There are many scenarios where probabilities are used to determine risk factors. Examples include Insurance, Casino, Lottery, Business, Medical, and other Sciences.

Random Variable is a variable whose value is a numerical outcome of a random phenomenon.

- Usually use **capital letters** at the end of the alphabet to denote a random variable like X or Y.
- A particular **value of a random variable** will be denoted with a **lower case letter**.

Two types of Random Variables:

1) **Discrete random variable** has a **finite number** of distinct outcomes
   Example: Number of books this term.
2) **Continuous random variable** can take on **any numerical value** within a range of values.
   Example: Cost of books this term.

Probability Model consists of all values, x, of a random variable, X, along with the probabilities for each value denoted $P(X = x)$.

Example: An insurance company offers a "death and disability" policy that pays $10,000 when you die or $5,000 if you are disabled. It charges a premium of only $50 a year for this benefit. Suppose the death rate is 1 one out of every 1,000 people, and that another 2 out of 1,000 suffer from some kind of disability. Is the company likely to make a profit selling such a plan?

- Come up with a probability model
- Find the expected value

The probability model for this insurance policy can be displayed as:

Policyholder Outcome	Payout x	Probability $P(X = x)$
Death	10,000	
Disability	5,000	
Neither	0	

The **Expected Value** of a random variable is the value we expect a random variable to take on or the theoretical long-run average value. It is denoted μ for population mean or $E(x)$ for expected value. The expected value for a discrete random variable is found by summing each value by its probability.

$$\mu = E(X) = \sum x \cdot P(X = x)$$

Note: Be sure that every possible outcome is included when finding $E(x)$ and verify that you have a valid probability model to start with.

Example: What is the expected value of the payout from the previous example?

$$\mu = E(X) = \sum x \cdot P(X = x)$$

$$= \$10{,}000\left(\qquad\right) + \$5{,}000\left(\qquad\right) + \$0\left(\qquad\right)$$

$$= \$20.$$

So our total payout comes to $20,000, or **$20 per policy**.

The **Variance** of a random variable is the expected value of the squared deviation from the mean. The variance of a discrete random variable is found by calculating

$$\sigma^2 = Var(X) = \sum (x - \mu)^2 \cdot P(X = x).$$

The **Standard Deviation** of a random variable is the square root of the variance.

$$SD(X) = \sqrt{Var(X)}.$$

Example: The expected value (or mean) is not what actually happens to any particular policyholder. No individual policy actually costs the company $20. Since we are dealing with random events, some policyholders receive big payouts, others nothing. Because the insurance company must anticipate this variability, it needs to know the standard deviation of the random variable.

Policyholder Outcome	Payout x	Probability $P(X = x)$	Deviation $(x - \mu)$
Death			
Disability			
Neither			

then compute for the standard deviation,

$$SD(X) = \sqrt{Var(X)}$$
$$= \sqrt{\quad\quad} \approx \$$$

The insurance company can expect an average payout of $ per policy, with a standard deviation of $.

More about Means and Variances:

- **Adding** or **subtracting a constant** to each value of a random variable **<u>shifts the mean</u>** but **<u>does not</u> change the variance** or **standard deviation:**

  $E(X \pm c) = E(X) \pm c$ $\quad\quad$ $Var(X \pm c) = Var(X)$

- **Multiplying** each value of a random variable by a constant **multiplies the mean by that constant** and the **variance by the square of the constant**:

  $E(aX) = aE(X)$ $\quad\quad$ $Var(aX) = a^2 Var(X)$

- The **mean of the sum** of two random variables is the **sum of the means**:

  $E(X + Y) = E(X) + E(Y)$

- The **mean of the difference** of two random variables is the **difference of the means**:

  $E(X - Y) = E(X) - E(Y)$

- If two random variables are independent, then **variance of their sum or difference** is <u>**always**</u> the **sum of the variances.**

  $Var(X \pm Y) = Var(X) + Var(Y)$

Continuous Variables

- Random variables that can take on **any value** in a range of values are called **continuous random variables.**
- Continuous random variables have means (expected values) and variances.
- We won't worry about how to calculate these means and variances in this course, but we can still work with models for continuous random variables when we're given the parameters.
- Nearly everything we've said about how discrete random variables behave is true of continuous random variables, as well.
- When two independent continuous random variables have Normal models, so does their sum or difference.
- This fact will let us apply our knowledge of Normal probabilities to questions about the sum or difference of independent random variables.

STEP-BY-STEP Example, p. 378: Packaging Stereos

A company manufactures small stereo systems. At the end of the production line the stereos are packaged and prepared for shipping. In stage 1, called "Packing", workers collect all the system components, put each in plastic bags and then place everything inside a protective Styrofoam form. The packed form then moved onto stage 2, called "Boxing". There, workers place the form and packing instructions in a cardboard box, close, seal and label for shipping. The company says that times required for the packing stage can be described by a Normal model with a mean of 9 minutes and a standard deviation of 1.5 minutes. The times for the boxing stage can also be modeled as Normal, with a mean of 6 minutes and a standard deviation of 1 minute.

a) What is the probability that packing two consecutive systems takes over 20 minutes?

Let P_1 = time for packing the first system
P_2 = time for packing the second
T = total time to pack two systems
$= P_1 + P_2$

✓**Normal Model Assumption:** We are told that both random variable follow Normal models.

✓ **Independence Assumption:** We can reasonably assume that the two packing times are independent.

Find the expected value:

$$E(T) = E(P_1 + P_2)$$

$=$

$=$ **minutes**

Since the times are independent,

$$Var(T) = Var(P_1 + P_2)$$

$=$

$=$

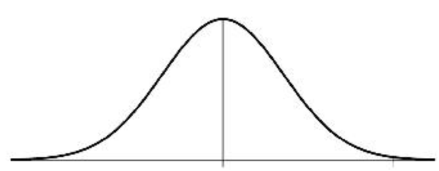
$=$

$SD(T) =$ $\approx$ **minutes**

Use a model T with N (,).

$z = \frac{}{} =$

$P(T > 20) = P(z \quad) =$



OR *normalcdf* ( ) =

There's a little than % chance that it will take a total of over 20 minutes to pack two consecutive stereo systems.

b) What percentage of the stereo systems take longer to pack than to box? (Hint: Let ***D*** = the difference in times to pack and box a system)

***P* =** time for packing a system
***B* =** time for boxing a system
***D* =** difference in times to pack and box a system
= *P* – *B*

The probability that it takes longer to pack than to box a system is the probability that the difference ***P* – *B*** is greater than zero.

✓**Normal Model Assumption:** We are told that both random variable follow Normal models.

✓ **Independence Assumption:** We can reasonably assume that the two packing times are independent.

Find the expected value:

$E(D) = E(P - B)$

$=$

$=$ $=$ **minutes**

Since the times are independent,

$Var(D) = Var(P - B)$

$= Var(P) \quad Var(B)$

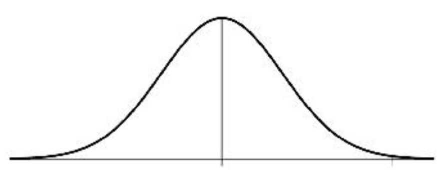
$=$

$=$

$SD(D) =$ $\approx$ **minutes**

Use a model ***D*** with ***N*** (,).

$z = \frac{\quad}{\quad} =$



$P(D > 0) = P(z \qquad) =$

OR *normalcdf* ( ) =

About % of all stereo systems will require more time for packing than for boxing.

Example, p. 368: Love and expected values

On Valentine's Day the *Quiet Nook* restaurant offers a *Lucky Lovers Special* that could save couples money on their romantic dinners. When the waiter brings the check, he'll also bring the four aces from a deck of cards. He'll shuffle them and lay them face down on the table. The couple will then get to turn one card over. If it's a black ace, they'll owe the full amount, but if it's the ace of hearts, the waiter will give them a $20 *Lucky Lovers* discount. If they first turn over the ace of diamonds, they'll then get to turn over one of the remaining cards, earning a $10 discount for finding the ace of hearts this time.

Based on a probability model for the size of the *Lucky Lovers* discounts the restaurant will award, what's the expected discount for the couple?

Outcome	A♥	A♦, then A♥	Black Ace
x	20	10	0
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{2}{3}$

$$E(X) = \sum x \bullet P(X = x)$$

Couples dining at the *Quiet Nook* can expect an average discount of $.

JUST CHECKING, p. 368: One of the authors of this book took his minivan in for repair recently because the air conditioner was cutting out intermittently. The mechanic identified the problem as dirt in a control unit. He said that in about 75% of such cases, drawing down and then recharging the coolant a couple of times cleans up the problem-and costs only $60. If that fails, then the control unit must be replaced at an additional cost of $100 for parts and $40 for labor.

a) Define the random variable and construct the probability model.

Outcome	**X = cost**	**Probability**
Recharging works	$60	0.75
Replace control unit	$200	0.25

b) What is the expected value of the cost of this repair?

$$E(X) = \sum x \bullet P(X = x)$$

c) What does that mean in context?

Car owners with this problem will .

Example, p. 370: Finding the standard deviation

Using the probability model for the *Lucky Lovers* restaurant discount, we found that couples can expect an average discount of μ = $5.83. What is the standard deviation of the discounts?

Outcome	A♥	A♦, then A♥	Black Ace
x	20	10	0
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{2}{3}$

$$\pm^2 = Var(X) = \sum (x - \text{«})^2 \bullet P(X = x)$$

then compute for the standard deviation,

$$SD(X) = \sqrt{Var(X)}$$

Couples can expect the *Lucky Lovers* discount to average $, with a standard deviation of $.

STEP-BY-STEP Example, p. 370: Expected Values and Standard Deviations for Discrete Random Variables

As the head of inventory for Knowway computer company, you were thrilled that you had managed to ship 2 computers to your biggest client the day the order arrived. You are horrified, though, to find out that someone had restocked refurbished computers in with the new computers in your storeroom. The shipped computers were randomly selected from the 15 computers in stock, but 4 of those were actually refurbished. If your client gets 2 new computers, things are fine. If the client gets one refurbished computer, it will be sent back at your expense ($100) and you can replace it. However, if both computers are refurbished, the client will cancel the order this month and you will lose $1000.

a) Find the probability model for the amount of loss.

Outcome	x	$P(X = x)$
Two refurbs	1000	$P(RR) =$
One refurbs	100	$P(RN \cup NR) =$
New / new	0	$P(NN) =$

b) Compute the expected value and the standard deviation of the company's loss.

$$E(X) = \sum x \bullet P(X = x)$$

$$Var(X) = \sum (x - \text{«})^2 \bullet P(X = x)$$

$$SD(X) = \sqrt{Var(X)}$$

I expect this mistake to cost the firm $, with a standard deviation of $. The large standard deviation reflects the fact that there's

More about Means and Variances

Example, p. 372: Adding a constant

We've determined that couples dining at the *Quiet Nook* can expect *Lucky Lovers* discounts averaging $5.83 with a standard deviation of $8.62. Suppose that for several weeks the restaurant has also been distributing coupons worth $5 off any one meal (one discount per table).

If every couple dining there on Valentine's Day brings a coupon, what will be the mean and standard deviation of the total discounts they'll receive?

Let D = total discount (Lucky Lovers plus the coupon); then $D = X + 5$

$$E(D) = E(X + 5) =$$

$$Var(D) = Var(X + 5) =$$

$$SD(D) = \sqrt{Var(X)} =$$

Couples with the coupon can expect the total discounts averaging $. The standard deviation is .

Example, p. 373: Double the love

At the *Quiet Nook* on Valentine's Day, when two couples dine together on a single check, the restaurant doubles the discount offer-$40 for the ace of hearts on the first card and $20 for the second. What are the mean and standard deviation of discounts for such a group of four?

$$E(2X) = 2E(X) =$$

$$Var(2X) = 2^2 Var(X) =$$

$$SD(2X) =$$

If the restaurant doubles the discount offer, two couples dining together can expect to save an average of $ with a standard deviation of $.

Example, p. 374: Adding the discounts

From the previous example, a group of four paying on one check can expect to save $11.66 with a standard deviation of $17.24. Some couples decide instead to get separate checks and pool their two discounts.

You and your amour go to this restaurant with another couple and agree to share any benefit from this promotion. Does it matter whether you pay separately or together?

Let X_1 and X_2 represent the two separate discounts, and T the total; then $T = X_1 + X_2$

$$E(T) = E(X_1 + X_2) =$$

So the expected savings is .

The cards are reshuffled for each couple's turn, so the discounts couples receive are independent. It's okay to add the variances.

$$Var(T) = Var(X_1 + X_2) =$$

$$SD(T) =$$

When two couples get separate checks, . The standard deviation is \$, compared to \$17.24 for couples who play for the double discount on a single check. .

Example, p. 375: Working with differences

From the previous example, we know the *Lucky Lovers* discount at the *Quiet Nook* averages \$5.83 with a standard deviation of \$8.62. Just up the street, the *Wise Fool* restaurant has a competing *Lottery of Love* promotion. There a couple can select a specially prepared chocolate from a large bowl and unwrap it to learn the size of their discount. The restaurant's manager says the discounts vary with an average of \$10.00 and a standard deviation of \$15.00.
How much more can you expect to save at the Wise Fool? With what standard deviation?

Let ***W*** = discount at the *Wise Fool*, ***X*** = discount at the *Quiet Nook*, and ***D*** = the difference: ***D*** = ***W*** – ***X***
These are different promotions at separate restaurants, so the outcomes are independent.

$$E(W - X) =$$

$$SD(W - X) = \sqrt{Var(W - X)}$$

$$=$$

$$=$$

$$=$$

Discounts at the *Wise Fool* will average \$ than at the *Quiet Nook*, with a standard deviation of \$.

Example, p. 375: Summing a series of outcomes

The *Quiet Nook* is planning to serve 40 couples on Valentine's Day. What's the expected total of the discounts the owner will give? With what standard deviation?

Let $X_1, X_2, X_3, X_4, \ldots, X_{40}$ represent the discounts to the 40 couples, and ***T*** the total of all discounts; Then

$$T = X_1 + X_2 + X_3 + \cdots + X_{40}$$

$$E(T) = E(X_1 + X_2 + X_3 + \cdots + X_{40})$$

$$=$$

$$=$$

$$= \$$$

Reshuffling cards between couples makes the discount independent, so;

$$SD(T) = \sqrt{Var(X_1 + X_2 + X_3 + \cdots + X_{40})}$$

$$E(T) =$$

$$=$$

$$= \$$$

The restaurant owner can expect 40 couples to win discounts totaling \$, with a standard deviation of \$.

JUST CHECKING, p. 376: Suppose the time it takes a customer to get and pay for seats at the ticket window of a baseball park is a random variable with a mean of 100 seconds and a standard deviation of 50 seconds. When you get there you find only two people in line in front of you.

a) How long do you expect to wait for your turn to get tickets?

$$E(X) =$$

$$= \quad \textbf{seconds}$$

b) What's the standard deviation of your wait time?

$$SD(X) = \sqrt{}$$

$$= \quad \textbf{seconds}$$

c) What assumption did you make about the two customers in finding the standard deviation?

The times for the two customers are .