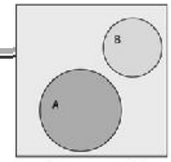
Chapter 15
General Probability Rules

Addition Rule



Two disjoint sets, A and B.

P (A or B)
Addition Rule

Are A and B Disjoint? — **Yes** → P (A or B) = P(A) + P(B)

No → P(A or B) = P(A) + P(B) – P(A and B)

Disjoint events cannot happen at the same time. They are separate, non-overlapping events

The First 3 Rules of Probability

- Rule #1:
  - Make a picture
- Rule #2:
  - Make a picture
- Rule # 3:
  - Make a picture
- You may recall that when we are dealing with data these are the same three rules! This doesn't change for probabilities. As the saying goes, "A picture is worth a thousand words."
- The most common picture in probability is a Venn diagram

The Addition Rule Revisited

- The following is the addition rule:
  - **P(A *or* B) =P(A U B) = P(A) + P(B)**
    - However, this only works with disjoint events.
- What happens if the events are not disjoint?
  - We need to use another formula:
  - **General Addition Rule**, for any 2 events, A and B:

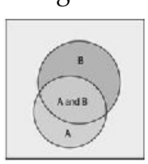
P(A *or* B) =P(A U B) = P(A) + P(B) – P(A ∩ B)

The Addition Rule Revisited

- Why?

P(A *or* B) =P(A U B) = P(A) + P(B) – P(A ∩ B)

- Well, if the events intersect and we apply the addition rule, we are adding the intersection twice. We need to get rid of our duplication.
- Will the general rule work if the events are disjoint? Why?



Conditional Probability

- Sometimes the knowledge that one event has occurred changes the probability that another event will occur.
  - ex: The probability of being in a car accident increases if you know that it is raining outside
  - ex: Suppose that the pass rate on the AP is exam is 80%. That is, for a randomly selected student, P(pass) = 0.80. However, if you know that the student got a D in the class, then the probability decreases 45%. That is, for a randomly selected student, P(pass given that you have a D) = 0.45.
- **New Notation:** P(pass | D) = 0.45
- When the probability of one event is conditioned upon the probability of another event, this is called **Conditional Probability**.
- The notation P(A | B) is the **probability that A will occur given that event B has occurred**.

Conditional Probability

- Probabilities in the form P(A | B) are called **conditional probabilities** and pronounced "the probability of A given B"
- To find the probability of the event **A** ***given*** the event **B**, we restrict our attention to the outcomes in **B**. We then find the fraction of ***those*** outcomes **A** that also occurred.

$$\mathbf{P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}} \text{ or } \mathbf{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

- Note: *P*(**B**) cannot equal 0, since we know that **B** has occurred.

Conditional Probability

- There were 2201 passengers on the Titanic. 367/1731 males survived and overall 711 survived.
- What is the best way to display this data?
  - The best way to display this data is in a **two-way table**. Recall that two-way tables help us to summarize the relationship between two categorical variables. It lists the outcomes of one variable down the left side, the outcomes of the other variable across the top, and the frequencies (or relative frequencies) in each cell.
- How do we display the relationship between two numerical variables?
  - A scatterplot.

Conditional Probability

	M	F	Total
Survived	367	**344**	711
Died	**1364**	**126**	**1490**
Total	1731	**470**	2201

Suppose you were to randomly select a name from the Titanic's passenger list. Find each of the following:

1. P(survived) = ?
2. P(male) = ?
3. P(female ∩ survived) = ?
4. P(survived | female) = ?
5. P(male | survived) = ?
6. P(died | female) = ?
7. P(male | died) = ?
8. P(female | survived) = ?

Drug Testing

- Suppose that in a certain company, 5% of the employees use drugs. The company decides to test all of its employees for drugs with a test that is 95% successful (it correctly identifies 95% of users as users and 95% of non-users as non-users).
- Let D = event that the employee uses drugs and P = event that the person tests positive.
  - Express the information given in symbolic notation.
  - Express the data in a tree diagram
  - If a person tests positive, what is the probability that they use drugs?
  - Does this result surprise you?

Drug Testing

- The false positive rate of a test is
  - P(positive | no drugs) =
- The false negative rate is
  - P(negative | drugs) =
  - Which of these errors is worse? For the employees? For the company?
  - If this were an AIDS test, which would be worse?

The Multiplication Rule Revisited

- We know that the multiplication rule, ***P*(A *and* B) = P(A ∩ B) = *P*(A) x *P*(B)**, only works if the events are independent. But what happens if the events are not independent?
- If the events are not independent, we need another rule.
  - The **General Multiplication Rule**:

P(A *and* B) = P(A) x P(B | A)

or

P(A *and* B) = P(B) x P(A | B)

The Multiplication Rule Revisited

- Will the General Multiplication Rule work if the events are independent?
  - Sure. If the events are independent, then the probability of B is not affected by the probability of A. Therefore, ***P*(B | A) = *P*(B)** or ***P*(A | B) = *P*(A)** if and only if the events are independent.

Independence ≠ Disjoint

- Disjoint events ***cannot*** be independent! Well, why not?
  - Since we know that disjoint events have no outcomes in common, knowing that one occurred means the other didn't.
  - Thus, the probability of the second occurring **changed** based on our knowledge that the first occurred.
  - It follows, then, that the two events are ***not*** independent because if one event occurs, it changes the probability of the other.
- A common error is to treat disjoint events as if they were independent and apply the Multiplication Rule for independent events—don't make that mistake.

Depending on Independence

- It's much easier to think about independent events than to deal with conditional probabilities.
  - It seems that most people's natural intuition for probabilities breaks down when it comes to conditional probabilities.
- Don't fall into this trap: **whenever you see probabilities multiplied together, stop and ask whether you think they are really independent.**

Drawing Without Replacement

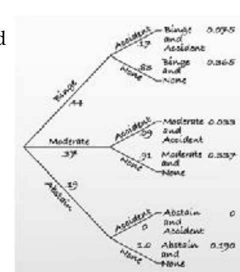
- Sampling **without replacement** means that once one individual is drawn it doesn't go back into the pool (this is a conditional event).
  - We often sample without replacement, which doesn't matter too much when we are dealing with a large population. Do you know why?
  - However, when drawing from a small population, we need to take note and adjust probabilities accordingly.
- Drawing without replacement is just another instance of working with conditional probabilities.

Tree Diagrams

- A **tree diagram** helps us think through conditional probabilities by showing sequences of events as paths that look like branches of a tree.
- Making a tree diagram for situations with conditional probabilities is consistent with our "make a picture" mantra.
- Consider the following example. There is a 44% chance that a college student is a binge drinker, 37% that they drink moderately, and 19% chance that they abstain from drinking alcohol. A study found that among binge drinkers, 17% are in alcohol involved accidents while only 9% of moderate drinkers are involved in accidents.
  1. What is the probability that a college student will be in an accident that involves a binge drinker?
  2. What is the probability of being a binge drinker and getting in an accident?
  3. If a student was in an accident, what is the probability that it was with a binge drinker?

Tree Diagrams (cont.)

- The following is a nice example of a tree diagram and shows how we multiply the probabilities of the branches together.
  - The first branch represents student habits with respect to drinking alcohol
  - The second branch represents alcohol related accidents – which, of course, are dependent on drinking habits
  - Each outcome can be determined by multiplying the probabilities of each branch



Reversing the Conditioning

- Reversing the conditioning of two events is rarely intuitive.
- Suppose we want to know $P(\mathbf{A} \mid \mathbf{B})$, but we know only $P(\mathbf{A})$, $P(\mathbf{B})$, and $P(\mathbf{B} \mid \mathbf{A})$.
- We also know $P(\mathbf{A}$ *and* $\mathbf{B})$, since:

P(A *and* B) = P(A) x P(B | A) or P(A and B) = P(B) x P(A | B)

- From this information, we can find $P(\mathbf{A} \mid \mathbf{B})$:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \text{ or } P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

- In the previous example, what is P(Binge | Accident)?

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.075}{0.075 + .033} = \frac{0.075}{.108} \approx .6944$$

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Reversing the Conditioning (cont.)

- What if we reverse the condition?
  - What is the probability of being in an accident if you are a binge drinker?
- When we reverse the probability from the conditional probability that you're originally given, you are actually using **Bayes's Rule.**
- Here is Bayes's Rule as a formal formula:

$$P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid B^C)P(B^C)}$$

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What Can Go Wrong?

- Don't use a simple probability rule where a general rule is appropriate:
  - Don't assume that two events are independent or disjoint without checking that they are.
- Don't find probabilities for samples drawn without replacement as if they had been drawn with replacement.
- Don't reverse conditioning naively.
- Don't confuse "disjoint" with "independent."

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What have we learned?

- The probability rules from the last lesson only work in special cases—when events are disjoint or independent.
- We now know the General Addition Rule and General Multiplication Rule which will always work.
- We also know about conditional probabilities and that reversing the conditioning can give surprising results.

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What have we learned? (cont.)

- Venn diagrams, tables, and tree diagrams help organize our thinking about probabilities.
- We now know more about independence—a sound understanding of independence will be important throughout the rest of this course.

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