 - \sqrt{2}\boldsymbol{i}}$

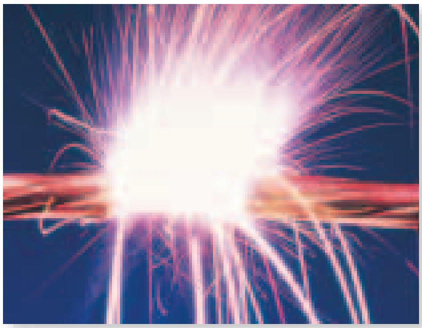
33. $\dfrac{2 - \sqrt{2}\boldsymbol{i}}{3 + \sqrt{6}\boldsymbol{i}}$

34. $\dfrac{3 + \boldsymbol{i}}{(2 + \boldsymbol{i})^2}$

35. $\dfrac{(1 + \boldsymbol{i})^2}{(-3 + 2\boldsymbol{i})^2}$

Applications and Problem Solving

36. Electricity *Impedance* is a measure of how much hindrance there is to the flow of charge in a circuit with alternating current. The impedance Z depends on the resistance R, the reactance due to capacitance X_C, and the reactance due to inductance X_L in the circuit. The impedance is written as the complex number $Z = R + (X_L - X_C)\boldsymbol{j}$. (Electrical engineers use $\boldsymbol{j}$ to denote the imaginary unit.) In the first part of a particular series circuit, the resistance is 10 ohms, the reactance due to capacitance is 2 ohms, and the reactance due to inductance is 1 ohm. In the second part of the circuit, the respective values are 3 ohms, 1 ohm, and 1 ohm.



a. Write complex numbers that represent the impedances in the two parts of the circuit.

b. Add your answers from part a to find the total impedance in the circuit.

c. The *admittance* of an AC circuit is a measure of how well the circuit allows current to flow. Admittance is the reciprocal of impedance. That is, $S = \frac{1}{Z}$. The units for admittance are siemens. Find the admittance in a circuit with an impedance of $6 + 3\boldsymbol{j}$ ohms.

37. Critical Thinking

a. Solve the equation $x^2 + 8\boldsymbol{i}x - 25 = 0$.

b. Are the solutions complex conjugates?

c. How does your result in part b compare with what you already know about complex solutions to quadratic equations?

d. Check your solutions.

38. Critical Thinking Sometimes it is useful to separate a complex function into its real and imaginary parts. Substitute $z = x + y\boldsymbol{i}$ into the function $f(z) = z^2$ to write the equation of the function in terms of x and y only. Simplify your answer.

39. Dynamical Systems Find the first five iterates for the given function and initial value.

a. $f(z) = \boldsymbol{i}z$, $z_0 = 2 - \boldsymbol{i}$

b. $f(z) = (0.5 - 0.866\boldsymbol{i})z$, $z_0 = 1 + 0\boldsymbol{i}$

40. Critical Thinking Simplify $(1 + 2\boldsymbol{i})^{-3}$.

41. Physics One way to derive the equation of motion in a spring-mass system is to solve a *differential equation.* The solutions of such a differential equation typically involve expressions of the form $\cos \beta t + \boldsymbol{i} \sin \beta t$. You generally expect solutions that are real numbers in such a situation, so you must use algebra to eliminate the imaginary numbers. Find a relationship between the constants c_1 and c_2 such that $c_1(\cos 2t + \boldsymbol{i} \sin 2t) + c_2(\cos 2t - \boldsymbol{i} \sin 2t)$ is a real number for all values of t.

Mixed Review

42. Write the equation $6x - 2y = -3$ in polar form. *(Lesson 9-4)*

43. Graph the polar equation $r = 4\theta$. *(Lesson 9-2)*

44. Write a vector equation of the line that passes through $P(-3, 6)$ and is parallel to $\vec{\mathbf{v}}\langle 1, -4\rangle$. *(Lesson 8-6)*

45. Find an ordered triple to represent $\vec{\mathbf{u}}$ if $\vec{\mathbf{u}} = \frac{1}{4}\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$, $\vec{\mathbf{v}} = \langle -8, 6, 4\rangle$, and $\vec{\mathbf{w}} = \langle 2, -6, 3\rangle$. *(Lesson 8-3)*

46. If α and β are measures of two first quadrant angles, find $\cos(\alpha + \beta)$ if $\tan \alpha = \frac{4}{3}$ and $\cot \beta = \frac{5}{12}$. *(Lesson 7-3)*

47. A twig floats on the water, bobbing up and down. The distance between its highest and lowest points is 7 centimeters. It moves from its highest point down to its lowest point and back up to its highest point every 12 seconds. Write a cosine function that models the movement of the twig in relationship to the equilibrium point. *(Lesson 6-6)*

48. **Surveying** A surveyor finds that the angle of elevation from a certain point to the top of a cliff is 60°. From a point 45 feet farther away, the angle of elevation to the top of the cliff is 52°. How high is the cliff to the nearest foot? *(Lesson 5-4)*

49. What type of polynomial function would be the best model for the set of data? *(Lesson 4-8)*

x	−3	−1	1	3	5	7	9	11
$f(x)$	−4	−2	3	8	6	1	−3	−8

50. **Construction** A community wants to build a second pool at their community park. Their original pool has a width 5 times its depth and a length 10 times its depth. They wish to make the second pool larger by increasing the width of the original pool by 4 feet, increasing the length by 6 feet, and increasing the depth by 2 feet. The volume of the new pool will be 3420 cubic feet. Find the dimensions of the original pool. *(Lesson 4-4)*

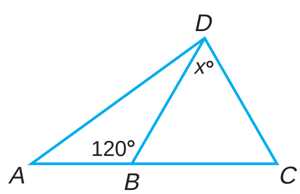
51. If y varies jointly as x and z and $y = 80$ when $x = 5$ and $z = 8$, find y when $x = 16$ and $z = 2$. *(Lesson 3-8)*

52. If $f(x) = 7 - x^2$, find $f^{-1}(x)$. *(Lesson 3-4)*

53. Find the maximum and minimum values of the function $f(x, y) = -2x + y$ for the polygonal convex set determined by the system of inequalities. *(Lesson 2-6)*
$x \leq 6$
$y \geq 1$
$y - x \leq 2$

54. Solve the system of equations. *(Lesson 2-2)*
$x + 2y - 7z = 14$
$-x - 3y + 5z = -21$
$5x - y + 2z = -7$

55. **SAT/ACT Practice** If $BC = BD$ in the figure, what is the value of $x + 40$?



A 100 **B** 80 **C** 60 **D** 40
E cannot be determined from the information given

9-6 The Complex Plane and Polar Form of Complex Numbers

OBJECTIVES

- Graph complex numbers in the complex plane.
- Convert complex numbers from rectangular to polar form and vice versa.

Real World Application

FRACTALS One of the standard ways to generate a fractal involves iteration of a quadratic function. If the function $f(z) = z^2$ is iterated using a complex number as the initial input, there are three possible outcomes. The terms of the sequence of outputs, called the *orbit,* may

- increase in absolute value,
- decrease toward 0 in absolute value, or
- always have an absolute value of 1.

One way to analyze the behavior of the orbit is to graph the numbers in the complex plane. Plot the first five members of the orbit of $z_0 = 0.9 + 0.3\boldsymbol{i}$ under iteration by $f(z) = z^2$. *This problem will be solved in Example 3.*

Recall that $a + b\boldsymbol{i}$ is referred to as the rectangular form of a complex number. The rectangular form is sometimes written as an ordered pair, (a, b). Two complex numbers in rectangular form are equal if and only if their real parts are equal and their imaginary parts are equal.

Example 1 **Solve the equation $2x + y + 3\boldsymbol{i} = 9 + x\boldsymbol{i} - y\boldsymbol{i}$ for x and y, where x and y are real numbers.**

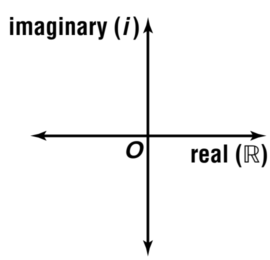
$2x + y + 3\boldsymbol{i} = 9 + x\boldsymbol{i} - y\boldsymbol{i}$

$(2x + y) + 3\boldsymbol{i} = 9 + (x - y)\boldsymbol{i}$ *On each side of the equation, group the real parts and the imaginary parts.*

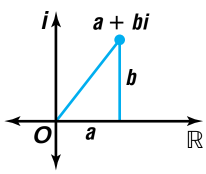
$2x + y = 9$ and $x - y = 3$ *Set the corresponding parts equal to each other.*

$x = 4$ and $y = 1$ *Solve the system of equations.*

Complex numbers can be graphed in the **complex plane.** The complex plane has a real axis and an imaginary axis. The real axis is horizontal, and the imaginary axis is vertical. The complex number $a + b\boldsymbol{i}$ is graphed as the ordered pair (a, b) in the complex plane. The complex plane is sometimes called the **Argand plane.**



Recall that the absolute value of a real number is its distance from zero on the number line. Similarly, the **absolute value** of a complex number is its distance from zero in the complex plane. When $a + b\boldsymbol{i}$ is graphed in the complex plane, the distance from zero can be calculated using the Pythagorean Theorem.

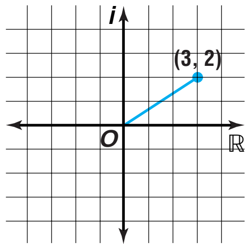


Absolute Value of a Complex Number	If $z = a + bi$, then $\lvert z \rvert = \sqrt{a^2 + b^2}$.

Examples

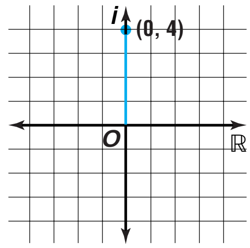
2 **Graph each number in the complex plane and find its absolute value.**

a. $z = 3 + 2i$



$$z = 3 + 2i$$
$$\lvert z \rvert = \sqrt{3^2 + 2^2}$$
$$= \sqrt{13}$$

b. $z = 4i$



$$z = 0 + 4i$$
$$\lvert z \rvert = \sqrt{0^2 + 4^2}$$
$$= 4$$

3 **FRACTALS** **Refer to the application at the beginning of the lesson. Plot the first five members of the orbit of $z_0 = 0.9 + 0.3i$ under iteration by $f(z) = z^2$.**

Real World Application

First, calculate the first five members of the orbit. Round the real and imaginary parts to the nearest hundredth.

$z_1 = 0.72 + 0.54i$ $z_1 = f(z_0)$

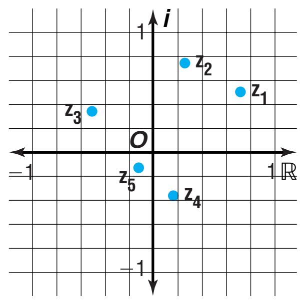
$z_2 = 0.23 + 0.78i$ $z_2 = f(z_1)$

$z_3 = -0.55 + 0.35i$ $z_3 = f(z_2)$

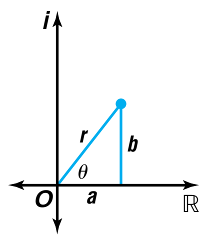
$z_4 = 0.18 - 0.39i$ $z_4 = f(z_3)$

$z_5 = -0.12 - 0.14i$ $z_5 = f(z_4)$

Then graph the numbers in the complex plane. The iterates approach the origin, so their absolute values decrease toward 0.



So far we have associated the complex number $a + bi$ with the rectangular coordinates (a, b). You know from Lesson 9-1 that there are also polar coordinates (r, θ) associated with the same point. In the case of a complex number, r represents the absolute value, or **modulus**, of the complex number. The angle θ is called the **amplitude** or **argument** of the complex number. Since θ is not unique, it may be replaced by $\theta + 2\pi k$, where k is any integer.



As with other rectangular coordinates, complex coordinates can be written in polar form by substituting $a = r\cos\theta$ and $b = r\sin\theta$.

$$\begin{aligned} z &= a + b\boldsymbol{i} \\ &= r\cos\theta + (r\sin\theta)\boldsymbol{i} \\ &= r(\cos\theta + \boldsymbol{i}\sin\theta) \end{aligned}$$

This form of a complex number is often called the **polar** or **trigonometric form.**

Polar Form of a Complex Number

The polar form or trigonometric form of the complex number $a + bi$ is

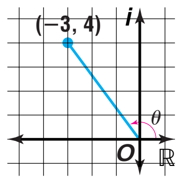
$$r(\cos\theta + \boldsymbol{i}\sin\theta).$$

$r(\cos\theta + \boldsymbol{i}\sin\theta)$ is often abbreviated as $r \text{ cis } \theta$.

Values for r and θ can be found by using the same process you used when changing rectangular coordinates to polar coordinates. For $a + b\boldsymbol{i}$, $r = \sqrt{a^2 + b^2}$ and $\theta = \text{Arctan}\,\frac{b}{a}$ if $a > 0$ or $\theta = \text{Arctan}\,\frac{b}{a} + \pi$ if $a < 0$. The amplitude θ is usually expressed in radian measure, and the angle is in standard position along the polar axis.

Example 4 **Express each complex number in polar form.**

a. $-3 + 4\boldsymbol{i}$



First, plot the number in the complex plane.

Then find the modulus.

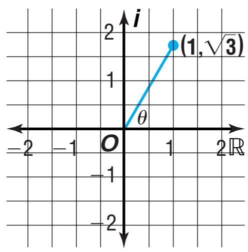
$r = \sqrt{(-3)^2 + 4^2}$ or 5

Now find the amplitude. Notice that θ is in Quadrant II.

$$\theta = \text{Arctan}\,\frac{4}{-3} + \pi$$
$$\approx 2.21$$

Therefore, $-3 + 4\boldsymbol{i} \approx 5(\cos 2.21 + \boldsymbol{i}\sin 2.21)$ or $5 \text{ cis } 2.21$.

b. $1 + \sqrt{3}\boldsymbol{i}$



First, plot the number in the complex plane.

Then find the modulus.

$r = \sqrt{1^2 + \left(\sqrt{3}\right)^2}$ or 2

Now find the amplitude. Notice that θ is in Quadrant I.

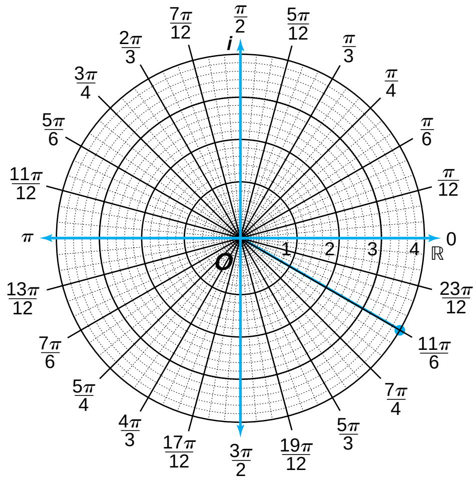
$$\theta = \text{Arctan}\,\frac{\sqrt{3}}{1} \text{ or } \frac{\pi}{3}$$

Therefore, $1 + \sqrt{3}\boldsymbol{i} = 2\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$ or $2 \text{ cis } \frac{\pi}{3}$.

You can also graph complex numbers in polar form.

Example 5 **Graph $4\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$. Then express it in rectangular form.**

In the polar form of this complex number, the value of r is 4, and the value of θ is $\frac{11\pi}{6}$. Plot the point with polar coordinates $\left(4, \frac{11\pi}{6}\right)$.



To express the number in rectangular form, simplify the trigonometric values:

$$4\left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)$$
$$= 4\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$
$$= 2\sqrt{3} - 2i$$

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how to find the absolute value of a complex number.
2. **Write** the polar form of i.
3. **Find a counterexample** to the statement $|z_1 + z_2| = |z_1| + |z_2|$ *for all complex numbers z_1 and z_2.*
4. *Math Journal* Your friend is studying complex numbers at another school at the same time that you are. She learned that the absolute value of a complex number is the square root of the product of the number and its conjugate. You know that this is not how you learned it. Write a letter to your friend explaining why this method gives the same answer as the method you know. Use algebra, but also include some numerical examples of both techniques.

Guided Practice

5. Solve the equation $2x + y + xi + yi = 5 + 4i$ for x and y, where x and y are real numbers.

Graph each number in the complex plane and find its absolute value.

6. $-2 - i$
7. $1 + \sqrt{2}i$

Express each complex number in polar form.

8. $2 - 2i$
9. $4 + 5i$
10. -2

Graph each complex number. Then express it in rectangular form.

11. $4\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$
12. $2(\cos 3 + i \sin 3)$
13. $\frac{3}{2}(\cos 2\pi + i \sin 2\pi)$

14. Graph the first five members of the orbit of $z_0 = -0.25 + 0.75i$ under iteration by $f(z) = z^2 + 0.5$.

15. **Vectors** The force on an object is represented by the complex number $10 + 15\boldsymbol{i}$, where the components are measured in newtons.
   a. What is the magnitude of the force?
   b. What is the direction of the force?

EXERCISES

Practice

Solve each equation for x and y, where x and y are real numbers.

16. $2x - 5y\boldsymbol{i} = 12 + 15\boldsymbol{i}$
17. $1 + (x + y)\boldsymbol{i} = y + 3x\boldsymbol{i}$
18. $4x + y\boldsymbol{i} - 5\boldsymbol{i} = 2x - y + x\boldsymbol{i} + 7\boldsymbol{i}$

Graph each number in the complex plane and find its absolute value.

19. $2 + 3\boldsymbol{i}$
20. $3 - 4\boldsymbol{i}$
21. $-1 - 5\boldsymbol{i}$
22. $-3\boldsymbol{i}$
23. $-1 + \sqrt{5}\boldsymbol{i}$
24. $4 + \sqrt{2}i$
25. Find the modulus of $z = -4 + 6\boldsymbol{i}$.

Express each complex number in polar form.

26. $3 + 3\boldsymbol{i}$
27. $-1 - \sqrt{3}\boldsymbol{i}$
28. $6 - 8\boldsymbol{i}$
29. $-4 + \boldsymbol{i}$
30. $20 - 21\boldsymbol{i}$
31. $-2 + 4\boldsymbol{i}$
32. 3
33. $-4\sqrt{2}$
34. $-2\boldsymbol{i}$

Graph each complex number. Then express it in rectangular form.

35. $3\left(\cos \frac{\pi}{4} + \boldsymbol{i} \sin \frac{\pi}{4}\right)$
36. $\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{6}\right)$
37. $2\left(\cos \frac{4\pi}{3} + \boldsymbol{i} \sin \frac{4\pi}{3}\right)$
38. $10(\cos 6 + \boldsymbol{i} \sin 6)$
39. $2\left(\cos \frac{5\pi}{4} + \boldsymbol{i} \sin \frac{5\pi}{4}\right)$
40. $2.5(\cos 1 + \boldsymbol{i} \sin 1)$
41. $5(\cos 0 + \boldsymbol{i} \sin 0)$
42. $3(\cos \pi + \boldsymbol{i} \sin \pi)$

Graph the first five members of the orbit of each initial value under iteration by the given function.

43. $z_0 = -0.5 + \boldsymbol{i}$, $f(z) = z^2 + 0.5$
44. $z_0 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\boldsymbol{i}$, $f(z) = z^2$
45. Graph the first five iterates of $z_0 = 0.5 - 0.5\boldsymbol{i}$ under $f(z) = z^2 - 0.5$.

Applications and Problem Solving

Real World Application

46. **Electrical Engineering** Refer to Exercise 44 in Lesson 9-3. Consider a circuit with alternating current that contains two voltage sources in series. Suppose these two voltages are given by $v_1(t) = 40 \sin(250t + 30°)$ and $v_2(t) = 60 \sin(250t + 60°)$, where t represents time, in seconds.
   a. The phasors for these two voltage sources are written as $40\angle 30°$ and $60\angle 60°$, respectively. Convert these phasors to complex numbers in rectangular form. (Use $\boldsymbol{j}$ as the imaginary unit, as electrical engineers do.)
   b. Add these two complex numbers to find the total voltage in the circuit.
   c. Write a sinusoidal function that gives the total voltage in the circuit.

47. **Critical Thinking** How are the polar forms of complex conjugates alike? How are they different?

48. **Electricity** A series circuit contains two sources of impedance, one of $10(\cos 0.7 + \boldsymbol{j} \sin 0.7)$ ohms and the other of $16(\cos 0.5 + \boldsymbol{j} \sin 0.5)$ ohms.

a. Convert these complex numbers to rectangular form.

b. Add your answers from part a to find the total impedance in the circuit.

c. Convert the total impedance back to polar form.

49. **Transformations** Certain operations with complex numbers correspond to geometric transformations in the complex plane. Describe the transformation applied to point z to obtain point w in the complex plane for each of the following operations.

a. $w = z + (2 - 3\boldsymbol{i})$

b. $w = \boldsymbol{i} \cdot z$

c. $w = 3z$

d. w is the conjugate of z

50. **Critical Thinking** Choose any two complex numbers, z_1 and z_2, in rectangular form.

a. Find the product z_1z_2.

b. Write z_1, z_2, and z_1z_2 in polar form.

c. Repeat this procedure with a different pair of complex numbers.

d. Make a conjecture about the product of two complex numbers in polar form.

Mixed Review

51. Simplify $(6 - 2\boldsymbol{i})(-2 + 3\boldsymbol{i})$. *(Lesson 9-5)*

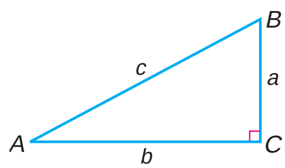
52. Find the rectangular coordinates of the point with polar coordinates $(-3, -135°)$. *(Lesson 9-3)*

53. Find the magnitude of the vector $\langle -3, 7 \rangle$, and write the vector as a sum of unit vectors. *(Lesson 8-2)*

54. Use a sum or difference identity to find tan 105°. *(Lesson 7-3)*

55. **Mechanics** A pulley of radius 18 centimeters turns at 12 revolutions per second. What is the linear velocity of the belt driving the pulley in meters per second? *(Lesson 6-2)*

56. If $a = 12$ and $c = 18$ in $\triangle ABC$, find the measure of angle A to the nearest tenth of a degree. *(Lesson 5-5)*



57. Solve $\sqrt{2a - 1} = \sqrt{3a - 5}$. *(Lesson 4-7)*

58. Without graphing, describe the end behavior of the graph of $y = 2x^2 + 2$. *(Lesson 3-5)*

59. **SAT/ACT Practice** A person is hired for a job that pays $500 per month and receives a 10% raise in each following month. In the fourth month, how much will that person earn?

A $550 **B** $600.50 **C** $650.50 **D** $665.50 **E** $700

Extra Practice See p. A43.

GRAPHING CALCULATOR EXPLORATION

9-6B Geometry in the Complex Plane

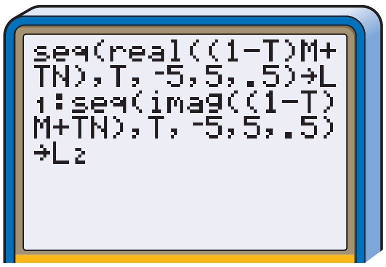
OBJECTIVE
- Explore geometric relationships in the complex plane.

An Extension of Lesson 9-6

Many geometric figures and relationships can be described by using complex numbers. To show points on figures, you can store the real and imaginary parts of the complex numbers that correspond to the points in lists **L1** and **L2** and use **STAT PLOT** to graph the points.

TRY THESE

1. Store $-1 + 2\boldsymbol{i}$ as M and $1 + 5\boldsymbol{i}$ as N. Now consider complex numbers of the form $(1 - T)M + TN$, where T is a real number. You can generate several numbers of this form and store their real and imaginary parts in **L1** and **L2**, respectively, by entering the following instructions on the home screen.



seq(is in the LIST OPS menu.
real(and imag(are in the MATH CPX menu.

Use a graphing window of $[-10, 10]$ scl:1 by $[-25, 25]$ scl:5. Turn on **Plot 1** and use a scatter plot to display the points defined in **L1** and **L2**. What do you notice about the points in the scatter plot?

2. Are the original numbers M and N shown in the scatter plot? Explain.

3. Repeat Exercise 1 storing $-1 + 1.5\boldsymbol{i}$ as M and $-2 - \boldsymbol{i}$ as N. Describe your results.

4. Repeat Exercises 1 and 2 for several complex numbers M and N of your choice. (You may need to change the window settings.) Then make a conjecture about where points of the form $(1 - T)M + TN$ are located in relation to M and N.

5. Suppose K, M, and N are three noncollinear points in the complex plane. Where will you find all the points that can be expressed in the form $aK + bM + cN$, where a, b, and c are nonnegative real numbers such that $a + b + c = 1$? Use the calculator to check your answer.

WHAT DO YOU THINK?

6. In Exercises 1-4, where is $(1 - T)M + TN$ in relation to M and N if the value of T is between 0 and 1?

7. Where in the complex plane will you find the complex numbers z that satisfy the equation $|z - (1 - \boldsymbol{i})| = 5$?

8. What equation models the points in the complex plane that lie on the circle of radius 2 that is centered at the point $2 + 3\boldsymbol{i}$?

9-7 Products and Quotients of Complex Numbers in Polar Form

OBJECTIVE
- Find the product and quotient of complex numbers in polar form.

Real World Application

ELECTRICITY Complex numbers can be used in the study of electricity, specifically alternating current (AC). There are three basic quantities to consider:

- the *current I*, measured in amperes,
- the *impedance Z* to the current, measured in ohms, and
- the *electromotive force E* or *voltage*, measured in volts.

These three quantities are related by the equation $E = I \cdot Z$. Current, impedance, and voltage can be expressed as complex numbers. Electrical engineers use $\boldsymbol{j}$ as the imaginary unit, so they write complex numbers in the form $a + b\boldsymbol{j}$. For the total impedance $a + b\boldsymbol{j}$, the real part a represents the opposition to current flow due to resistors, and the imaginary part b is related to the opposition due to inductors and capacitors. If a circuit has a total impedance of $2 - 6\boldsymbol{j}$ ohms and a voltage of 120 volts, find the current in the circuit. *This problem will be solved in Example 3.*

Multiplication and division of complex numbers in polar form are closely tied to geometric transformations in the complex plane. Let $r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)$ and $r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)$ be two complex numbers in polar form. A formula for the product of the two numbers can be derived by multiplying the two numbers directly and simplifying the result.

$$
\begin{aligned}
& r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1) \cdot r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2) \\
&\quad = r_1 r_2(\cos \theta_1 \cos \theta_2 + \boldsymbol{i} \cos \theta_1 \sin \theta_2 + \boldsymbol{i} \sin \theta_1 \cos \theta_2 + \boldsymbol{i}^2 \sin \theta_1 \sin \theta_2) \\
&\quad = r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + \boldsymbol{i}(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \quad \textit{i}^2 = -1 \\
&\quad = r_1 r_2[\cos (\theta_1 + \theta_2) + \boldsymbol{i} \sin (\theta_1 + \theta_2)] \quad \textit{Sum identities for cosine and sine}
\end{aligned}
$$

Product of Complex Numbers in Polar Form

$$r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1) \cdot r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2) = r_1 r_2[\cos (\theta_1 + \theta_2) + \boldsymbol{i} \sin (\theta_1 + \theta_2)]$$

Notice that the modulus (r_1r_2) of the product of the two complex numbers is the product of their moduli. The amplitude $(\theta_1 + \theta_2)$ of the product is the sum of the amplitudes.

Example 1 **Find the product $3\left(\cos \frac{7\pi}{6} + \boldsymbol{i} \sin \frac{7\pi}{6}\right) \cdot 2\left(\cos \frac{2\pi}{3} + \boldsymbol{i} \sin \frac{2\pi}{3}\right)$. Then express the product in rectangular form.**

Find the modulus and amplitude of the product.

$$r = r_1r_2 = 3(2) = 6 \qquad \theta = \theta_1 + \theta_2 = \frac{7\pi}{6} + \frac{2\pi}{3} = \frac{11\pi}{6}$$

The product is $6\left(\cos \frac{11\pi}{6} + \boldsymbol{i} \sin \frac{11\pi}{6}\right)$.

Now find the rectangular form of the product.

$$6\left(\cos \frac{11\pi}{6} + \boldsymbol{i} \sin \frac{11\pi}{6}\right) = 6\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}\right) \qquad \cos \frac{11\pi}{6} = \frac{\sqrt{3}}{2},\ \sin \frac{11\pi}{6} = -\frac{1}{2}$$

$$= 3\sqrt{3} - 3\boldsymbol{i}$$

The rectangular form of the product is $3\sqrt{3} - 3\boldsymbol{i}$.

Suppose the quotient of two complex numbers is expressed as a fraction. A formula for this quotient can be derived by rationalizing the denominator. To rationalize the denominator, multiply both the numerator and denominator by the same value so that the resulting new denominator does not contain imaginary numbers.

$$\frac{r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)}{r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)}$$

$$= \frac{r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)}{r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)} \cdot \frac{(\cos \theta_2 - \boldsymbol{i} \sin \theta_2)}{(\cos \theta_2 - \boldsymbol{i} \sin \theta_2)} \qquad \textit{$\cos \theta_2 - \boldsymbol{i} \sin \theta_2$ is the conjugate of $\cos \theta_2 + \boldsymbol{i} \sin \theta_2$.}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + \boldsymbol{i}(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2 \theta_2 + \sin^2 \theta_2}$$

$$= \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + \boldsymbol{i} \sin (\theta_1 - \theta_2)] \qquad \textit{Trigonometric identities}$$

Quotient of Complex Numbers in Polar Form

$$\frac{r_1(\cos \theta_1 + \boldsymbol{i} \sin \theta_1)}{r_2(\cos \theta_2 + \boldsymbol{i} \sin \theta_2)} = \frac{r_1}{r_2}[\cos (\theta_1 - \theta_2) + \boldsymbol{i} \sin (\theta_1 - \theta_2)]$$

Notice that the modulus $\left(\frac{r_1}{r_2}\right)$ of the quotient of two complex numbers is the quotient of their moduli. The amplitude $(\theta_1 - \theta_2)$ of the quotient is the difference of the amplitudes.

Example 2 **Find the quotient $12\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \div 4\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$. Then express the quotient in rectangular form.**

Find the modulus and amplitude of the quotient.

$$r = \frac{r_1}{r_2} = \frac{12}{4} = 3 \qquad \theta = \theta_1 - \theta_2 = \frac{\pi}{4} - \frac{3\pi}{2} = -\frac{5\pi}{4}$$

The quotient is $3\left[\cos\left(-\frac{5\pi}{4}\right) + i \sin\left(-\frac{5\pi}{4}\right)\right]$.

Now find the rectangular form of the quotient.

$$3\left[\cos\left(-\frac{5\pi}{4}\right) + i \sin\left(-\frac{5\pi}{4}\right)\right] = 3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) \qquad \cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \sin\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$= -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i$$

The rectangular form of the quotient is $-\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2} i$.

You can use products and quotients of complex numbers in polar form to solve the problem presented at the beginning of the lesson.

Example 3 **ELECTRICITY** **If a circuit has an impedance of $2 - 6j$ ohms and a voltage of 120 volts, find the current in the circuit.**

Real World Application

Express each complex number in polar form.

$$120 = 120(\cos 0 + j \sin 0)$$

$$2 - 6j \approx \sqrt{40}[\cos(-1.25) + j \sin(-1.25)] \qquad r = \sqrt{2^2 + (-6)^2} = \sqrt{40} \text{ or } 2\sqrt{10},$$

$$\approx 2\sqrt{10}[\cos(-1.25) + j \sin(-1.25)] \qquad \theta = \text{Arctan } \frac{-6}{2} \text{ or } -1.25$$

Substitute the voltage and impedance into the equation $E = I \cdot Z$.

$$E = I \cdot Z$$

$$120(\cos 0 + j \sin 0) = I \cdot 2\sqrt{10}[\cos(-1.25) + j \sin(-1.25)]$$

$$\frac{120(\cos 0 + j \sin 0)}{2\sqrt{10}[\cos(-1.25) + j \sin(-1.25)]} = I$$

$$6\sqrt{10}(\cos 1.25 + j \sin 1.25) = I$$

Now express the current in rectangular form.

$$I = 6\sqrt{10}(\cos 1.25 + j \sin 1.25)$$

$$\approx 5.98 + 18.01j \qquad \text{Use a calculator. } 6\sqrt{10} \cos 1.25 \approx 5.98,\ 6\sqrt{10} \sin 1.25 \approx 18.01$$

The current is about $6 + 18j$ amps.

CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Explain** how to find the quotient of two complex numbers in polar form.
2. **Describe** how to square a complex number in polar form.
3. **List** which operations with complex numbers you think are easier in rectangular form and which you think are easier in polar form. Defend your choices with examples.

Guided Practice

Find each product or quotient. Express the result in rectangular form.

4. $2\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right) \cdot 2\left(\cos\frac{3\pi}{2} + \boldsymbol{i}\sin\frac{3\pi}{2}\right)$
5. $3\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right) \div 4\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)$
6. $4\left(\cos\frac{9\pi}{4} + \boldsymbol{i}\sin\frac{9\pi}{4}\right) \div 2\left[\cos\left(-\frac{\pi}{2}\right) + \boldsymbol{i}\sin\left(-\frac{\pi}{2}\right)\right]$
7. $\frac{1}{2}\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right) \cdot 6\left(\cos\frac{5\pi}{6} + \boldsymbol{i}\sin\frac{5\pi}{6}\right)$
8. Use polar form to find the product $\left(2 + 2\sqrt{3}\boldsymbol{i}\right) \cdot \left(-3 + \sqrt{3}\boldsymbol{i}\right)$. Express the result in rectangular form.
9. **Electricity** Determine the voltage in a circuit when there is a current of $2\left(\cos\frac{11\pi}{6} + \boldsymbol{j}\sin\frac{11\pi}{6}\right)$ amps and an impedance of $3\left(\cos\frac{\pi}{3} + \boldsymbol{j}\sin\frac{\pi}{3}\right)$ ohms.

EXERCISES

Practice

Find each product or quotient. Express the result in rectangular form.

10. $4\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right) \cdot 7\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)$
11. $6\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right) \div 2\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)$
12. $\frac{1}{2}\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right) \div 3\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)$
13. $5(\cos\pi + \boldsymbol{i}\sin\pi) \cdot 2\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right)$
14. $6\left[\cos\left(-\frac{\pi}{3}\right) + \boldsymbol{i}\sin\left(-\frac{\pi}{3}\right)\right] \cdot 3\left(\cos\frac{5\pi}{6} + \boldsymbol{i}\sin\frac{5\pi}{6}\right)$
15. $3\left(\cos\frac{7\pi}{3} + \boldsymbol{i}\sin\frac{7\pi}{3}\right) \div \left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)$
16. $2(\cos 240° + \boldsymbol{i}\sin 240°) \cdot 3(\cos 60° + \boldsymbol{i}\sin 60°)$
17. $\sqrt{2}\left(\cos\frac{7\pi}{4} + \boldsymbol{i}\sin\frac{7\pi}{4}\right) \div \frac{\sqrt{2}}{2}\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right)$
18. $3(\cos 4 + \boldsymbol{i}\sin 4) \cdot 0.5(\cos 2.5 + \boldsymbol{i}\sin 2.5)$

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CONTENTS

19. $4[\cos(-2) + \boldsymbol{i}\sin(-2)] \div (\cos 3.6 + \boldsymbol{i}\sin 3.6)$

20. $20\left(\cos\frac{7\pi}{6} + \boldsymbol{i}\sin\frac{7\pi}{6}\right) \div 15\left(\cos\frac{11\pi}{3} + \boldsymbol{i}\sin\frac{11\pi}{3}\right)$

21. $2\left(\cos\frac{3\pi}{4} + \boldsymbol{i}\sin\frac{3\pi}{4}\right) \cdot \sqrt{2}\left(\cos\frac{\pi}{2} + \boldsymbol{i}\sin\frac{\pi}{2}\right)$

22. Find the product of $2\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$ and $6\left[\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i}\sin\left(-\frac{\pi}{6}\right)\right]$. Write the answer in rectangular form.

23. If $z_1 = 4\left(\cos\frac{5\pi}{3} + \boldsymbol{i}\sin\frac{5\pi}{3}\right)$ and $z_2 = \frac{1}{2}\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$, find $\frac{z_1}{z_2}$ and express the result in rectangular form.

Use polar form to find each product or quotient. Express the result in rectangular form.

24. $(2 - 2\boldsymbol{i}) \cdot (-3 + 3\boldsymbol{i})$

25. $\left(\sqrt{2} - \sqrt{2}\boldsymbol{i}\right) \cdot \left(-3\sqrt{2} - 3\sqrt{2}\boldsymbol{i}\right)$

26. $\left(\sqrt{3} - \boldsymbol{i}\right) \div \left(2 - 2\sqrt{3}\boldsymbol{i}\right)$

27. $\left(-4\sqrt{2} + 4\sqrt{2}\boldsymbol{i}\right) \div (6 + 6\boldsymbol{i})$

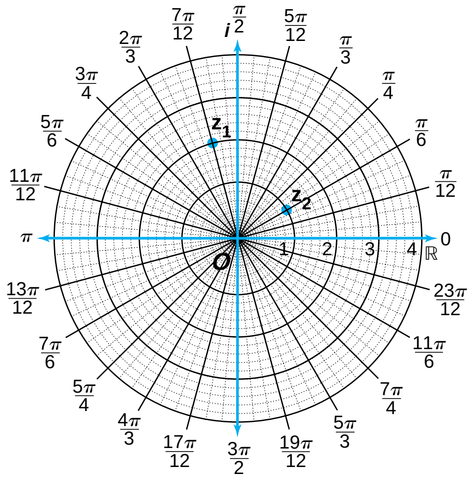
Applications and Problem Solving

Real World Application

28. **Electricity** Find the current in a circuit with a voltage of 13 volts and an impedance of $3 - 2\boldsymbol{j}$ ohms.

29. **Electricity** Find the impedance in a circuit with a voltage of 100 volts and a current of $4 - 3\boldsymbol{j}$ amps.

30. **Critical Thinking** Given z_1 and z_2 graphed at the right, graph z_1z_2 and $\frac{z_1}{z_2}$ without actually calculating them.



31. **Transformations**
   a. Describe the transformation applied to the graph of the complex number z if z is multiplied by $\cos\theta + \boldsymbol{i}\sin\theta$.
   b. Describe the transformation applied to the graph of the complex number z if z is multiplied by $\frac{1}{2} + \frac{\sqrt{3}}{2}\boldsymbol{i}$.

32. **Critical Thinking** Find the quadratic equation $az^2 + bz + c = 0$ such that $a = 1$ and the solutions are $3\left(\cos\frac{\pi}{3} + \boldsymbol{i}\sin\frac{\pi}{3}\right)$ and $2\left(\cos\frac{5\pi}{6} + \boldsymbol{i}\sin\frac{5\pi}{6}\right)$.

Mixed Review

33. Express $5 - 12\boldsymbol{i}$ in polar form. *(Lesson 9-6)*

34. Write the equation $r = 5\sec\left(\theta - \frac{5\pi}{6}\right)$ in rectangular form. *(Lesson 9-4)*

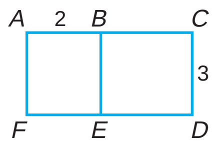
35. **Physics** A prop for a play is supported equally by two wires suspended from the ceiling. The wires form a 130° angle with each other. If the prop weighs 23 pounds, what is the tension in each of the wires? *(Lesson 8-5)*

Extra Practice See p. A43.

36. Solve $\cos 2x + \sin x = 1$ for principal values of x. *(Lesson 7-5)*

37. Write the equation for the inverse of $y = \cos x$. *(Lesson 6-8)*

38. **SAT/ACT Practice** In the figure, the perimeter of square $BCDE$ is how much smaller than the perimeter of rectangle $ACDF$?



A 2 **B** 3 **C** 4

D 6 **E** 16

CAREER CHOICES

Astronomer

Have you ever gazed into the sky at night hoping to spot a constellation? Do you dream of having your own telescope? If you enjoy studying about the universe, then a career in astronomy may be just for you. Astronomers collect and analyze data about the universe including stars, planets, comets, asteroids, and even artificial satellites. As an astronomer, you may collect information by using a telescope or spectrometer here on earth, or you may use information collected by spacecraft and satellites.

Most astronomers specialize in one branch of astronomy such as astrophysics or celestial mechanics. Astronomers often teach in addition to conducting research. Astronomers located throughout the world are prime sources of information for NASA and other countries' space programs.

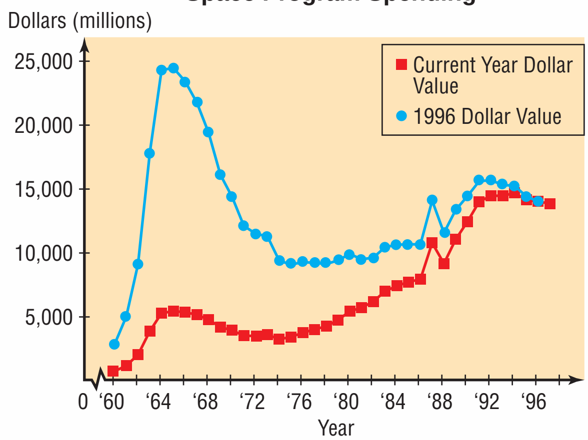
Career Overview

Degree Preferred:
at least a bachelor's degree in astronomy or physics

Related Courses:
mathematics, physics, chemistry, computer science

Outlook:
average through the year 2006

Space Program Spending



Source: National Aeronautics and Space Administration

interNET CONNECTION For more information on careers in astronomy, visit: **www.amc.glencoe.com**

9-8

Powers and Roots of Complex Numbers

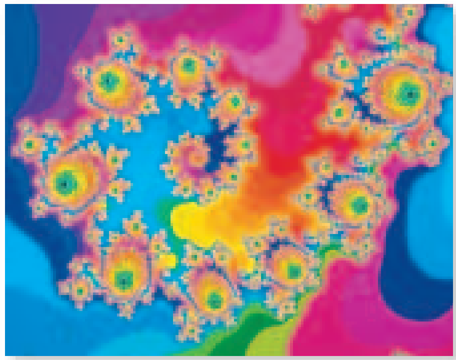
OBJECTIVE

- Find powers and roots of complex numbers in polar form using De Moivre's Theorem.

Real World Application

COMPUTER GRAPHICS Many of the computer graphics that are referred to as fractals are graphs of Julia sets, which are named after the mathematician Gaston Julia. When a function like $f(z) = z^2 + c$, where c is a complex constant, is iterated, points in the complex plane can be classified according to their behavior under iteration.

- Points that escape to infinity under iteration belong to the **escape set** of the function.
- Points that do not escape belong to the **prisoner set.**



The **Julia set** is the boundary between the escape set and the prisoner set. Is the number $w = 0.6 - 0.5\boldsymbol{i}$ in the escape set or the prisoner set of the function $f(z) = z^2$? *This problem will be solved in Example 6.*

You can use the formula for the product of complex numbers to find the square of a complex number.

$$\begin{aligned}[r(\cos\theta + \boldsymbol{i}\sin\theta)]^2 &= [r(\cos\theta + \boldsymbol{i}\sin\theta)]\cdot[r(\cos\theta + \boldsymbol{i}\sin\theta)]\\ &= r^2[\cos(\theta+\theta) + \boldsymbol{i}\sin(\theta+\theta)]\\ &= r^2(\cos 2\theta + \boldsymbol{i}\sin 2\theta)\end{aligned}$$

Other powers of complex numbers can be found using De Moivre's Theorem.

De Moivre's Theorem	$[r(\cos\theta + \boldsymbol{i}\sin\theta)]^n = r^n(\cos n\theta + \boldsymbol{i}\sin n\theta)$

You will be asked to prove De Moivre's Theorem in Chapter 12.

Example 1 Find $\left(2 + 2\sqrt{3}\boldsymbol{i}\right)^6$.

First, write $2 + 2\sqrt{3}\boldsymbol{i}$ in polar form. Note that its graph is in the first quadrant of the complex plane.

$$\begin{aligned} r &= \sqrt{2^2 + \left(2\sqrt{3}\right)^2} \\ &= \sqrt{4+12} \\ &= 4 \end{aligned} \qquad \begin{aligned} \theta &= \operatorname{Arctan}\frac{2\sqrt{3}}{2} \\ &= \operatorname{Arctan}\sqrt{3} \\ &= \frac{\pi}{3} \end{aligned}$$

(continued on the next page)

The polar form of $2 + 2\sqrt{3}\boldsymbol{i}$ is $4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right)$.

Now use De Moivre's Theorem to find the sixth power.

$$\begin{aligned}
(2 + 2\sqrt{3}\boldsymbol{i})^6 &= \left[4\left(\cos \frac{\pi}{3} + \boldsymbol{i} \sin \frac{\pi}{3}\right)\right]^6 \\
&= 4^6\left[\cos 6\left(\frac{\pi}{3}\right) + \boldsymbol{i} \sin 6\left(\frac{\pi}{3}\right)\right] \\
&= 4096(\cos 2\pi + \boldsymbol{i} \sin 2\pi) \\
&= 4096(1 + 0\boldsymbol{i}) \quad \textit{Write the result in rectangular form.} \\
&= 4096
\end{aligned}$$

Therefore, $\left(2 + 2\sqrt{3}\boldsymbol{i}\right)^6 = 4096$.

De Moivre's Theorem is valid for all rational values of n. Therefore, it is also useful for finding negative powers of complex numbers and roots of complex numbers.

Example 2 **Find $\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}\right)^{-5}$.**

First, write $\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}$ in polar form. Note that its graph is in the fourth quadrant of the complex plane.

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} \qquad \theta = \text{Arctan}\,\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \sqrt{\frac{3}{4} + \frac{1}{4}} \text{ or } 1 \qquad = \text{Arctan}\left(-\frac{\sqrt{3}}{3}\right) \text{ or } -\frac{\pi}{6}$$

The polar form of $\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}$ is $1\left[\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{6}\right)\right]$.

Use De Moivre's Theorem to find the negative 5th power.

$$\begin{aligned}
\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\boldsymbol{i}\right)^{-5} &= \left[1\left(\cos\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin\left(-\frac{\pi}{6}\right)\right)\right]^{-5} \\
&= 1^{-5}\left[\cos(-5)\left(-\frac{\pi}{6}\right) + \boldsymbol{i} \sin(-5)\left(-\frac{\pi}{6}\right)\right] \quad \textit{De Moivre's Theorem} \\
&= 1\left(\cos \frac{5\pi}{6} + \boldsymbol{i} \sin \frac{5\pi}{6}\right) \quad \textit{Simplify.} \\
&= -\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i} \quad \textit{Write the answer in rectangular form.}
\end{aligned}$$

Recall that positive real numbers have two square roots and that the positive one is called the principal square root. In general, all nonzero complex numbers have p distinct pth roots. That is, they each have two square roots, three cube roots, four fourth roots, and so on. The principal pth root of a complex number is given by:

$$(a + b\boldsymbol{i})^{\frac{1}{p}} = [r(\cos \theta + \boldsymbol{i} \sin \theta)]^{\frac{1}{p}}$$
$$= r^{\frac{1}{p}}\left(\cos \frac{\theta}{p} + \boldsymbol{i} \sin \frac{\theta}{p}\right).$$

When finding a principal root, the interval $-\pi < \theta \leq \pi$ is used.

Example 3 **Find $\sqrt[3]{8\boldsymbol{i}}$.**

$\sqrt[3]{8\boldsymbol{i}} = (0 + 8\boldsymbol{i})^{\frac{1}{3}}$ *$a = 0$, $b = 8$*

$= \left[8\left(\cos \frac{\pi}{2} + \boldsymbol{i} \sin \frac{\pi}{2}\right)\right]^{\frac{1}{3}}$ *Polar form; $r = \sqrt{0^2 + 8^2}$ or 8, $\theta = \frac{\pi}{2}$ since $a = 0$.*

$= 8^{\frac{1}{3}}\left[\cos \left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) + \boldsymbol{i} \sin \left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)\right]$ *De Moivre's Theorem*

$= 2\left(\cos \frac{\pi}{6} + \boldsymbol{i} \sin \frac{\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\boldsymbol{i}\right)$ or $\sqrt{3} + \boldsymbol{i}$ *This is the principal cube root.*

The following formula generates all of the pth roots of a complex number. It is based on the identities $\cos \theta = \cos (\theta + 2n\pi)$ and $\sin \theta = \sin (\theta + 2n\pi)$, where n is any integer.

The p Distinct pth Roots of a Complex Number

The p distinct pth roots of $a + bi$ can be found by replacing n with 0, 1, 2, ..., $p - 1$, successively, in the following equation.

$$(a + bi)^{\frac{1}{p}} = (r[\cos (\theta + 2n\pi) + i \sin (\theta + 2n\pi)])^{\frac{1}{p}}$$
$$= r^{\frac{1}{p}}\left(\cos \frac{\theta + 2n\pi}{p} + i \sin \frac{\theta + 2n\pi}{p}\right)$$

Example 4 **Find the three cube roots of $-2 - 2\boldsymbol{i}$.**

First, write $-2 - 2\boldsymbol{i}$ in polar form.

$r = \sqrt{(-2)^2 + (-2)^2}$ or $2\sqrt{2}$ $\qquad \theta = \text{Arctan } \frac{-2}{-2} + \pi$ or $\frac{5\pi}{4}$

$-2 - 2\boldsymbol{i} = 2\sqrt{2}\left[\cos \left(\frac{5\pi}{4} + 2n\pi\right) + \boldsymbol{i} \sin \left(\frac{5\pi}{4} + 2n\pi\right)\right]$ *n is any integer.*

Now write an expression for the cube roots.

$(-2 - 2\boldsymbol{i})^{\frac{1}{3}} = \left(2\sqrt{2}\left[\cos \left(\frac{5\pi}{4} + 2n\pi\right) + \boldsymbol{i} \sin \left(\frac{5\pi}{4} + 2n\pi\right)\right]\right)^{\frac{1}{3}}$

$= \sqrt{2}\left[\cos \left(\frac{\frac{5\pi}{4} + 2n\pi}{3}\right) + \boldsymbol{i} \sin \left(\frac{\frac{5\pi}{4} + 2n\pi}{3}\right)\right]$ *$\left(2\sqrt{2}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{2}}$ or $\sqrt{2}$*

(continued on the next page)

Let $n = 0$, 1, and 2 successively to find the cube roots.

Let $n = 0$. $$\sqrt{2}\left[\cos\left(\frac{\frac{5\pi}{4} + 2(0)\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\frac{5\pi}{4} + 2(0)\pi}{3}\right)\right]$$
$$= \sqrt{2}\left(\cos\frac{5\pi}{12} + \boldsymbol{i}\sin\frac{5\pi}{12}\right)$$
$$\approx 0.37 + 1.37\boldsymbol{i}$$

Let $n = 1$. $$\sqrt{2}\left[\cos\left(\frac{\frac{5\pi}{4} + 2(1)\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\frac{5\pi}{4} + 2(1)\pi}{3}\right)\right]$$
$$= \sqrt{2}\left(\cos\frac{13\pi}{12} + \boldsymbol{i}\sin\frac{13\pi}{12}\right)$$
$$\approx -1.37 - 0.37i$$

Let $n = 2$. $$\sqrt{2}\left[\cos\left(\frac{\frac{5\pi}{4} + 2(2)\pi}{3}\right) + \boldsymbol{i}\sin\left(\frac{\frac{5\pi}{4} + 2(2)\pi}{3}\right)\right]$$
$$= \sqrt{2}\left(\cos\frac{21\pi}{12} + \boldsymbol{i}\sin\frac{21\pi}{12}\right)$$
$$= 1 - \boldsymbol{i}$$

The cube roots of $-2 - 2\boldsymbol{i}$ are approximately $0.37 + 1.37\boldsymbol{i}$, $-1.37 - 0.37\boldsymbol{i}$, and $1 - \boldsymbol{i}$. *These roots can be checked by multiplication.*

GRAPHING CALCULATOR EXPLORATION

The p distinct pth roots of a complex number can be approximated using the parametric mode on a graphing calculator. For a particular complex number $r(\cos\theta + \boldsymbol{i}\sin\theta)$ and a particular value of p:

- Select the **Radian** and **Par** modes.
- Select the viewing window.

  Tmin $= \frac{\theta}{p}$, Tmax $= \frac{\theta}{p} + 2\pi$, Tstep $= \frac{2\pi}{p}$,

  Xmin $= -r^{\frac{1}{p}}$, Xmax $= r^{\frac{1}{p}}$, Xscl $= 1$,

  Ymin $= -r^{\frac{1}{p}}$, Ymax $= r^{\frac{1}{p}}$, and Yscl $= 1$.
- Enter the parametric equations $X_{1T} = r^{\frac{1}{p}}\cos T$ and $Y_{1T} = r^{\frac{1}{p}}\sin T$.
- Graph the equations.
- Use TRACE to locate the roots.

TRY THESE

1. Approximate the cube roots of 1.

2. Approximate the fourth roots of $\boldsymbol{i}$.

3. Approximate the fifth roots of $1 + \boldsymbol{i}$.

WHAT DO YOU THINK?

4. What geometric figure is formed when you graph the three cube roots of a complex number?

5. What geometric figure is formed when you graph the fifth roots of a complex number?

6. Under what conditions will the complex number $a + b\boldsymbol{i}$ have a root that lies on the positive real axis?

You can also use De Moivre's Theorem to solve some polynomial equations.

Examples

5 **Solve $x^5 - 32 = 0$. Then graph the roots in the complex plane.**

The solutions to this equation are the same as those of the equation $x^5 = 32$. That means we have to find the fifth roots of 32.

$32 = 32 + 0\boldsymbol{i}$ *$a = 32$, $b = 0$*

$= 32(\cos 0 + \boldsymbol{i} \sin 0)$ *Polar form; $r = \sqrt{32^2 + 0^2}$ or 32, $\theta = \text{Arctan } \frac{0}{32}$ or 0*

Now write an expression for the fifth roots.

$$32^{\frac{1}{5}} = [32(\cos(0 + 2n\pi) + \boldsymbol{i} \sin(0 + 2n\pi))]^{\frac{1}{5}}$$

$$= 2\left(\cos \frac{2n\pi}{5} + \boldsymbol{i} \sin \frac{2n\pi}{5}\right)$$

Let $n = 0, 1, 2, 3$, and 4 successively to find the fifth roots, x_1, x_2, x_3, x_4, and x_5.

Let $n = 0$. $\quad x_1 = 2(\cos 0 + \boldsymbol{i} \sin 0) = 2$

Let $n = 1$. $\quad x_2 = 2\left(\cos \frac{2\pi}{5} + \boldsymbol{i} \sin \frac{2\pi}{5}\right) \approx 0.62 + 1.90\boldsymbol{i}$

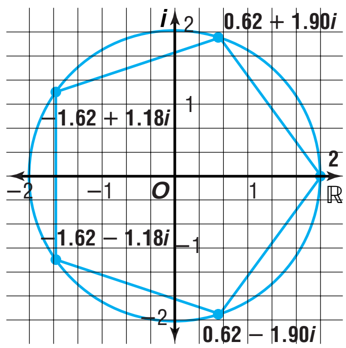
Let $n = 2$. $\quad x_3 = 2\left(\cos \frac{4\pi}{5} + \boldsymbol{i} \sin \frac{4\pi}{5}\right) \approx -1.62 + 1.18\boldsymbol{i}$

Let $n = 3$. $\quad x_4 = 2\left(\cos \frac{6\pi}{5} + \boldsymbol{i} \sin \frac{6\pi}{5}\right) \approx -1.62 - 1.18\boldsymbol{i}$

Let $n = 4$. $\quad x_5 = 2\left(\cos \frac{8\pi}{5} + \boldsymbol{i} \sin \frac{8\pi}{5}\right) \approx 0.62 - 1.90\boldsymbol{i}$

The solutions of $x^5 - 32 = 0$ are 2, $0.62 \pm 1.90\boldsymbol{i}$, and $-1.62 \pm 1.18\boldsymbol{i}$.

The solutions are graphed at the right. Notice that the points are the vertices of a regular pentagon. The roots of a complex number are cyclical in nature. That means, when the roots are graphed on the complex plane, the roots are equally spaced around a circle.



6 **COMPUTER GRAPHICS** **Refer to the application at the beginning of the lesson. Is the number $w = 0.6 - 0.5i$ in the escape set or the prisoner set of the function $f(z) = z^2$?**

Real World Application

Iterating this function requires you to square complex numbers, so you can use De Moivre's Theorem.

Write w in polar form.   *$r = \sqrt{0.6^2 + (-0.5)^2}$ or about 0.78*

$w = 0.78[\cos(-0.69) + \boldsymbol{i} \sin(-0.69)]$ *$\theta = \text{Arctan } \frac{-0.5}{0.6}$ or about -0.69*

(continued on the next page)

interNET CONNECTION

Graphing Calculator Programs
For a program that draws Julia sets, visit: **www.amc.glencoe.com**

Now iterate the function.

$w_1 = f(w)$

$= w^2$

$= (0.78[\cos(-0.69) + \boldsymbol{i}\sin(-0.69)])^2$

$= 0.78^2[\cos 2(-0.69) + \boldsymbol{i}\sin 2(-0.69)]$ *De Moivre's Theorem*

$= 0.12 - 0.60\boldsymbol{i}$ *Use a calculator to approximate the rectangular form.*

$w_2 = f(w_1)$

$= {w_1}^2$

$= (0.78^2[\cos 2(-0.69) + \boldsymbol{i}\sin 2(-0.69)])^2$ *Use the polar form of w_1.*

$= 0.78^4[\cos 4(-0.69) + \boldsymbol{i}\sin 4(-0.69)]$

$= -0.34 - 0.14\boldsymbol{i}$

$w_3 = f(w_2)$

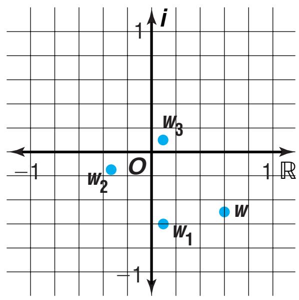
$= {w_2}^2$

$= (0.78^4[\cos 4(-0.69) + \boldsymbol{i}\sin 4(-0.69)])^2$ *Use the polar form of w_2.*

$= 0.78^8[\cos 8(-0.69) + \boldsymbol{i}\sin 8(-0.69)]$

$= 0.10 + 0.09\boldsymbol{i}$

The moduli of these iterates are 0.78^2, 0.78^4, 0.78^8, and so on. These moduli will approach 0 as the number of iterations increases. This means the graphs of the iterates approach the origin in the complex plane, so $w = 0.6 - 0.5\boldsymbol{i}$ is in the prisoner set of the function.



CHECK FOR UNDERSTANDING

Communicating Mathematics

Read and study the lesson to answer each question.

1. **Evaluate** the product $(1 + \boldsymbol{i})(1 + \boldsymbol{i})(1 + \boldsymbol{i})(1 + \boldsymbol{i})(1 + \boldsymbol{i})$ by traditional multiplication. Compare the results with the results using De Moivre's Theorem on $(1 + \boldsymbol{i})^5$. Which method do you prefer?
2. **Explain** how to use De Moivre's Theorem to find the reciprocal of a complex number in polar form.
3. **Graph** all the fourth roots of a complex number if $a + a\boldsymbol{i}$ is one of the fourth roots. Assume a is positive.
4. **You Decide** Shembala says that if $a \neq 0$, then $(a + a\boldsymbol{i})^2$ must be a pure imaginary number. Arturo disagrees. Who is correct? Use polar form to explain.

Guided Practice

Find each power. Express the result in rectangular form.

5. $\left(\sqrt{3} - \boldsymbol{i}\right)^3$

6. $(3 - 5\boldsymbol{i})^4$

Find each principal root. Express the result in the form $a + bi$ with a and b rounded to the nearest hundredth.

7. $\boldsymbol{i}^{\frac{1}{6}}$

8. $(-2 - \boldsymbol{i})^{\frac{1}{3}}$

Solve each equation. Then graph the roots in the complex plane.

9. $x^4 + \boldsymbol{i} = 0$

10. $2x^3 + 4 + 2\boldsymbol{i} = 0$

11. **Fractals** Refer to the application at the beginning of the lesson. Is $w = 0.8 - 0.7\boldsymbol{i}$ in the prisoner set or the escape set for the function $f(z) = z^2$? Explain.

EXERCISES

Practice

Find each power. Express the result in rectangular form.

12. $\left[3\left(\cos\frac{\pi}{6} + \boldsymbol{i}\sin\frac{\pi}{6}\right)\right]^3$

13. $\left[2\left(\cos\frac{\pi}{4} + \boldsymbol{i}\sin\frac{\pi}{4}\right)\right]^5$

14. $(-2 + 2\boldsymbol{i})^3$

15. $\left(1 + \sqrt{3}\boldsymbol{i}\right)^4$

16. $(3 - 6\boldsymbol{i})^4$

17. $(2 + 3\boldsymbol{i})^{-2}$

18. Raise $2 + 4\boldsymbol{i}$ to the fourth power.

Find each principal root. Express the result in the form $a + bi$ with a and b rounded to the nearest hundredth.

19. $\left[32\left(\cos\frac{2\pi}{3} + \boldsymbol{i}\sin\frac{2\pi}{3}\right)\right]^{\frac{1}{5}}$

20. $(-1)^{\frac{1}{4}}$

21. $(-2 + \boldsymbol{i})^{\frac{1}{4}}$

22. $(4 - \boldsymbol{i})^{\frac{1}{3}}$

23. $(2 + 2\boldsymbol{i})^{\frac{1}{3}}$

24. $(-1 - \boldsymbol{i})^{\frac{1}{4}}$

25. Find the principal square root of $\boldsymbol{i}$.

Solve each equation. Then graph the roots in the complex plane.

26. $x^3 - 1 = 0$

27. $x^5 + 1 = 0$

28. $2x^4 - 128 = 0$

29. $3x^4 + 48 = 0$

30. $x^4 - (1 + \boldsymbol{i}) = 0$

31. $2x^4 + 2 + 2\sqrt{3}\boldsymbol{i} = 0$

Graphing Calculator

Use a graphing calculator to find all of the indicated roots.

32. fifth roots of $10 - 9\boldsymbol{i}$

33. sixth roots of $2 + 4\boldsymbol{i}$

34. eighth roots of $36 + 20\boldsymbol{i}$

Applications and Problem Solving

Real World Application

35. **Fractals** Is the number $\frac{1}{2} + \frac{3}{4}\boldsymbol{i}$ in the escape set or the prisoner set for the function $f(z) = z^2$? Explain.

36. **Critical Thinking** Suppose $w = a + b\boldsymbol{i}$ is one of the 31st roots of 1.
   a. What is the maximum value of a?
   b. What is the maximum value of b?

37. **Design** Gloribel works for an advertising agency. She wants to incorporate a hexagon design into the artwork for one of her proposals. She knows that she can locate the vertices of a regular hexagon by graphing the solutions to the equation $x^6 - 1 = 0$ in the complex plane. What are the solutions to this equation?

38. **Computer Graphics** Computer programmers can use complex numbers and the complex plane to implement geometric transformations. If a programmer starts with a square with vertices at (2, 2), (−2, 2), (−2, −2), and (2, −2), each of the vertices can be stored as a complex number in polar form. Complex number multiplication can be used to rotate the square 45° counterclockwise and dilate it so that the new vertices lie at the midpoints of the sides of the original square.
   a. What complex number should the programmer multiply by to produce this transformation?
   b. What happens if the original vertices are multiplied by the square of your answer to part a?

39. **Critical Thinking** Explain why the sum of the imaginary parts of the p distinct pth roots of any positive real number must be zero.

Mixed Review

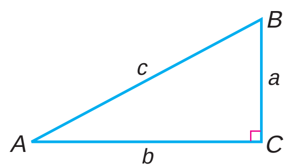
40. Find the product $2\left(\cos \frac{\pi}{6} + \boldsymbol{i} \sin \frac{\pi}{6}\right) \cdot 3\left(\cos \frac{5\pi}{3} + \boldsymbol{i} \sin \frac{5\pi}{3}\right)$. Express the result in rectangular form. *(Lesson 9-7)*

41. Simplify $(2 - 5\boldsymbol{i}) + (-3 + 6\boldsymbol{i}) - (-6 + 2\boldsymbol{i})$. *(Lesson 9-5)*

42. Write parametric equations of the line with equation $y = -2x + 7$. *(Lesson 8-6)*

43. Use a half-angle identity to find the exact value of $\cos 22.5°$. *(Lesson 7-4)*

44. Solve triangle ABC if $A = 81°15'$ and $b = 28$. Round angle measures to the nearest minute and side measures to the nearest tenth. *(Lesson 5-4)*



45. **Manufacturing** The Precious Animal Company must produce at least 300 large stuffed bears and 400 small stuffed bears per day. At most, the company can produce a total of 1200 bears per day. The profit for each large bear is \$9.00, and the profit for each small bear is \$5.00. How many of each type of bear should be produced each day to maximize profit? *(Lesson 2-7)*

46. **SAT/ACT Practice** Six quarts of a 20% solution of alcohol in water are mixed with 4 quarts of a 60% solution of alcohol in water. The alcoholic strength of the mixture is

A 36% **B** 40% **C** 48% **D** 60% **E** 80%

Extra Practice See p. A43.

VOCABULARY

absolute value of a complex number (p. 586)
amplitude of a complex number (p. 587)
Argand plane (p. 586)
argument of a complex number (p. 587)
cardioid (p. 563)
complex conjugates (p. 582)
complex number (p. 580)
complex plane (p. 586)
escape set (p. 599)
imaginary number (p. 581)
imaginary part (p. 581)
iteration (p. 581)
Julia set (p. 599)
lemniscate (p. 564)
limaçon (p. 562)
modulus (p. 587)
polar axis (p. 553)
polar coordinates (p. 553)
polar equation (p. 556)
polar form of a complex number (p. 588)
polar graph (p. 556)
polar plane (p. 553)
pole (p. 553)
prisoner set (p. 599)
pure imaginary number (p. 581)
real part (p. 581)
rectangular form of a complex number (p. 581)
rose (p. 563)
spiral of Archimedes (p. 564)
trigonometric form of a complex number (p. 588)

UNDERSTANDING AND USING THE VOCABULARY

Choose the correct term to best complete each sentence.

1. The (absolute value, conjugate) of a complex number is its distance from zero in the complex plane.
2. (Complex, Polar) coordinates give the position of an object using distances and angles.
3. Points that do not escape under iteration belong to the (escape, prisoner) set.
4. The process of repeatedly applying a function to the output produced by the previous input is called (independent, iteration).
5. A complex number in the form $b\boldsymbol{i}$ where $b \neq 0$ is a (pure imaginary, real) number.
6. A (cardioid, rose) is a special type of limaçon.
7. The complex number $a + b\boldsymbol{i}$, where a and b are real numbers, is in (polar, rectangular) form.
8. The (spiral of Archimedes, lemniscate) has a polar equation of the form $r = a\theta$.
9. The complex plane is sometimes called the (Argand, polar) plane.
10. The polar form of a complex number is given by $r(\cos\theta + \boldsymbol{i}\sin\theta)$, where r represents the (modulus, amplitude) of the complex number.

interNET CONNECTION For additional review and practice for each lesson, visit: **www.amc.glencoe.com**

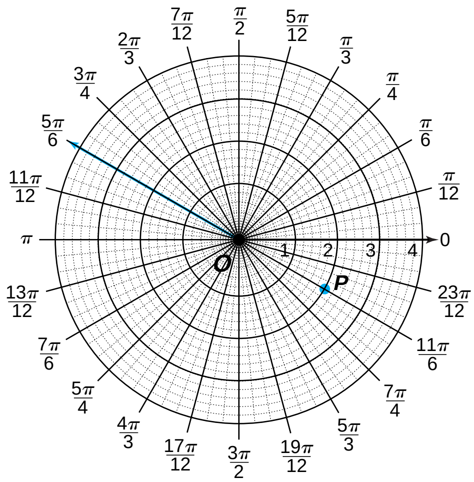
SKILLS AND CONCEPTS

OBJECTIVES AND EXAMPLES

Lesson 9-1 Graph points in polar coordinates.

Graph $P\left(-2, \frac{5\pi}{6}\right)$.

Extend the terminal side of the angle measuring $\frac{5\pi}{6}$. Find the point P that is 2 units from the pole along the ray opposite the terminal side.



REVIEW EXERCISES

Graph each point.

11. $A(-3, 50°)$

12. $B(1.5, -110°)$

13. $C\left(2, \frac{\pi}{4}\right)$

14. $D\left(-3, \frac{\pi}{2}\right)$

15. Name four other pairs of polar coordinates for the point at $(4, 225°)$.

Graph each polar equation.

16. $r = \sqrt{7}$

17. $r = -2$

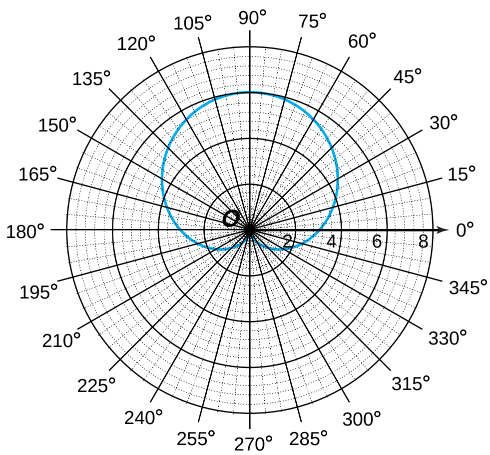
18. $\theta = -80°$

19. $\theta = \frac{3\pi}{4}$

Lesson 9-2 Graph polar equations.

Graph $r = 3 + 3 \sin \theta$.

The graph below can be made from a table of values. This graph is a cardioid.



Graph each polar equation. Identify the type of curve each represents.

20. $r = 7 \cos \theta$

21. $r = 5\theta$

22. $r = 2 + 4 \cos \theta$

23. $r = 6 \sin 2\theta$

OBJECTIVES AND EXAMPLES

Lesson 9-3 Convert between polar and rectangular coordinates.

- Find the rectangular coordinates of $C\left(-3, \frac{\pi}{6}\right)$.

For $C\left(-3, \frac{\pi}{6}\right)$, $r = -3$ and $\theta = \frac{\pi}{6}$.

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= -3\cos\frac{\pi}{6} \qquad = -3\sin\frac{\pi}{6}$$
$$= -\frac{3\sqrt{3}}{2} \qquad = -\frac{3}{2}$$

The rectangular coordinates of C are $\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$, or about $(-2.6, -1.5)$.

REVIEW EXERCISES

Find the rectangular coordinates of each point with the given polar coordinates.

24. $(6, 45°)$
25. $(2, 330°)$
26. $\left(-2, \frac{3\pi}{4}\right)$
27. $\left(1, \frac{\pi}{2}\right)$

Find the polar coordinates of each point with the given rectangular coordinates. Use $0 \le \theta < 2\pi$ and $r \ge 0$.

28. $\left(-\sqrt{3}, -3\right)$
29. $(5, 5)$
30. $(-3, 1)$
31. $(4, 2)$

Lesson 9-4 Write the polar form of a linear equation.

- Write $\sqrt{2}x - \sqrt{2}y = 6$ in polar form.

The standard form of the equation is $\sqrt{2}x - \sqrt{2}y - 6 = 0$.
Since $\sqrt{A^2 + B^2} = \sqrt{2 + 2}$ or 2, the normal form is $\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y - 3 = 0$.

$$\tan\phi = \frac{\sin\phi}{\cos\phi}$$
$$\tan\phi = -1$$
$$\phi = -45° \text{ or } 315°$$

The polar form is $3 = r\cos(\theta - 315°)$.

Write each equation in polar form. Round ϕ to the nearest degree.

32. $2x + y = -3$

33. $y = -3x - 4$

Write each equation in rectangular form.

34. $3 = r\cos\left(\theta - \frac{\pi}{3}\right)$

35. $4 = r\cos\left(\theta + \frac{\pi}{2}\right)$

Lesson 9-5 Add, subtract, multiply, and divide complex numbers in rectangular form.

- Simplify $(2 - 4\boldsymbol{i})(5 + 2\boldsymbol{i})$.

$$\begin{aligned}(2 - 4i)(5 + 2i) &= 2(5 + 2i) - 4i(5 + 2i)\\ &= 10 + 4i - 20i - 8i^2\\ &= 10 - 16i - 8(-1)\\ &= 18 - 16i\end{aligned}$$

Simplify.

36. $i^{10} + i^{25}$

37. $(2 + 3i) - (4 - 4i)$

38. $(2 + 7i) + (-3 - i)$

39. $i^3(4 - 3i)$

40. $(i - 7)(-i + 7)$

41. $\frac{4 + 2i}{5 - 2i}$

42. $\frac{5 + i}{1 - \sqrt{2}i}$

OBJECTIVES AND EXAMPLES

Lesson 9-6 Convert complex numbers from rectangular to polar form and vice versa.

- Express $5 - 2i$ in polar form.

  Find the modulus.

  $r = \sqrt{5^2 + (-2)^2}$ or $\sqrt{29}$

  Find the amplitude. Since $5 - 2i$ is in Quadrant IV in the complex plane, θ is in Quadrant IV.

  $\theta = \text{Arctan}\,\frac{-2}{5}$ or about -0.38

  $5 - 2i = \sqrt{29}[\cos(-0.38) + i\sin(-0.38)]$

REVIEW EXERCISES

Express each complex number in polar form

43. $2 + 2i$
44. $1 - 3i$
45. $-1 + \sqrt{3}i$
46. $-6 - 4i$
47. $-4 - i$
48. 4
49. $-2\sqrt{2}$
50. $3i$

Graph each complex number. Then express it in rectangular form.

51. $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
52. $3\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

Lesson 9-7 Find the product and quotient of complex numbers in polar form.

- Find the product of $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ and $3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. Then express the product in rectangular form.

  $r = r_1r_2 \qquad \theta = \theta_1 + \theta_2$

  $= 4(3) \qquad = \frac{\pi}{2} + \frac{3\pi}{4}$

  $= 12 \qquad = \frac{5\pi}{4}$

  The product is $12\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$.

  Now find the rectangular form of the product.

  $$12\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 12\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -6\sqrt{2} - 6\sqrt{2}i$$

Find each product or quotient. Express the result in rectangular form.

53. $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \cdot 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$
54. $8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$
55. $2(\cos 2 + i\sin 2) \cdot 5(\cos 0.5 + i\sin 0.5)$
56. $8\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) \div 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$
57. $6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \div 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
58. $2.2(\cos 1.5 + i\sin 1.5) \div 4.4(\cos 0.6 + i\sin 0.6)$

Lesson 9-8 Find powers and roots of complex numbers in polar form using De Moivre's Theorem.

- Find $(-3 + 3i)^4$.

  $$(-3 + 3i)^4 = \left[3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]^4$$
  $$= \left(3\sqrt{2}\right)^4(\cos 3\pi + i\sin 3\pi)$$
  $$= 324(-1 + 0) \text{ or } -324$$

Find each power. Express the result in rectangular form.

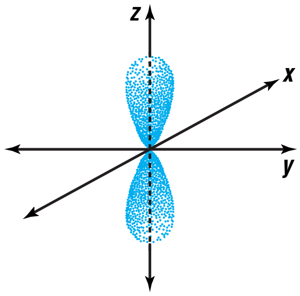
59. $(2 + 2i)^8$
60. $\left(\sqrt{3} - i\right)^7$
61. $(-1 + i)^4$
62. $(-2 - 2i)^3$

Find each principal root. Express the result in the form $a + bi$ with a and b rounded to the nearest hundredth.

63. $i^{\frac{1}{4}}$
64. $\left(\sqrt{3} + i\right)^{\frac{1}{3}}$

APPLICATIONS AND PROBLEM SOLVING

65. Chemistry An electron moves about the nucleus of an atom at such a high speed that if it were visible to the eye, it would appear as a cloud. Identify the classical curve represented by the electron cloud below. *(Lesson 9-2)*



66. Surveying A surveyor identifies a landmark at the point with rectangular coordinates (75, 125). What are the polar coordinates of this point? *(Lesson 9-3)*

67. Navigation A submarine sonar is tracking a ship. The path of the ship is being coded as the equation $r \cos\left(\theta - \frac{\pi}{2}\right) + 5 = 0$. Find the rectangular equation of the path of the ship. *(Lesson 9-4)*

68. Electricity Find the current in a circuit with a voltage of $50 + 180\boldsymbol{j}$ volts and an impedance of $4 + 5\boldsymbol{j}$ ohms. *(Lesson 9-7)*

ALTERNATIVE ASSESSMENT

OPEN-ENDED ASSESSMENT

1. The simplest form of the sum of two complex numbers is $7 - 4\boldsymbol{i}$.
   a. Give examples of two complex numbers in which $a \neq 0$ and $b \neq 0$ with this sum.
   b. Are the two complex numbers you chose in part a the only two with this sum? Explain.

2. The absolute value of a complex number is $\sqrt{17}$.
   a. Give a complex number in which $a \neq 0$ and $b \neq 0$ with this absolute value.
   b. Is the complex number you chose in part a the only one with this absolute value? Explain.

Additional Assessment See p. A64 for Chapter 9 practice test.

Unit 3 interNET Project

SPACE—THE FINAL FRONTIER

Coordinates in Space—What do scientists use?

- Search the Internet to find types of coordinate systems that are used to locate objects in space. Find at least two different types.
- Write a summary that describes each coordinate system that you found. Compare them to rectangular and polar coordinates. Include diagrams illustrating how to use each coordinate system and any information you found about converting between systems.

PORTFOLIO

Choose one of the classical curves you studied in this chapter. Give the possible general forms of the polar equation for your curve and sketch the general graph. Then write and graph a specific polar equation.

Geometry Problems—Lines, Angles, and Arcs

SAT and ACT geometry problems often combine triangles and quadrilaterals with angles and parallel lines. Problems that deal with circles often include arcs and angles.

Review these concepts.
- Angles: vertical, supplementary, complementary
- Parallel Lines: transversal, alternate interior angles
- Circles: inscribed angle, central angle, arc length, tangent line

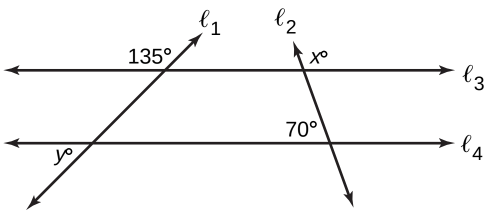
THE PRINCETON REVIEW

TEST-TAKING TIP

Use the square corner of a sheet of paper to estimate angle measure. Fold the corner in half to form a 45° angle.

SAT EXAMPLE

1. If four lines intersect as shown in the figure, $x + y =$



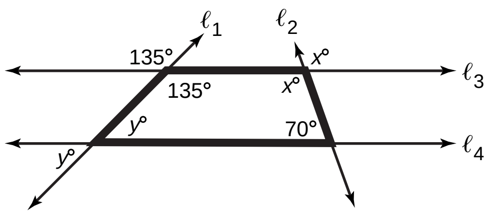
A 65 **B** 110

C 155 **D** 205

E It cannot be determined from the information given.

HINT Do not assume anything from a figure. In this figure, ℓ_3 and ℓ_4 look parallel, but that information is not given.

Solution Notice that the lines intersect to form a quadrilateral. The sum of the measures of the interior angles of a quadrilateral is 360°. Use vertical angles to determine the angles in the quadrilateral.



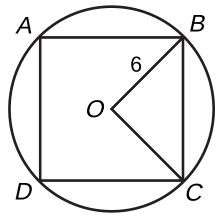
Write an equation for the sum of the angles.

$$135 + x + 70 + y = 360$$
$$205 + x + y = 360$$
$$x + y = 155$$

The answer is choice **C**.

ACT EXAMPLE

2. In the figure below, $ABCD$ is a square inscribed in the circle centered at O. If $\overline{OB}$ is 6 units long, how many units long is minor arc $\overset{\frown}{BC}$?



A $\frac{3\pi}{2}$ **B** 3π **C** 6π **D** 12π **E** 36π

HINT Consider *all* of the information given and implied. For example, a square's diagonals are perpendicular.

Solution Figure $ABCD$ is a square, so its angles are each 90°, and its diagonals are perpendicular. So $\angle BOC = 90°$. This means that minor arc $\overset{\frown}{BC}$ is one-fourth of the circle.

Look at the answer choices. They all include π. Use the formula for the circumference of a circle.

$$C = 2\pi r$$
$$= 2\pi(6) \text{ or } 12\pi$$

The length of minor arc $\overset{\frown}{BC}$ is one-fourth of the circumference of the circle.

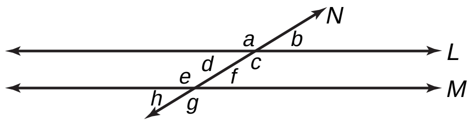
$$\frac{1}{4}C = \frac{1}{4}(12\pi) \text{ or } 3\pi$$

The answer is choice **B**.

After you work each problem, record your answer on the answer sheet provided or on a piece of paper.

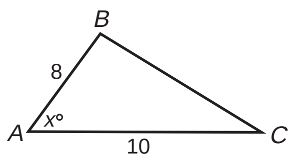
Multiple Choice

1. In the figure below, line L is parallel to line M. Line N intersects both L and M, with angles a, b, c, d, e, f, g, and h as shown. Which of the following lists includes all of the angles that are supplementary to $\angle a$?



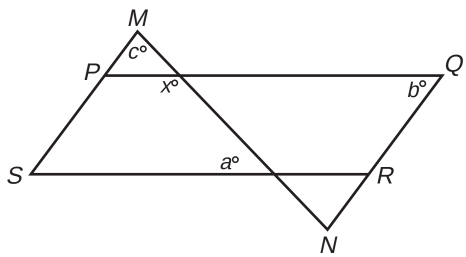
A b, d, f, h **B** c, e, g **C** b, d, c
D e, f, g, h **E** d, c, h, g

2. In the figure below, what is the area of $\triangle ABC$ in terms of x?



A $10 \sin x$ **B** $40 \sin x$ **C** $80 \sin x$
D $40 \cos x$ **E** $80 \cos x$

3. If $PQRS$ is a parallelogram and $\overline{MN}$ is a line segment, then x must equal



A $180 - b$ **B** $180 - c$ **C** $a + b$
D $a + c$ **E** $b + c$

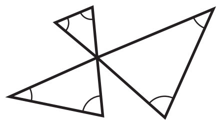
4. If a rectangular swimming pool has a volume of 16,500 cubic feet, a depth of 10 feet, and a length of 75 feet, what is the width of the pool, in feet?

A 22 **B** 26 **C** 32 **D** 110 **E** 1650

5. $\frac{1}{10^{100}} - \frac{1}{10^{99}} =$

A $\frac{-9}{10^{100}}$ **B** $\frac{-1}{10^{100}}$ **C** $\frac{1}{10^{100}}$ **D** $\frac{1}{10}$ **E** $\frac{9}{10}$

6. In the figure, what is the sum of the degree measures of the marked angles?

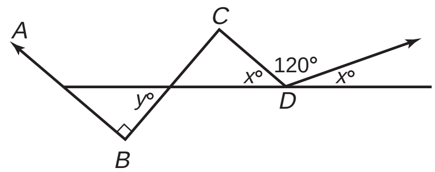


A 180 **B** 270 **C** 360 **D** 540
E It cannot be determined from the information given.

7. If $5x^2 + 6x = 70$ and $5x^2 - 6y = 10$, then what is the value of $10x + 10y$?

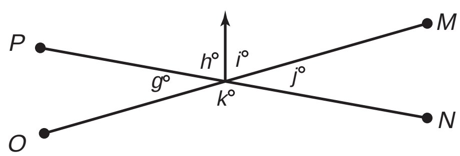
A 10 **B** 20 **C** 60 **D** 80 **E** 100

8. In the figure below, if $\overline{AB} \parallel \overline{CD}$, then what is the value of y? *Figure not drawn to scale.*



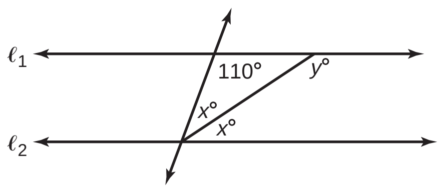
A 30 **B** 60 **C** 90 **D** 120 **E** 150

9. Which pair must be equal?



A h and i
B $(g + h)$ and $(i + j)$
C $(g + i)$ and $(h + j)$
D g and j
E $(g + j)$ and $(h + i)$

10. **Grid-In** If ℓ_1 is parallel to ℓ_2 in the figure below, what is the value of y?



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